Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 132457, 17 pages
doi:10.1155/2011/132457

Research Article

Steady Thermal Analysis of Two-Dimensional Cylindrical Pin Fin with a Nonconstant Base Temperature

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Received 31 March 2011; Revised 13 June 2011; Accepted 28 June 2011

Academic Editor: Jerzy Warminski

Steady heat transfer through a pin fin is studied. Thermal conductivity, heat transfer coefficient, and the source or sink term are assumed to be temperature dependent. In the model considered, the source or sink term is given as an arbitrary function. We employ symmetry techniques to determine forms of the source or sink term for which the extra Lie point symmetries are admitted. Method of separation of variables is used to construct exact solutions when the governing equation is linear. Symmetry reductions result in reduced ordinary differential equations when the problem is nonlinear and some invariant solution for the linear case. Furthermore, we analyze the heat flux, fin efficiency, and the entropy generation.

1. Introduction

Fins play an important role in increasing the efficiency of heating systems which is achieved by increased (extended) surface area. In particular, fins are used in power generators, air conditioning, semiconductors, refrigeration, cooling of computer processor, exothermic reactors, and many other devices in which heat is generated and must be transported. Theory, solutions and applications of problems on extended surfaces may be found in texts such as [1].

One-dimensional steady state numerical analysis have been considered, for example, by [2–5], and exact solutions were constructed via symmetry techniques in [6]. The transient one-dimensional fin problem has attracted sizeable interest from the Lie symmetry analysts (see, e.g., [7–11]). Two-dimensional transient analysis have been carried out for fins without

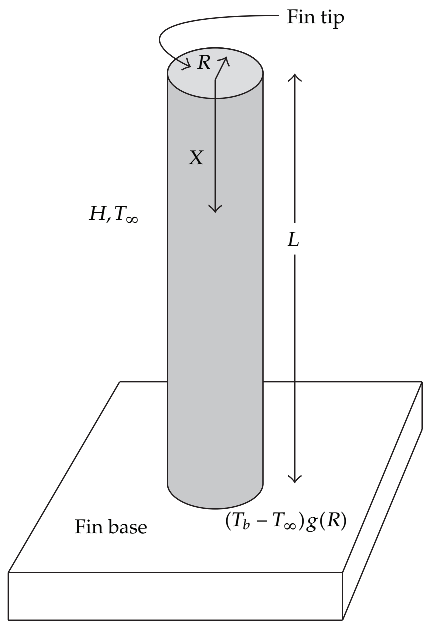

Figure 1: Schematic representation of pin fin.

heat source or sink in [12]. Solutions for two-dimensional fin models exist for the constant thermal conductivity (see e.g., [13–16]). Recently, heat transfer and entropy generation in two-dimensional orthotropic pin fin has been studied in [17]. In [18], the authors combined the Laplace transformation and the finite difference methods to determine solutions for the two-dimensional pin fins with nonconstant base heat flux. A search for exact and numerical solutions for heat transfer in extended surfaces continues to be of scientific interest. perhaps the interest is instilled by frequent encounters in many engineering applications. To cite a few, some other contributions of heat flow particularly in pin fins may be found, for example, in [19, 20].

In this paper, we consider the steady heat flow through a two-dimensional pin fin with a temperature-dependent internal heat generating or extracting function and thermal conductivity. Furthermore, heat is transferred at the boundary through the temperature-dependent heat transfer coefficient. Symmetry analysis is employed to determine all possible forms of the source or sink term for which the problem is at least reducible to ordinary differential equations. The paper is arranged as follows: in Section 2, we provide mathematical formulation of the problem. Section 3 deals with the symmetry analysis, and exact solutions are constructed in Section 4. The fin efficiency and heat flux are given in Section 5. Entropy analysis is carried out in Section 6. Lastly, we provide the discussions in Section 7 and concluding remarks in Section 8.

2. Mathematical Formulation

We consider a two-dimensional pin fin with length L and radius R. The fin is attached to a base surface of temperature $(T_b - T_\infty)g(R)$ and extended into the fluid of temperature T_∞. The tip of the fin is insulated (i.e., heat transfer at the tip is negligibly small). The fin is measured from the tip to the base. A schematic representation of a pin fin is given in Figure 1. We assume that the heat transfer coefficient along the fin is nonuniform and temperature dependent and that the fin has internal heat source or sink. Furthermore, the temperature-dependent thermal conductivity is assumed to be the same in both radial and axial directions. The base temperature is assumed to be nonconstant (see, e.g., [19, 20]). The energy balance equation is written as

$$\frac{1}{R}\frac{\partial}{\partial R}\left(RK(T)\frac{\partial T}{\partial R}\right)+\frac{\partial}{\partial X}\left(K(T)\frac{\partial T}{\partial X}\right)=S(T). \tag{2.1}$$

The boundary conditions are

$$\begin{gathered}\frac{\partial T}{\partial X}=0, \quad X=0, \qquad 0\leq R\leq R_a,\\ T-T_\infty=(T_b-T_\infty)g(R), \qquad X=L, \qquad 0\leq R\leq R_a,\\ \frac{\partial T}{\partial R}=0, \quad R=0, \qquad 0\leq X\leq L,\\ K(T)\frac{\partial T}{\partial R}=-H(T)[T-T_\infty], \quad R=R_a, \qquad 0\leq X\leq L.\end{gathered} \tag{2.2}$$

Here, R_a is radial distance from the center to the surface of the pin fin.

Introducing the dimensionless variables

$$\begin{gathered}\theta=\frac{T-T_\infty}{T_b-T_\infty}, \qquad x=\frac{X}{L}, \qquad r=\frac{R}{R_a}, \qquad k=\frac{K}{K_a}, \qquad E^2=\left(\frac{L}{R_a}\right)^2,\\ s(\theta)=\frac{L^2S(T)}{K_a(T_b-T_\infty)}, \qquad h=\frac{H}{h_b},\end{gathered} \tag{2.3}$$

we obtain

$$E^2\frac{1}{r}\frac{\partial}{\partial r}\left(rk(\theta)\frac{\partial\theta}{\partial r}\right)+\frac{\partial}{\partial x}\left(k(\theta)\frac{\partial\theta}{\partial x}\right)=s(\theta), \tag{2.4}$$

and the boundary conditions are

$$\begin{gathered}\frac{\partial\theta}{\partial x}=0, \quad x=0,\\ \theta=g(r), \qquad x=1,\\ \frac{\partial\theta}{\partial r}=0, \quad r=0,\\ k(\theta)\frac{\partial\theta}{\partial r}=-Bih(\theta)\theta, \quad r=1,\end{gathered} \tag{2.5}$$

where E is the fin extension factor, h_b is the fin base heat transfer coefficient, and Bi is the Biot number given by

$$Bi = \frac{h_b R_a}{K_a}. \tag{2.6}$$

Three physically realistic functions of thermal conductivity are (i) K linearly dependent on temperature, (ii) the power law case, and (iii) the exponential case. We focus on two cases: (i) $K(T)$ depending linearly on temperature; that is,

$$K(T) = K_a\left(1 + \beta(T - T_\infty)\right), \tag{2.7}$$

and (ii) $K(T)$ is given by the power law (nonlinear)

$$K(T) = K_a\left(\frac{T - T_\infty}{T_b - T_\infty}\right)^n. \tag{2.8}$$

In dimensionless variables, we have

$$\text{(i)}\ k(\theta) = 1 + B\theta, \qquad \text{(ii)}\ k(\theta) = \theta^n, \tag{2.9}$$

where $B = \beta(T_b - T_\infty)$. Here, K_a is the thermal conductivity of the fin at ambient temperature, β is the thermal conductivity gradient, n is an exponent, and B is the thermal conductivity parameter. Applying the Kirchoff's transformation (see, e.g., [16]),

$$w(x, r) = \int_0^\theta k(\theta_*)\,d\theta_*. \tag{2.10}$$

The boundary value problem reduces to

$$E^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial x^2} = s(w), \tag{2.11}$$

$$\frac{\partial w}{\partial x} = 0, \quad x = 0, \tag{2.12}$$

$$w(1, r) = g(r) + \frac{B}{2}\left(g(r)\right)^2 = F(r), \quad \text{given } k = 1 + B\theta, \tag{2.13}$$

or

$$w(1, r) = \frac{g(r)^{n+1}}{n+1} = G(r), \quad \text{given } k = \theta^n, \quad n \neq -1, \tag{2.14}$$

$$\frac{\partial w}{\partial r} = 0, \quad r = 0, \tag{2.15}$$

$$\frac{\partial w}{\partial r} = -Bi\,w, \quad r = 1, \tag{2.16}$$

where in

$$h(\theta) = 1 + \frac{B}{2}\theta, \quad \text{when } k(\theta) = 1 + B\theta,$$
$$h(\theta) = \frac{\theta^n}{n+1}, \quad \text{when } k(\theta) = \theta^n, \quad n \neq -1. \tag{2.17}$$

Note that $w = \ln\theta$ when $n = -1$ so that (2.4) holds.

3. Classical Lie Point Symmetry Analysis

In brief, a symmetry of a differential equation is an invertible transformation of the dependent and independent variables that leaves the original equation invariant (unchanged). Furthermore, a symmetry of differential equations maps an arbitrary solution to another solution of the same differential equation. Symmetries depend continuously on a parameter and form a group, the *one-parameter group of transformations*. The classical Lie groups of point invariance transformations act on the system's graph space that is coordinated by the independent and dependent variables. This group may be determined algorithmically (by Lie's classical method), and there are a number of computer algebraic packages developed to construct symmetries (see, e.g., [21–23]), or one may use interactive packages such as REDUCE [24].

Differential equations arising in modelling real-world problems often involve one or more functions depending on either the independent variable or on the dependent variables. It is possible by symmetry techniques to determine the cases which allow the equation in question to admit extra symmetries. The exercise of searching for the forms of arbitrary functions for which extra symmetries are admitted is called *group classification*. The notion of group classification is pioneered by Lie [25].

The theory and applications of symmetry analysis may be found in excellent text such as those of [26–29]. We adopt the direct methods in [27] (which exclude explicit equivalence transformation analysis) to determine possible forms of the source term for which (2.11) admits extra point symmetries.

In essence, determining classical Lie point symmetries for the governing Equation (2.11) implies seeking transformation of the form

$$\begin{aligned} r_* &= r + \epsilon\xi^1(r,x,w) + O(\epsilon^2), \\ x_* &= x + \epsilon\xi^2(r,x,w) + O(\epsilon^2), \\ w_* &= w + \epsilon\eta(r,x,w) + O(\epsilon^2), \end{aligned} \tag{3.1}$$

generated by the vector field

$$\Gamma = \xi^1(r,x,w)\frac{\partial}{\partial r} + \xi^2(r,x,w)\frac{\partial}{\partial x} + \eta(r,x,w)\frac{\partial}{\partial w}, \tag{3.2}$$

which leaves the governing equation invariant. Note that we seek symmetries that leave the single (2.11) invariant rather than the entire boundary value problem. This is because the number of symmetries admitted by the governing equation and the imposed boundary conditions is less than for those admitted by the single governing equation. One may apply the boundary condition to the obtained invariant solutions.

The action of Γ is extended to all the derivatives appearing in the governing equation through the second prolongation

$$\Gamma^{[2]} = \Gamma + \zeta^{x}\frac{\partial}{\partial w_x} + \zeta^{r}\frac{\partial}{\partial w_r} + \zeta^{xx}\frac{\partial}{\partial w_{xx}} + \zeta^{rr}\frac{\partial}{\partial w_{rr}} + \cdots, \tag{3.3}$$

where

$$\begin{aligned}\zeta^{x} &= D_x(\eta) - w_r D_x(\xi^1) - w_x D_x(\xi^2),\\ \zeta^{r} &= D_r(\eta) - w_r D_r(\xi^1) - w_x D_r(\xi^2),\\ \zeta^{xx} &= D_x(\zeta^{x}) - w_{xr} D_x(\xi^1) - w_{xx} D_x(\xi^2),\\ \zeta^{rr} &= D_r(\zeta^{r}) - w_{rr} D_r(\xi^1) - w_{rx} D_r(\xi^2),\end{aligned} \tag{3.4}$$

with

$$\begin{aligned}D_r &= \frac{\partial}{\partial r} + w_r\frac{\partial}{\partial w} + w_{rr}\frac{\partial}{\partial w_r} + w_{xr}\frac{\partial}{\partial w_x} + \cdots,\\ D_x &= \frac{\partial}{\partial x} + w_x\frac{\partial}{\partial w} + w_{xx}\frac{\partial}{\partial w_x} + w_{xr}\frac{\partial}{\partial w_r} + \cdots,\end{aligned} \tag{3.5}$$

being the operators of total derivatives. The generator Γ is a Lie point symmetry of (2.11), if

$$\Gamma^{[2]}(\text{Equation (2.11)})\big|_{\text{Equation (2.11)}} = 0. \tag{3.6}$$

The invariance condition (3.6) yields the determining equations

$$-E^2\frac{1}{r^2}w_r\xi^1 + E^2\frac{1}{r}\zeta^{r} + E^2\zeta^{rr} + \zeta^{xx} = \eta s'(w), \tag{3.7}$$

on solutions of (2.11). Here prime implies differentiation with respect to w. Since the coefficient of Γ (the infinitesimals) does not involve the derivatives of the dependent variable, we can separate (3.7) with respect to these derivatives and solve the resulting overdetermined system of linear homogeneous partial differential equations. Further calculations are omitted at this stage since they were facilitated by the freely available interactive computer algebra package REDUCE [24].

In the initial symmetry analysis of (2.11) where s is arbitrary, we obtained nothing beyond translation in x. The cases of the sink or source term for which the principal Lie algebra is extended are listed in Table 1. Wherever they appear in Table 1, a, b, p, and q are arbitrary constants.

Table 1: Extra point symmetries admitted by (2.11).

$s(w)$	Symmetries
0	$\Gamma_2 = -(x\partial_x + r\partial_r)$, $\Gamma_3 = w\partial_w - 2x\partial_x - 2r\partial_r$, $\Gamma_4 = xw\partial_w + (r^2 - x^2)\partial_x - 2xr\partial_r$, $\Gamma_5 = \digamma(r,x)\partial_w$, where $\digamma_{rr} + (\digamma_r/r) + \digamma_{xx} = 0$.
w	$\Gamma_2 = -w\partial_w$, $\Gamma_3 = \digamma(r,x)\partial_w$, where $\digamma_{rr} + \digamma_r/r + \digamma_{xx} = 0$.
$w^p, p \neq 5$	$\Gamma_2 = 1/(p-5)(2w\partial_w + (1-p)x\partial_x + (1-p)r\partial_r)$
$p = 5$	$\Gamma_3 = -xw\partial_w + (x^2 - r^2)\partial_x + 2xr\partial_r$
e^{pw}	$\Gamma_2 = (1/p)\partial_w - (x/2)\partial_x - (r/2)\partial_r$
$(a+bw)^q$	$\Gamma_2 = 1/(q-5)((2(a+bw)/n)\partial_w + (1-q/q-5)x\partial_x + ((1-q)/(q-5))r\partial_r)$
$q = 5$	$\Gamma_3 = -(r(a+bw)/2n)\partial_w + ((x^2-r^2)/2)\partial_x + xr\partial_r$

4. Exact Solutions

In this section we construct exact solutions, first using method of separation of variables and secondly using symmetry techniques. Note that $s = 0$ and $s = w$ renders (2.11) linear and hence solvable by method of separation of variable. In fact, the solutions have been constructed, for example, in [16] when $s = 0$ (source or sink term is neglected).

4.1. Exact Solutions by Separation of Variables

We consider (2.11) with a linear source term. Three cases arise for the separation constant, σ. Note that $\sigma = 0$ leads to trivial solutions.

4.1.1. Case $\sigma = -\lambda^2$, $\lambda > 0$

Exact solution to (2.11) is given by

$$w(x,r) = \sum_{m=1}^{\infty} d_m \cosh(\lambda_m x) J_0(\delta_m r), \tag{4.1}$$

where eigenvalues λ_m satisfy

$$\delta_m J_1(\delta_m) = -Bi\, J_0(\delta_m), \tag{4.2}$$

with $\delta_m = \sqrt{\lambda_m^2 - 1}/E$, $\lambda_m^2 \neq 1$, for all $m = 1, 2, 3, \ldots$, and

$$d_m = \frac{\int_0^1 rF(r) J_0(\lambda_m r)\,dr}{\cosh(\lambda_m) \int_0^1 r J_0^2(\lambda_m r)\,dr}, \tag{4.3}$$

given boundary condition (2.13); otherwise, $F(r)$ may simply be replaced by $G(r)$ if condition (2.14) is given. Here, J_0 and J_1 are Bessel functions of order 0 and 1, respectively [30]. Solution (4.1) is depicted in Figures 2, 3, 4, 5, 6, 7, 8, and 9. In Figures 2–5, we have plotted solution (4.1) given thermal conductivity which is linearly dependent on temperature, whereas in

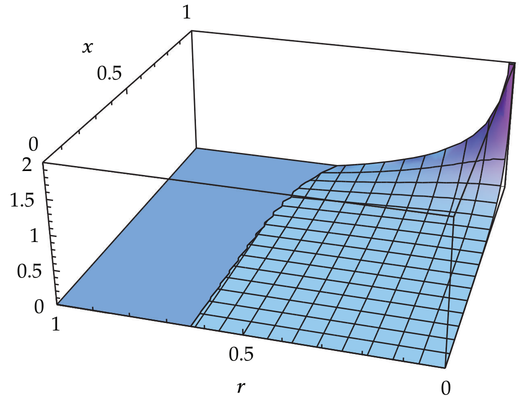

Figure 2: Temperature distribution using 550 terms of solution (4.1) with $E = 2, Bi = 0.2$ and $g(r) = r^2$. Here, k is linear in θ.

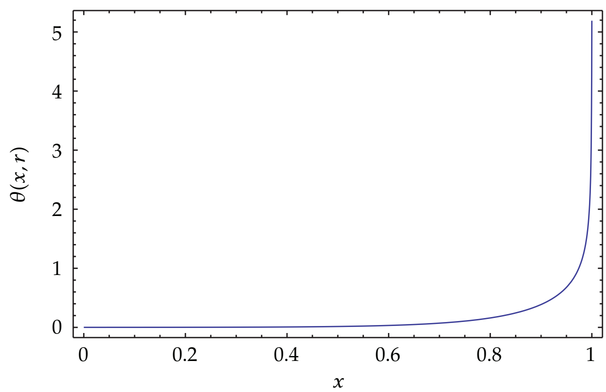

Figure 3: Temperature profile at $r = 0$ using 550 terms of the series (4.1) with $E = 2$, $Bi = 0.2$ and $g(r) = r^2$. Here, k is linear in θ.

Figures 6–9, we considered a power law thermal conductivity. Note that for all $\lambda_m^2 = 1$, we obtain trivial solutions.

4.1.2. Case $\sigma = \lambda^2$, $\lambda > 0$

In this case, we obtain the exact solution

$$w(x,r) = \sum_{m=1}^{\infty} c_m \cos(\lambda_m x) I_0(\alpha_m r), \tag{4.4}$$

where eigenvalues λ_m satisfy

$$\alpha_m I_1(\alpha_m) = -Bi I_0(\alpha_m), \tag{4.5}$$

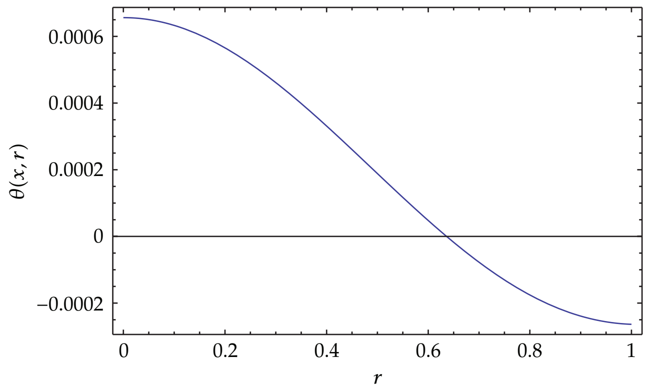

Figure 4: Temperature profile at $x = 0$ using 550 terms of the series (4.1). Here, we used parameters $E = 2$, $Bi = 0.2$ and $g(r) = r^2$. Here, k is linear in θ.

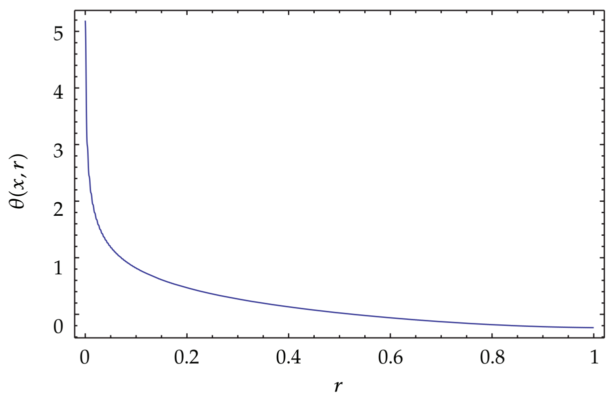

Figure 5: Temperature profile at $x = 1$ using 550 terms of the series (4.1) with $E = 2$, $Bi = 0.2$ and $g(r) = r^2$. Here, k is linear in θ.

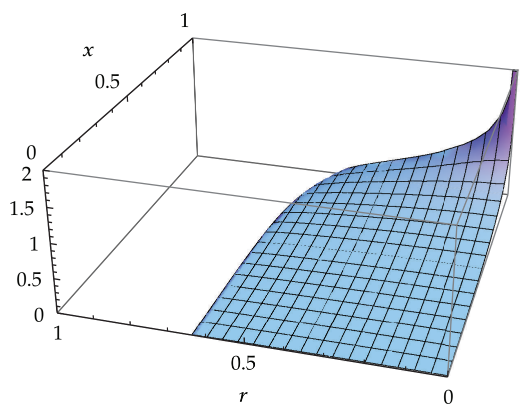

Figure 6: Temperature distribution using 550 terms of solution (4.1) with $E = 2$, $Bi = 0.2$ and $g(r) = r^2$. Here, k is given by the power law.

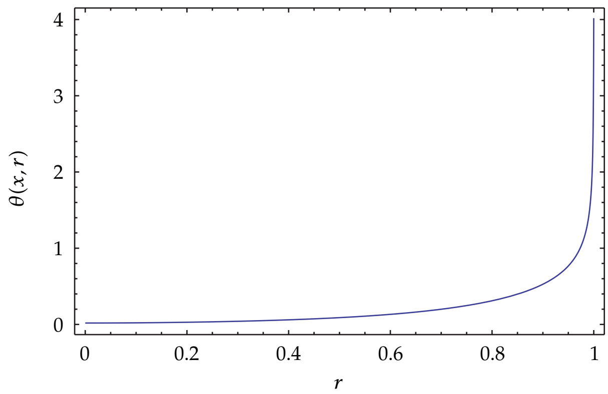

Figure 7: Temperature profile at $r = 0$ using 550 terms of the series (4.1) with $E = 2$, $Bi = 0.2$ and $g(r) = r^2$. Here, k is given by the power law.

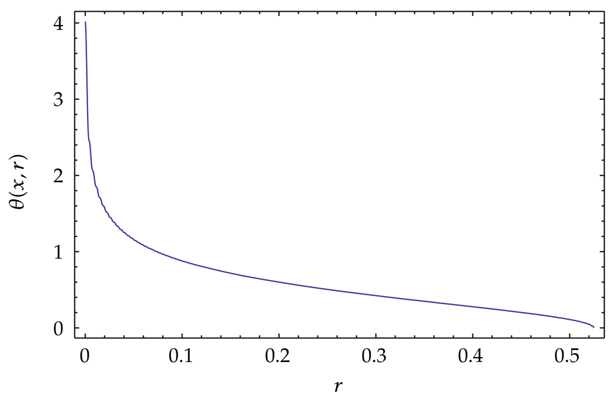

Figure 8: Temperature profile at $x = 1$ using 550 terms of the series (4.1) with $E = 2$, $Bi = 0.2$ and $g(r) = r^2$. Here, k is given by the power law.

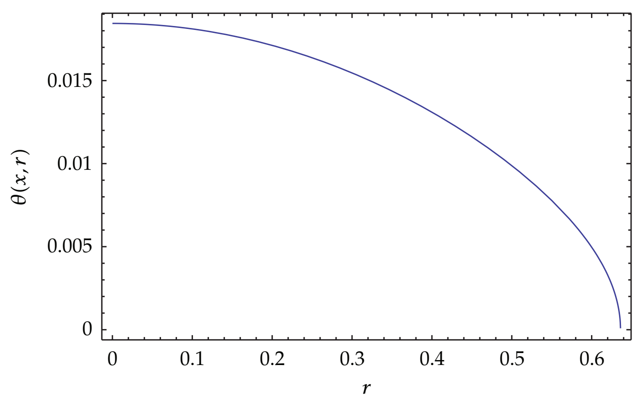

Figure 9: Temperature profile at $x = 0$ using 550 terms of the series (4.1). Here, we used parameters $E = 2$, $Bi = 0.2$ and $g(r) = r^2$. Here, k is given by the power law.

with $\alpha_m = \sqrt{1 + \lambda_m^2 / E}$ and

$$c_m = \frac{\int_0^1 rF(r) I_0(\lambda_m r) dr}{\cos(\lambda_m) \int_0^1 r I_0^2(\lambda_m r) dr}. \tag{4.6}$$

Here, I_0 and I_1 are modified Bessel functions of order 0 and 1, respectively [30]. We omit further analysis of solution 25, since similar observations to solution 24 are obtained. Note that in terms of the original variables, we obtain solutions

$$\begin{gathered} \theta = \frac{-1 \pm \sqrt{1 + 2Bw}}{B} \quad \text{given } k(\theta) = 1 + B, \\ \theta = w^{1/(n+1)} \quad \text{given } k(\theta) = \theta^n, \; n \neq -1, \\ \theta = e^w n = -1. \end{gathered} \tag{4.7}$$

with w given in Sections 4.1.1 and 4.1.2.

4.2. Symmetry Reductions and Invariant Solutions

If a differential equation is invariant under a Lie point symmetry, then one can reduce the order of the ordinary differential equation or the number of variables of the partial differential equation by one. The reduced equation may or may not be solved exactly. The exact (similarity) solutions obtained via symmetries are referred to as invariant solutions. In this section, we consider two cases as illustrative examples.

4.2.1. Linear Sink Term

We consider the linear combination of the symmetry generators Γ_1 and Γ_2; namely,

$$\Gamma_1 \pm \alpha_1 \Gamma_2 = \frac{\partial}{\partial x} \pm \alpha_1 w \frac{\partial}{\partial w}, \quad \alpha_1 \neq 0. \tag{4.8}$$

The basis for the invariants is constructed by the corresponding characteristic equations in Pfaffian form

$$\frac{dw}{w} = \pm\alpha_1 \frac{dx}{1} = \frac{dr}{0}. \tag{4.9}$$

Thus, we obtain the functional form of the solution

$$w = e^{\pm\alpha_1 x} U(r), \tag{4.10}$$

where $U(r)$ satisfy

$$E^2U'' + E^2\frac{1}{r}U' + \left(\pm\alpha_1^2 - 1\right)U = 0, \tag{4.11}$$

where prime indicates the derivative with respect to r. The general exact solution to (4.11) is given by

$$U(r) = a_1 J_0\left(\frac{\sqrt{\pm\alpha_1^2 - 1}}{E}r\right) + a_2 Y_0\left(\frac{\sqrt{\pm\alpha_1^2 - 1}}{E}r\right), \tag{4.12}$$

where a_1 and a_2 are arbitrary constants and J_0 and Y_0 are Bessel J and Y functions of order 0, respectively. In terms of the transformed variable w, we obtain the general exact solution

$$w = a_1 e^{\pm\alpha_1 x} J_0(\gamma r), \quad \text{with } \gamma J_1(\gamma) + Bi J_0(\gamma) = 0, \quad \gamma = \frac{\sqrt{\pm\alpha_1^2 - 1}}{E}, \tag{4.13}$$

for (2.11) with $s = w$. We observe that this solution does not satisfy the boundary conditions at $x = 0$ and $x = 1$. One may consider semifinite fins where the temperature gradient vanishes at large x values and with a appropriate choice of F, a_1 may be given by an exponential constant. Note that $\alpha_1 = 1$ leads to trivial solutions.

4.2.2. Nonlinear Sink Term

The power law source term $s = w^p$, $p \neq 1$ renders (2.11) nonlinear and separation of variables is inapplicable. Following the techniques outlined in Section 4.2.1, we observe that the symmetry generator Γ_2 listed in Table 1 leads to the reduction

$$w = x^{2/1-p}G(\gamma), \quad \text{where } \gamma = \frac{x}{r} \text{ and } G \text{ satisfy the O.D.E.} \tag{4.14}$$

$$\left(1 + E^2\gamma^2\right)G'' + \left\{E^2\gamma + \left(\frac{4}{1-p}\right)\frac{1}{\gamma}\right\}G' + \left\{\left(\frac{2}{1-p}\right)\left(\frac{2}{1-p} - 1\right)\frac{1}{\gamma^2}\right\}G - \frac{1}{\gamma^2}G^p = 0, \quad p \neq 1. \tag{4.15}$$

We observe that (4.15) is harder to solve exactly. Furthermore, the boundary conditions are not invariant under Γ_2 given a nonlinear source term.

5. Fin Efficiency and Heat Flux

5.1. Heat Flux

The heat transfer from the fin base may be constructed by evaluating heat conduction rate at the base (see, e.g., [31])

$$q_b = 2\pi \int_0^{R_a} K(T) \left.\frac{\partial T}{\partial X}\right|_{X=L} R dR = \frac{2\pi R_a^2 (T_b - T_\infty) K_a}{L} \int_0^1 k(\theta) \left.\frac{\partial \theta}{\partial x}\right|_{x=1} r dr. \tag{5.1}$$

The dimensionless heat transfer rate from the base of the fin is defined by [31]

$$Q = \frac{q_b L}{2\pi K_a (T_b - T_\infty) R_a^2} = \int_0^1 k(\theta) \left.\frac{\partial \theta}{\partial x}\right|_{x=1} r dr. \tag{5.2}$$

5.2. Fin Efficiency

Fin efficiency (overall fin performance) is defined as the ratio of the actual heat transferred from the fin surface to the surrounding fluid to the heat which would be transferred if the entire fin area were kept at the base temperature [2, 32]. For the pin fin, analogous to the definition in [33], the local fin efficiency is defined by

$$\eta = \frac{q_b}{Q_i} = \frac{\left(2\pi R_a^2 (T_b - T_\infty) K_a / L\right) \int_0^1 k(\theta)(\partial\theta/\partial x)\,|_{x=1} r dr}{2\pi R_a h_b (T_b - T_\infty) L}, \tag{5.3}$$

or simply

$$\eta = \frac{1}{E^2 Bi} \int_0^1 k(\theta) \left.\frac{\partial \theta}{\partial x}\right|_{x=1} r dr. \tag{5.4}$$

5.2.1. *Flux and Fin Efficiency Given* (4.1)

Given the solution (4.1) with linear thermal conductivity $k = 1 + B\theta$ as an example, we obtain heat flux for w

$$Q = \int_0^1 \left.\frac{\partial w}{\partial x}\right|_{x=1} r dr = \sum_{m=1}^{\infty} \frac{d_m \lambda_m \sinh(\lambda_m) J_1(\delta_m)}{\delta_m}, \tag{5.5}$$

and fin efficiency

$$\eta = \frac{1}{E^2 Bi} \int_0^1 \left.\frac{\partial w}{\partial x}\right|_{x=1} r dr = \frac{1}{E^2 Bi} \sum_{m=1}^{\infty} \frac{d_m \lambda_m \sinh(\lambda_m) J_1(\delta_m)}{\delta_m}. \tag{5.6}$$

6. Entropy Generation Analysis

Entropy generation results from the nonequilibrium conditions arising due to the exchange of energy within the fluid (in case of a flow between two plates) and the solid boundaries [34]. In fact, entropy generation analysis has been limited to vertical cylindrical annulus (or channels) (see, e.g., [35–37]), and studies in pure conduction may be found in the literature such as in [38–40]. The local volumetric rate of entropy generation is given in dimensionless variable (see, e.g., [17])

$$N_L = \left(\frac{\partial \theta}{\partial r}\right)^2 + \left(\frac{\partial \theta}{\partial x}\right)^2. \tag{6.1}$$

The total dimensionless entropy generated in a pin fin is given by [17]

$$N_T = \iint_0^1 \left[\left(\frac{\partial \theta}{\partial r}\right)^2 + \left(\frac{\partial \theta}{\partial x}\right)^2\right] r dr dx. \tag{6.2}$$

Given $k = 1 + B\theta$, we have in terms of w

$$\begin{gathered} N_L = \left(\frac{1}{1+2Bw}\right)\left\{\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\}, \\ N_T = \iint_0^1 \left(\frac{1}{1+2Bw}\right)\left[\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] r dr dx. \end{gathered} \tag{6.3}$$

7. Discussions

We follow the analysis in [17]. The number of eigenvalues required to calculate the temperature distribution, heat flux, and fin efficiency accurately depend on the Biot number Bi. We observe in Table 2 below that Biot number is inversely proportional to the eigenvalues. The expression for the temperature distribution is given explicitly in (4.1) and (4.4). However, in further analysis, we focus on solution (4.4). The temperature distribution depends on a number of variables including Bi, eigenvalues, and the arbitrary function of r describing the temperature at the base of the fin. We may choose any function g such that $g'(0) = 0$. In fact, the nonuniform base temperature is modeled by cosine function of r, namely, $1 + p\cos r$ for some parameter p [13], and base temperature may be given in general [41] by the power law $1 + p\cos^{\kappa} r$, κ being an exponent (see also Chapter 15 in [1]). In Figure 2, not surprisingly, we observe that the temperature is higher at the center of the pin, that is, at $r = 0$. Figure 3 depicts the temperature profile along the $r = 0$, and temperature decreases from the base to the tip of the fin. We observe in Figures 4 and 5 that the temperature is much higher at the center of the pin than at its surface. Similar results are recorded in the literature (see, e.g., [17]). Similar profiles are observed in Figures 6–9, wherein thermal conductivity is assumed to be given by the power law. We note the difference at surface of the fin tip in Figures 4 and 9. In both cases, the temperature drops even to the negative values when k is linear in temperature. The drop in the temperature is due to the presence of the sink term.

Table 2: Eigenvalues λ_n for different values of the Biot number with $E = 2$.

Bi	0.2	0.4	0.6	0.8	1	2
λ_1	7.6243	7.5193	7.4141	7.3093	7.2056	6.7249
λ_2	14.0098	13.9527	13.8955	13.8384	13.7815	13.5023
λ_3	20.3322	20.2928	20.2535	20.2141	20.1748	19.9801
λ_4	26.6361	26.6060	26.5760	26.5460	26.5160	26.3667
λ_5	32.9322	32.9079	32.8836	32.8593	32.8350	32.7140
λ_6	39.2241	39.2037	39.1833	39.1629	39.1425	39.0408
λ_7	45.5136	45.4960	45.4784	45.4608	45.4433	45.3556
λ_8	51.8016	51.7861	51.7707	51.7552	51.7398	51.6627
λ_9	58.0885	58.0747	58.0609	58.0472	58.0334	57.9646
λ_{10}	64.3747	64.3623	64.3498	64.3374	64.3250	64.2629

The classical Lie point symmetry analysis resulted in a number of admitted symmetries. In general, symmetries yield self-similar or similarity solutions known also as invariant solutions. It is more difficult to construct invariant (similarity) solutions for the boundary value problem defined by characteristic length. However, one may consider the semiinfinite fins (see also [1]), whereby either temperature or temperature gradient vanishes at large spatial variable. In our case, at large x. Note that construction of invariant solutions for steady nonlinear one-dimensional problems is easier (see, e.g., [6]). On the other hand, symmetry techniques may be use to reduce the boundary value problem in partial differential equation to the boundary value problem in ordinary differential equations, as such the reduced problem may be solve exactly or easily by numerical schemes.

8. Concluding Remarks

We have considered a steady state problem describing heat dissipation in a pin fin. We have successfully applied the Kirchoff transformation to partly linearize the resulting nonlinear diffusion equation. To the best of our knowledge, symmetry methods have not yet been applied to two-dimensional fin problems. Some cases of the source term for which the governing equation admits extra symmetries have been obtained. Unfortunately, admitted symmetries do not leave the entire boundary value problem invariant. Some new solutions are constructed by separation of variables. Heat transfer analysis is carried out following the work in [17]. Further analysis into the influence of a larger number of terms may reveal a more in depth understanding of the underlying dynamics of the system under consideration.

Acknowledgments

R. J. Moitsheki wishes to thank the National Research Foundation of South Africa under Thuthuka program for the continued generous financial support. The authors thank the anonymous referees on pointing out a number of references and for the valuable comments which led to clarifications and improvements to this paper.

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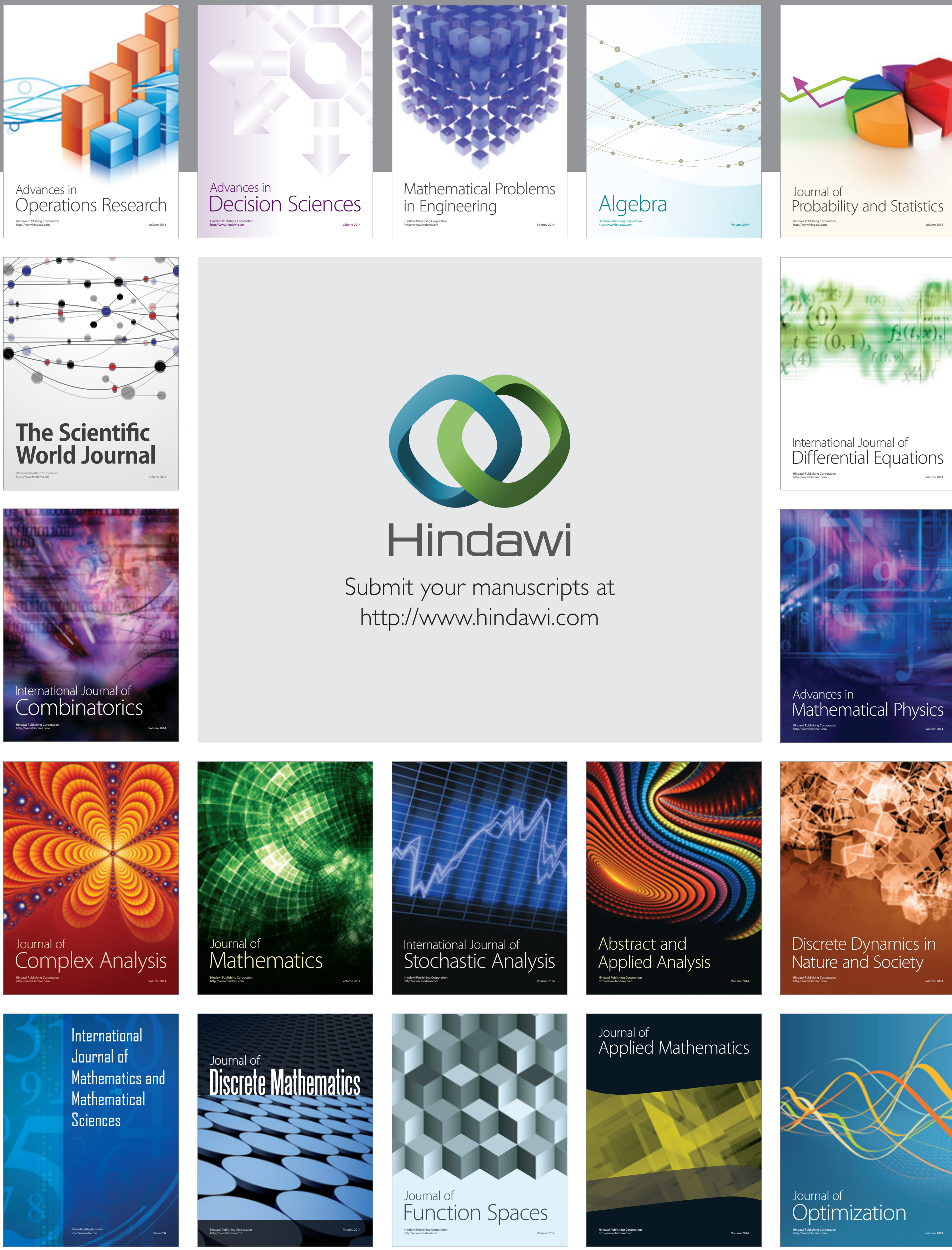

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