Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 984927, 12 pages
doi:10.1155/2010/984927

Research Article

Feedback Controller Stabilizing Vibrations of a Flexible Cable Related to an Overhead Crane

Abdelhadi Elharfi

Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University, B.P. 2390-40000, Marrakesh, Morocco

Correspondence should be addressed to Abdelhadi Elharfi, a.elharfi@ucam.ac.ma

Received 30 April 2010; Accepted 27 August 2010

Academic Editor: Horst Ecker

The problem of stabilizing vibrations of flexible cable related to an overhead crane is considered. The cable vibrations are described by a hyperbolic partial differential equation (HPDE) with an update boundary condition. We provide in this paper a systematic way to derive a boundary feedback law which restores in a closed form the cable vibrations to the desired zero equilibrium. Such a control law is explicitly constructed in terms of the solution of an appropriate kernel PDE. The pursued approach combines the “backstepping method” and “semigroup theory”.

1. Introduction

In this paper, we are concerned with the problem of boundary feedback stabilization of a second-order HPDE describing vibrations of a flexible cable related to an overhead crane. As illustrated in Figure 1, the rigid load with mass M is related to cart of the overhead crane by a flexible cable.

The cable displacement $z(t,x)$, at time t and height x, is mathematically modeled by the following hyperbolic equation:

$$\begin{gathered}
\ddot{z}(t,x) = [\varepsilon(x)z_x(t,x)]_x + b(x)z_x(t,x) + a(x)z(t,x), \quad \text{in } (0,\infty)\times(0,1), \\
M\ddot{z}(t,0) = \varepsilon(0)z_x(t,0), \quad \text{in } (0,\infty), \\
m\ddot{z}(t,1) = -\varepsilon(1)z_x(t,1) + \beta z(t,1) - u(t), \quad \text{in } (0,\infty), \\
z(0,x) = z^0(x), \quad \dot{z}(0,x) = z^1(x), \quad \text{in } (0,1),
\end{gathered} \tag{1.1}$$

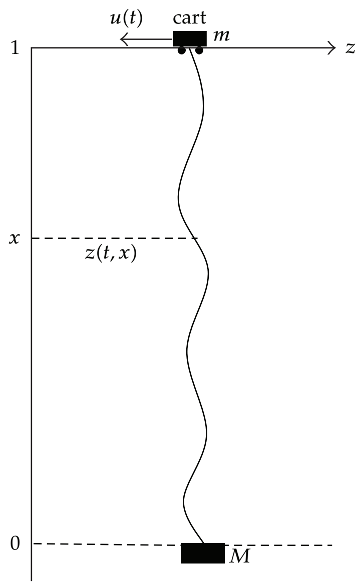

Figure 1

coupled with the update boundary condition imposed at the level $x = 0$,

$$z_x(t,0) = \rho z(t,0). \tag{1.2}$$

The parameter $\varepsilon(x) = g(M + x)$ denotes the tension force of the cable at the height x, g being the gravitational acceleration, m the mass of the cart, and M the mass of the rigid load. It is assumed that the line density of the cable is homogeneous and equal to 1. The vibrations in system (1.1) are not only being diffused and bifurcated $((\varepsilon z_x)_x + bz_x)$ but also a destabilizing displacement (az) is generated. Here, β, ρ are two constants and $u(t)$ is a control placed at the extremity $x = 1$. The boundary condition (1.2) corresponds to situations where the displacement z is subject to a dispositive effect when the rigid load is arrived to the soil, that is, $x = 0$. Such effect arises in (1.2) as an external force which depends on the displacements. System (1.1)-(1.2) serves also as a linearized model of strings. Hereafter, we assume that the parameters a, b and the initial data z^0, z^1 satisfy the regularity conditions

$$z^0 \in H^2,\ z^1 \in H^1,\ a \in C^1[0,1],\ b \in C^2[0,1], \quad \text{with } b(0) = 0, \tag{H}$$

where H^1, H^2 are the usual Sobolev spaces on $(0,1)$, see Section 2.

The control objective that we are interested in, is to construct a feedback controller u which restores the displacements $z(t)$ to the equilibrium $z \equiv 0$ (as $t \to \infty$). From a practical point of view, Rao [1] treated the stabilization problem of suppressing vibrations of the distributed overhead crane model with one rigid load, when $a = b = \beta = 0$. The exponential stability of the closed loop is proved by exploiting an energy functional. In the study by Rahn et al. in [2], a study has been conducted to develop control algorithms for flexible cable crane models. An appropriate coupling amplification controller which asymptotically stabilizes all modes of a linear gantry crane model is constructed. Sano and Otanaka [3] generalized the stabilization problem of a flexible cable with two rigid loads. The model is described by two HPDEs, but the model contained a defect by neglecting the mass of the cart, that is, $m = 0$. The defect of [3] is surmounted by H. Sano in [4] by using the LaSallse's invariance principle. Kim and Hong [5] augmented the simple model with an axially moving system concept. The crane was modeled as an axially moving string system. The dynamics of the moving string is derived using Hamilton's principle for systems with changing mass. Simplified versions of the concerned model was the subject, with respect to the stability, of several works by deferent approaches, see, for example, [6–10].

In comparison with the existing works, the model treated in this paper generalizes HPDEs describing vibrations of overhead crane cable, ($a \neq 0$, $b \neq 0$, $\beta \neq 0$). Moreover, the concerned model contains a perturbing actuation due to the update boundary condition (1.2) which has a nonneglected effect on the behavior analysis of the cable displacements. The proposed method provides a systematic way to construct a boundary feedback law which restores the cable displacements $z(t)$, described by the HPDE (1.1)-(1.2), to the desired equilibrium $z \equiv 0$, as $t \to \infty$. Further, the boundary feedback law is explicitly represented in terms of the solution of an adequate kernel PDE.

The paper is organized as follows: in Section 2, we derive an appropriate kernel PDE, and we convert system (1.1)-(1.2) into a well-known open-loop system. A control law for the new system is constructed, and the well-posedness of the resulting closed-loop system is shown in Section 3. In Section 4, we derive a feedback controller which asymptotically stabilizes the solution of the closed-loop system associated with (1.1)-(1.2).

2. Preliminaries

To simplify the reading, we denote $\Delta := \{(x, y) : 0 \le y \le x \le 1\}$. H^i, $i = 0, 1, 2$, are the usual Sobolev spaces on the interval $(0, 1)$. $\langle \cdot, \cdot \rangle$ will denote the inner product on the Hilbert space $H^0 = L^2$. If $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a C_0-semigroup $\mathfrak{T}$ on a Hilbert space $\mathcal{X}$, we denote by $\mathcal{X}_1$ the space $D(\mathcal{A})$ endowed with the graph norm $\|x\| := \|x\|_{\mathcal{X}} + \|\mathcal{A}x\|_{\mathcal{X}}$.

First, using the transformation

$$\tilde{z}(t, x) := e^{\int_0^x (b(s)/2\varepsilon(s)) ds} z(t, x), \tag{2.1}$$

with the compatible changes of parameters

$$\begin{gathered} \tilde{\varepsilon}(x) := \varepsilon(x), \qquad \tilde{a}(x) := a(x) - \frac{b'(x)}{2} - \frac{b^2(x)}{4\varepsilon(x)}, \\ \tilde{\beta} := \beta - \frac{b(1)}{2}, \qquad \tilde{\rho} = \rho, \qquad \tilde{u}(t) := e^{\int_0^1 (b(s)/2\varepsilon(s)) ds} u(t), \end{gathered} \tag{2.2}$$

one can eliminate the bifurcation term (bz) from (1.1). In fact, direct computations give

$$\begin{gathered} \ddot{\tilde{z}} - (\varepsilon \tilde{z}_x)_x - \tilde{a}\tilde{z} = \{\ddot{z} - (\varepsilon z_x)_x - b z_x - a z\} e^{\int_0^x (b(s)/2\varepsilon(s)) ds}, \\ M\ddot{\tilde{z}}(t, 0) - \varepsilon(0)\tilde{z}_x = 0, \\ m\ddot{\tilde{z}}(t, 1) + \varepsilon(1)\tilde{z}_x(t, 1) = \tilde{\beta}\tilde{z}(t, 1) - \tilde{u}(t). \end{gathered} \tag{2.3}$$

Then, z satisfies (1.1) if and only if $\tilde{z}$ satisfies (1.1) with the parameters 0, $\tilde{a}$, $\tilde{\beta}$, and $\tilde{u}$, instead of b, a, β, and u. Moreover, provided that $b \in C^2$, the parameters $\tilde{a} \in C^1$. So, without loss of generality, we set in what follows $b \equiv 0$.

The following lemma is due to [11, Lemma 2.4]. It describes an integral transformation which will be used to convert (1.1)-(1.2) into a well-known one.

Lemma 2.1. *Let $k \in H^2(\Delta)$, and define the bounded operator $T_k : H^i \to H^i$ by*

$$(T_k\varphi)(x) := \varphi(x) + \int_0^x k(x,y)\varphi(y)dy. \tag{2.4}$$

Then, T_k has a linear bounded inverse $T_k^{-1} : H^i \to H^i$, $i = 0,1,2$.

Next, assume that $z(t)$ satisfies (1.1)-(1.2) and set for $t \geq 0$, $x \in [0,1]$

$$w(t,x) := (T_k z(t))(x) = z(t,x) + \int_0^x k(x,y)z(t,y)dy. \tag{2.5}$$

By integrating by parts from 0 to x, we get for $t > 0$,

$$\begin{aligned}
\ddot{w}(t,x) &= \ddot{z}(t,x) + \int_0^x k(x,y)\ddot{z}(t,y)dy \\
&= \ddot{z}(t,x) + \int_0^x k(x,y)\Big[\left[\varepsilon(y)z_y(t,y)\right]_y + a(y)z(t,y)\Big]dy \\
&= \ddot{z}(t,x) + \varepsilon(x)k(x,x)z_x(t,x) - \varepsilon(0)k(x,0)z_x(t,0) - \varepsilon(x)k_y(x,x)z(t,x) \\
&\quad + \varepsilon(0)k_y(x,0)z(t,0) + \int_0^x \Big[\left[\varepsilon(y)k_y(x,y)\right]_y + a(y)k(x,y)\Big]z(t,y)dy.
\end{aligned} \tag{2.6}$$

Moreover,

$$\begin{aligned}
[\varepsilon w_x]_x &= [\varepsilon(x)z_x(t,x)]_x + \varepsilon(x)k(x,x)z_x(t,x) + [\varepsilon(x)k(x,x)]_x z(t,x) \\
&\quad + \varepsilon(x)k_x(x,x)z(t,x) + \int_0^x [\varepsilon(x)k_x(x,y)]_x z(t,y)dy.
\end{aligned} \tag{2.7}$$

Taking into account of (1.2), we obtain

$$\begin{aligned}
\ddot{w} - [\varepsilon w_x]_x &= \left[a(x) - 2\varepsilon(x)\frac{d}{dx}(k(x,x)) - \varepsilon'(x)k(x,x)\right]z(t,x) \\
&\quad + \int_0^x \Big[a(y)k(x,y) + \Big(\left[\varepsilon(y)k_y(x,y)\right]_y - \left[\varepsilon(x)k_x(x,y)\right]_x\Big)\Big]z(t,y)dy \\
&\quad + \left[k_y(x,0) - \rho k(x,0)\right]\varepsilon(0)z(t,0).
\end{aligned} \tag{2.8}$$

Then, $\ddot{w} - [\varepsilon(x)w_x]_x = 0$, in $(0,\infty)\times(0,1)$, if and only if the kernel k verifies the PDE

$$\begin{gathered}
\left[\varepsilon(x)k_x(x,y)\right]_x - \left[\varepsilon(y)k_y(x,y)\right]_y = a(y)k(x,y), \quad 0 \le y \le x \le 1, \\
k_y(x,0) = \rho k(x,0), \quad 0 \le x \le 1, \\
k(x,x) = \frac{1}{2\sqrt{\varepsilon(x)}}\int_0^x \frac{a(s)}{\sqrt{\varepsilon(s)}}ds, \quad 0 \le x \le 1.
\end{gathered} \tag{2.9}$$

We note that the third (boundary) equation of (2.9) is obtained by solving the first order differential equation

$$2\varepsilon(x)\frac{d}{dx}(k(x,x)) + \varepsilon'(x)k(x,x) = a(x), \tag{2.10}$$

with the initial condition $k(0,0) = 0$. Due to [12], for a given C^2-function ε, the kernel PDE (2.9) has a unique solution $k \in H^2(\Delta)$, see also [13] for $\varepsilon = \text{const}$. Further, the function k can be approximated numerically via scheme of successive approximations.

Now, let k be the solution of (2.9). In view of (2.8), the new state w satisfies

$$\begin{gathered}
\ddot{w}(t,x) = [\varepsilon(x)w_x(t,x)]_x, \quad \text{in } (0,\infty)\times(0,1), \\
M\ddot{w}(t,0) = \varepsilon(0)w_x(t,0), \quad \text{in } (0,\infty), \\
m\ddot{w}(t,1) = -\varepsilon(1)w_x(t,1) - U(t), \quad \text{in } (0,\infty), \\
w(0,x) =: w^0(x), \qquad \dot{w}(0,x) =: w^1(x), \quad \text{in } (0,1),
\end{gathered} \tag{2.11}$$

with $w^0 = T_k z^0$, $w^1 = T_k z^1$, and

$$U(t) =: u(t) - c_0 z(t,1) - c_1 z_x(t,1) - \langle p, z(t)\rangle, \tag{2.12}$$

where

$$\begin{gathered}
c_0 := \beta + \varepsilon(1)k(1,1) - m\varepsilon(1)k_y(1,1), \\
c_1 := m\varepsilon(1)k(1,1), \\
p(y) := (\varepsilon(1) + mg)k_x(1,y) + m\varepsilon(1)k_{xx}(1,y), \qquad y \in (0,1).
\end{gathered} \tag{2.13}$$

We note here that the expression of $U(t)$ is obtained using (1.1) and the first equation of (2.9). We summarize these results in the following.

Lemma 2.2. *Let k be the solution of* (2.9)*, and consider c_0, c_1, p with representation* (2.13)*. Then, the isomorphism T_k converts* (1.1)-(1.2) *into* (2.11)-(2.12).

3. Stabilization of the Transformed System

We proceed in this section to construct an appropriate control $U(t)$ which stabilizes the new open-loop system (2.11), and we show the well-posedness of the resulting closed-loop system. To do so, it is logical to think about the energy of the system as Lyapunov function. So, let us introduce the following energy associated with (2.11)

$$E(t) = \frac{1}{2}\left\{\int_0^1 \left[\varepsilon(x) w_x(t,x)^2 + \dot{w}(t,x)^2\right]dx + M\dot{w}(t,0)^2 + m\dot{w}(t,1)^2 + \alpha w(t,1)^2\right\}, \quad (3.1)$$

where α is a positive constant. The integral term corresponds to the inner energy of the cable. The coefficient $\dot{w}^2(t,0)+\dot{w}^2(t,1)$ is proportional to the kinetic energy of the cart. However, the term $w^2(t,1)$ guarantees the position convergence, it can be replaced by $(w - w_d)^2$ in order to reach any desired position w_d by the cart, see [7] and the reference therein for a more discussions on the functional energy E associated with hybrid systems.

Differentiating (3.1) with respect to t, we get by using (2.11)

$$\dot{E}(t) = -\dot{w}(t,1)(U(t) - \alpha w(t,1)). \quad (3.2)$$

To cause $E(t)$ to decrease, a simple choice of the feedback law is

$$U(t) = \alpha w(t,1) + \gamma \dot{w}(t,1), \qquad \gamma > 0. \quad (3.3)$$

Substituting (3.3) in (3.4), we obtain

$$\dot{E}(t) = -\gamma \dot{w}(t,1)^2. \quad (3.4)$$

This means that under the boundary feedback law (3.3), the energy E decreases with time t.

Now, let us consider the Hilbert space $\mathcal{X} := H^1 \times L^2 \times \mathbb{R} \times \mathbb{R}$ endowed with the inner product

$$\left\langle (f,g,\xi,\eta), \left(\tilde{f},\tilde{g},\tilde{\xi},\tilde{\eta}\right) \right\rangle := \int_0^1 \left[\varepsilon f_x \tilde{f}_x + g\tilde{g}\right]dx + M\xi\tilde{\xi} + m\eta\tilde{\eta} + \alpha f(1)\tilde{f}(1), \quad (3.5)$$

and introduce the operator

$$\mathfrak{D}(\mathcal{A}_0) := \left\{(f,g,\xi,\eta)^\top \in H^2 \times H^1 \times \mathbb{R} \times \mathbb{R} : g(0) = \xi,\ g(1) = \eta\right\},$$
$$\mathcal{A}_0 \begin{pmatrix} f \\ g \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} g \\ (\varepsilon f_x)_x \\ \dfrac{\varepsilon(0) f_x(0)}{M} \\ \dfrac{-[\varepsilon(1) f_x(1) + \alpha f(1) + \gamma\eta]}{m} \end{pmatrix}, \quad \text{for} \begin{pmatrix} f \\ g \\ \xi \\ \eta \end{pmatrix} \in \mathfrak{D}(\mathcal{A}_0). \quad (3.6)$$

Setting now $v(t) := (w(t), \dot{w}(t), \dot{w}(t,0), \dot{w}(t,1))^{\top}$, for $t \geq 0$. Then, (2.11)-(3.3) can be represented on the state space $\mathcal{X}$ by the abstract Cauchy problem

$$\dot{v}(t) = \mathcal{A}_0 v(t), \quad t \geq 0, \; v(0) = v^0, \tag{3.7}$$

where $v^0 := (w^0, w^1, w^1(0), w^1(1))^{\top}$. In the following lemma, we will confirm the well-posedness of (2.11)-(3.3).

Lemma 3.1. *$\mathcal{A}_0$ generates a C_0-semigroup of contraction $(\mathcal{T}_t^0)_{t\geq 0}$ on $\mathcal{X}$.*

Proof. Obviously, $\mathcal{A}_0$ is densely defined. Moreover, by integrating by parts, we get

$$\langle \mathcal{A}_0 v, v \rangle_{\mathcal{X}} = -\gamma \eta^2, \tag{3.8}$$

for $v = (f, g, \xi, \eta) \in \mathfrak{D}(\mathcal{A}_0)$. Therefore, $\mathcal{A}_0$ is dissipative. By the Lumer-Philips theorem [14, page 85], the proof will be accomplished if one can show that $(I - \mathcal{A}_0)$ is surjective. In fact, for a given $v_0 = (f_0, g_0, \xi_0, \eta_0) \in \mathcal{X}$, we have to solve the functional equation

$$(I - \mathcal{A}_0)v = v_0, \quad v = (f, g, \xi, \eta) \in \mathfrak{D}(\mathcal{A}_0), \tag{3.9}$$

which means that

$$\begin{gathered} f \in H^2, \qquad g \in H^1, \\ g = f - f_0, \qquad \xi = g(0), \qquad \eta = g(1), \end{gathered} \tag{3.10}$$

$$\begin{gathered} g - (\varepsilon f_x)_x = g_0, \\ M\xi - \varepsilon(0) f_x(0) = M\xi_0, \\ \varepsilon(1) f_x(1) + \alpha f(1) + (m + \gamma)\eta = m\eta_0. \end{gathered} \tag{3.11}$$

Substituting (3.10) in (3.11), we obtain the PDE

$$\begin{gathered} (\varepsilon f_x)_x - f = -(f_0 + g_0), \\ \varepsilon(0) f_x(0) - M f(0) = -M(\xi_0 + f_0(0)), \\ \varepsilon(1) f_x(1) + (\alpha + m + \gamma) f(1) = m\eta_0 + (m + \gamma) f_0(1). \end{gathered} \tag{3.12}$$

Following the method of [15, VIII.4], one can see that the system (3.12) has a unique solution $\phi \in H^2$. Of course, the vector $v = (\phi, \varphi; r_0; r_1)^\top$ is given by

$$\begin{gathered}
\phi \text{ solution of } (3.12), \\
\varphi = \phi - f_0, \\
r_0 := \frac{\left[M\xi_0 + \varepsilon(0)\phi_x(0)\right]}{M}, \\
r_1 := \frac{\left[m\eta_0 + (m - \alpha - \beta) f_0(1) - \varepsilon(1)\phi_x(1)\right]}{(\alpha + \beta + \gamma)},
\end{gathered} \tag{3.13}$$

which is a $\mathfrak{D}(\mathcal{A}_0)$-solution of the functional equation (3.9). This leads us to conclude that $I - \mathcal{A}_0$ is surjective. □

Since $\mathcal{A}_0$ generates a C_0-semigroup of contraction, then $(0, \infty) \subseteq \varrho(A_0)$, and the resolvent $\mathcal{R}(\mu, \mathcal{A}_0)$ is well-defined for all $\mu > 0$.

Lemma 3.2. *$\mathcal{R}(\mu, \mathcal{A}_0)$ is compact for all $\mu > 0$. In particular, $\mathcal{T}_0$ is relatively compact.*

Proof. In view of [14, page 117], it remains to show that the injection $j : \mathcal{X}_1 \to X$ is compact. To do so, we introduce the auxiliary Hilbert space $\mathcal{V} := H^2 \times H^1 \times \mathbb{R} \times \mathbb{R}$ with the inner product

$$\langle v, \tilde{v} \rangle_{\mathcal{V}} := \left\langle f, \tilde{f} \right\rangle_{H^2} + \langle g, \tilde{g} \rangle_{L^2} + \xi\tilde{\xi} + \eta\tilde{\eta} + f(1)\tilde{f}(1), \tag{3.14}$$

for $v = (f, g, \xi, \eta) \in \mathcal{V}$ and $\tilde{v} = (\tilde{f}, \tilde{g}, \tilde{\xi}, \tilde{\eta}) \in \mathcal{V}$. Obviously, $\mathcal{X}_1 \subseteq \mathcal{V} \subseteq \mathcal{X}$. Moreover, from the Sobolev's embedding theorem, the embedding from H^1 in L^2 and the one of H^2 in H^1 are compact. It follows that the injection $j_1 : \mathcal{V} \leftrightarrow \mathcal{X}$ is compact. On the other hand, direct computations show that

$$\|v\|_{\mathcal{V}} \le C\|v\|_{\mathcal{X}_1}, \tag{3.15}$$

for some constant C. Therefore, the injection $j_2 : \mathcal{X}_1 \to \mathcal{V}$ is continuous. Consequently, $j := j_1 \circ j_2$ is compact from $\mathcal{X}_1$ into $\mathcal{X}$. Thus, $\mathcal{A}_0$ has a compact resolvent, and by [14, Corollary V.2.15], we conclude that the semigroup $\mathcal{T}^0$ is relatively compact. □

Taking into account of [14, page 318], and the fact that $\mathcal{T}^0$ is relatively compact, the following decomposition holds

$$\mathcal{X} = \mathcal{X}_s \oplus \mathcal{X}_r, \tag{3.16}$$

where

$$\begin{gathered}
\mathcal{X}_s = \left\{ v \in \mathcal{X} : \left\| \mathcal{T}_t^0 v \right\|_{\mathcal{X}} \longrightarrow 0 \text{ as } (t \longrightarrow \infty) \right\}, \\
\mathcal{X}_r = \overline{\operatorname{lin}}\{ v \in \mathcal{X} : \exists \sigma \in \mathbb{R},\ \mathcal{A}v = i\sigma v \}.
\end{gathered} \tag{3.17}$$

We point out here that, due to (H), the initial data v^0 belongs to $\mathfrak{D}(\mathcal{A}_0)$. Since $\mathcal{A}_0$ generates a C_0 semigroup $\mathfrak{T}^0$, then, by [14, page 145], the evolution equation (3.7) has a unique solution $v \in C^1([0,\infty),\mathcal{X}) \cap C([0,\infty),\mathfrak{D}(\mathcal{A}_0))$, given by $v(t) = \mathfrak{T}_t^0 v^0$. Which implies that the closed-loop system (2.11)-(3.3) has a unique solution w satisfying

$$\begin{gathered} w \in C^1\left([0,\infty),H^1\right) \cap C\left([0,\infty),H^2\right), \\ \dot{w} \in C^1\left([0,\infty),L^2\right) \cap C\left([0,\infty),H^1\right), \\ (w(t),\dot{w}(t),\dot{w}(t,0),\dot{w}(t,1))^\top = v(t) = \mathfrak{T}_t^0 v^0. \end{gathered} \tag{3.18}$$

Equation (3.18) explains that the solution of (2.11)-(3.3) is represented by the semigroup $\mathfrak{T}^0$. For that reason, we will adopt in this work the concept of stability associated with semigroup theory as defined in [14]. One says that (2.11)-(3.3) is asymptotically stable, if $\mathfrak{T}^0$ is strongly stable, that is, for all $v \in \mathcal{X}$, $\|\mathfrak{T}_t^0 v\|_{\mathcal{X}} \to 0$ as $(t \to 0)$. In the following theorem, we show how the controller (3.3) affects on either the stability and the energy of (2.11).

Theorem 3.3. (i) $\mathfrak{T}^0$ *is strongly stable. In particular,* (2.11)-(3.3) *is asymptotically stable.*

(ii) *The energy* $E(t)$ *decreases with time t:* $E(t) \searrow 0$ *as* $t \to \infty$.

Proof. In view of (3.16), it suffices to show that $\mathcal{X}_r = \{0\}$. In fact, let $v = (f,g,g(0),g(1))^\top \in \mathfrak{D}(\mathcal{A}_0)$ satisfying

$$\mathcal{A}_0 v = i\sigma v, \tag{3.19}$$

for some $\sigma \in \mathbb{R}$. Equation (3.19) is equivalent to

$$f \in H^2, \qquad g = i\sigma f, \tag{3.20}$$

$$\begin{gathered} (\varepsilon f_x)_x + \sigma^2 f = 0, \\ \varepsilon(0) f_x(0) + M\sigma^2 f(0) = 0, \\ \varepsilon(1) f_x(1) + \left(\alpha + i\sigma\gamma - m\sigma^2\right) f(1) = 0. \end{gathered} \tag{3.21}$$

By using the method of [15, VIII.4] and taking into consideration of the regularity condition $f \in H^2$, one can see that $f \equiv 0$ is the unique solution of (3.21). From (3.20), we conclude that $g \equiv 0$. Therefore, $\mathcal{X}_r = \{0\}$ and $\mathcal{X} = \mathcal{X}s$.

On the other hand, in view of the third equation of (3.18), we have

$$E(t) = \frac{1}{2}\left\|\mathfrak{T}_t^0 v^0\right\|_{\mathcal{X}}^2 \longrightarrow 0, \quad (t \longrightarrow \infty). \tag{3.22}$$

This proves the statement (ii). □

4. Closed-Loop Stability of the Cable Displacements

We will derive in this section a controller u which restores, in a closed form, the cable displacements $z(t)$ of the concerned system (1.1)-(1.2) to the equilibrium $z \equiv 0$. In fact, by substituting (2.12) in (3.3), we reach, by using (2.5), the following expression of the control $u(t)$

$$u(t) = (\alpha + c_0)z(t,1) + c_1 z_x(t,1) + \gamma \dot{z}(t,1) + \langle q, z(t)\rangle + \langle \gamma k_0, \dot{z}(t)\rangle, \tag{4.1}$$

where $k_0(y) := k(1,y)$, $q := \alpha k_0 + p$.

The system (1.1)-(1.2), (4.1) is well posed, since it can be obtained from the well-posed system (2.11)-(3.3) via the isomorphism $(T_k)^{-1}$. Which means that the closed-loop system (1.1)-(1.2), (4.1) has a unique solution z satisfying, in view of (3.18), the following regularity conditions:

$$z \in C^1\Big([0,\infty), H^1\Big) \cap C\Big([0,\infty), H^2\Big), \qquad \dot{z} \in C^1\Big([0,\infty), L^2\Big) \cap C\Big([0,\infty), H^1\Big). \tag{4.2}$$

Consider now the operator

$$\mathfrak{D}(\mathcal{A}) := \mathfrak{D}(\mathcal{A}_0),$$
$$\mathcal{A}\begin{pmatrix} f \\ g \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} g \\ (\varepsilon f_x)_x + af \\ \dfrac{\varepsilon(0) f_x(0)}{M} \\ \dfrac{-\left[(\varepsilon(1) + c_1) f_x(1) + (\alpha + c_0 - \beta) f(1) + \gamma\eta + \langle q, f\rangle + \langle \gamma k_0, g\rangle\right]}{m} \end{pmatrix}, \tag{4.3}$$

for $(f, g, \xi, \eta)^\top \in \mathfrak{D}(\mathcal{A})$.

By direct computations one can prove that the function ζ defined by

$$\begin{gathered} z \text{ solution of (1.1)-(1.2), (4.1)}, \\ \zeta(t) := (z(t), \dot{z}(t), \dot{z}(t,0), \dot{z}(t,1))^\top, \quad t \geq 0, \end{gathered} \tag{4.4}$$

is the unique classical solution of the evolution equation

$$\dot{\mathcal{Z}}(t) = \mathcal{A}\mathcal{Z}(t), \quad t \geq 0, \ \mathcal{Z}(0) = \zeta^0, \tag{4.5}$$

where $\zeta^0 = (z^0, z^1, z^1(0), z^1(1))^\top$. This means that the operator $\mathcal{A}$ generates a C_0-semigroup $\mathfrak{T}_t$ given by $\mathfrak{T}_t\zeta^0 := \zeta(t)$. Moreover, the fact that $(T_k)^{-1}$ is bounded, then there exists a constant $C > 0$ such that

$$\begin{gathered} \|z(t)\|_{H^1} \leq C\|w(t)\|_{H^1}, \qquad \|\dot{z}(t)\|_{L^2} \leq C\|\dot{w}(t)\|_{L^2}, \\ |\dot{z}(t,0)| \leq C|\dot{w}(t,0)|, \qquad |\dot{z}(t,1)| \leq C|\dot{w}(t,1)|, \end{gathered} \tag{4.6}$$

for $t \geq 0$. Thus, $\|\zeta(t)\|_{\mathcal{X}} \leq C\|v(t)\|_{\mathcal{X}}$, and so $\|\mathcal{T}_t\zeta^0\|_{\mathcal{X}} \leq C\|\mathcal{T}_t^0 v^0\|_{\mathcal{X}}$. By Theorem 3.3, we deduce the strong stability of $\mathcal{T}_t$. Therefore, the closed-loop system (1.1)-(1.2), (4.1) is asymptotically stable. This proves the main result of this paper which can be reformulated in the following theorem.

Theorem 4.1. *The semigroup $\mathcal{T}_t$ is strongly stable. In particular, the closed-loop system* (1.1)-(1.2), (4.1) *is asymptotically stable.*

Remark 4.2. One can express the controller (4.1) using the solution w of (2.11)-(3.3). In fact, let us denote by l the kernel of the inverse transformation

$$z(t,x) = w(t,x) + \int_0^x l(x,y)w(t,y), \tag{4.7}$$

where w is the solution of (2.11)-(3.3). Substituting (4.7) in (1.1)-(1.2), we find the PDE governing the kernel l

$$\begin{gathered} \left[\varepsilon(x)l_x(x,y)\right]_x - \left[\varepsilon(y)l_y(x,y)\right]_y = -a(x)l(x,y), \quad 0 \leq y \leq x \leq 1, \\ l_y(x,0) = \rho l(x,0), \quad 0 \leq x \leq 1, \\ l(x,x) = -\frac{1}{2\sqrt{\varepsilon(x)}}\int_0^x \frac{a(s)}{\sqrt{\varepsilon(s)}}ds, \quad 0 \leq x \leq 1. \end{gathered} \tag{4.8}$$

The PDE (4.8) is in the same class of (2.9). Hence, the PDE (4.8) has a unique solution $l \in H^2(\Delta)$. Let now l be the solution of (4.8). Substituting (4.7) in (4.1), one obtains an expression of the controller (4.1) in terms of w and l.

5. Conclusion

The proposed approach represents a blinding of the so-called "backsteeping method" and "semigroup theory" to construct a controller which asymptotically stabilizes the solution of the HPDE (1.1)-(1.2). Various properties of parabolic PDEs and hyperbolic PDEs can be treated by using similar techniques. The idea of the study is to convert a complicated (parabolic or hyperbolic) PDE into a well-known one by using the famous integral transformation (2.4) with a kernel required to satisfy an adequate PDE. We also note that the isomorphism (2.4) transforms such PDEs without effects on their topological properties. Therefore, one can deal with others topological properties of complicated systems such as: regularity, controllability, and observability, and so forth.

References

[1] B. Rao, "Exponential stabilization of a hybrid system by dissipative boundary damping," in *Proceedings of the 2nd European Control Conference*, pp. 314–317, Groningen, The Netherlands, 1993.

[2] C. D. Rahn, F. Zhang, S. Joshi, and D. M. Dawson, "Asymptotically stabilizing angle feedback for a flexible cable gantry crane," *Journal of Dynamic Systems, Measurement and Control*, vol. 121, no. 3, pp. 563–566, 1999.

[3] H. Sano and M. Otanaka, "Stabilization of a flexible cable with two rigid loads," *Japan Journal of Industrial and Applied Mathematics*, vol. 23, no. 3, pp. 225–237, 2006.
[4] H. Sano, "Boundary stabilization of hyperbolic systems related to overhead cranes," *IMA Journal of Mathematical Control and Information*, vol. 25, no. 3, pp. 353–366, 2008.
[5] C.-S. Kim and K.-S. Hong, "Boundary control of container cranes from the perspective of controlling an axially moving string system," *International Journal of Control, Automation and Systems*, vol. 7, no. 3, pp. 437–445, 2009.
[6] K. Ammari, M. Jellouli, and M. Khenissi, "Stabilization of generic trees of strings," *Journal of Dynamical and Control Systems*, vol. 11, no. 2, pp. 177–193, 2005.
[7] B. D'Andréa-Novel and J. M. Coron, "Exponential stabilization of an overhead crane with flexible cable via a back-stepping approach," *Automatica*, vol. 36, no. 4, pp. 587–593, 2000.
[8] D. L. Russell, "Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions," *SIAM Review*, vol. 20, no. 4, pp. 639–739, 1978.
[9] J. A. Wickert and C. D. Mote Jr., "Classical vibration analysis of axially moving continua," *Journal of Applied Mechanics*, vol. 57, no. 3, pp. 738–744, 1990.
[10] K.-J. Yang, K.-S. Hong, and F. Matsuno, "Robust boundary control of an axially moving string by using a PR transfer function," *IEEE Transactions on Automatic Control*, vol. 50, no. 12, pp. 2053–2058, 2005.
[11] W. Liu, "Boundary feedback stabilization of an unstable heat equation," *SIAM Journal on Control and Optimization*, vol. 42, no. 3, pp. 1033–1043, 2003.
[12] A. Elharfi, "Explicit construction of a boundary feedback law to stabilize a class of parabolic equations," *Differential and Integral Equations*, vol. 21, no. 3-4, pp. 351–362, 2008.
[13] A. Smyshlyaev and M. Krstic, "Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations," *IEEE Transactions on Automatic Control*, vol. 49, no. 12, pp. 2185–2202, 2004.
[14] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, vol. 194 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2000.
[15] H. Brezis, *Analyse Fonctionnelle*, Collection Mathématiques Appliquées pour la Maîtrise, Masson, Paris, France, 1983.