Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 724531, 22 pages
doi:10.1155/2010/724531

Research Article

Linear Robust Output Regulation in a Class of Switched Power Converters

Josep M. Olm[1] and Domingo Biel[2]

[1] *Department of Applied Mathematics IV, Technical University of Catalonia, 08800 Vilanova, i la Geltrú, Spain*
[2] *Institute of Industrial and Control Engineering, Technical University of Catalonia, 08028 Barcelona, Spain*

Correspondence should be addressed to Josep M. Olm, josep.olm@upc.edu

Received 19 November 2009; Accepted 14 June 2010

Academic Editor: John Burns

This article addresses the robust output regulation problem for a class of nonlinear switched power converters after its linearization by means of a change of the control vector variable. The methodology employs a dynamic state feedback control law and considers parametric uncertainty due to unknown values of resistive loads. Restrictions arising from the fact that the control gains exhibit fixed values are taken into account. The proposed technique is exemplified with the output voltage regulation of a Noninverting Buck-Boost converter and tested through realistic numerical simulations.

1. Introduction

The output regulation—or servomechanism—problem deals with the design of feedback control laws that provide output tracking of any reference belonging to a family of command profiles and, at the same time, are able to reject any perturbation from a certain set, with both references and disturbances being generated by a known, autonomous system of ordinary differential equations, the so-called exosystem. The control design is robust when the objective is achieved despite the presence of parametric uncertainties.

This capital subject, with long trajectory in control theory, was solved for linear systems in the early 1970's with the introduction of the well-known *internal model principle* [1]; there is also an interesting algebraic approach contained in [2]. The extension of the solution to nonlinear systems appeared almost two decades later in the celebrated paper [3]. The reader is referred to [4] for a summary of main topics in Output Regulation Theory. On the other hand, any of the excellent monographic text-books [5, 6] offer a complete overview of the subject.

In this article we address the robust output regulation problem for a family of nonlinear power converters with two control switches that includes the NonInverting Buck-Boost (NIBB), the Watkins-Johnson (WJ), the Inverse of Watkins-Johnson (IWJ) and the Full-Bridge NonInverting Buck-Boost (FBNIBB). The NIBB, the WJ and the IWJ are among the class of eight elementary single-input (i.e., possessing a single voltage source) single-output converters containing a single inductor [7], while the FBNIBB is derived through the substitution of the original switches by a full-bridge in a NIBB [7, 8].

The converters being nonlinear, the first thought may be to face the problem by means of nonlinear output regulation techniques. However, performing a change of control variable the resulting system appears to be linear. Thus, linear output regulation tools are used from this stage on.

The approach, which considers resistive loads with uncertain output resistance, proves the existence of a dynamic state feedback law that solves the linear robust output regulation problem and provides an algorithmic-like construction of the regulator for the general case, that is, either output voltage regulation or tracking. Nevertheless, the eventual achievement of control objectives in the physical system is limited by a possible control action saturation due to the fixed values of the control gains. Hence, guaranteeing a dynamical evolution of the converter in an unsaturated region of the phase plane involves restrictions on the system parameters, state variables and reference profiles that are also studied.

The article is structured as follows. Section 2 contains the main results of linear robust output regulation by means of dynamic state feedback. Section 3 introduces a family of switched power converters and establishes the solution of its output regulation problem. Section 4 is specifically devoted to the construction of a dynamic state feedback regulator for the target system. A numerical example based on the methods outlined in this section is presented in Section 5, while the corresponding simulation results are in Section 6. Finally, conclusions are outlined in Section 7.

2. Linear Robust Output Regulation

This section is focused on the major highlights of Linear Robust Output Regulation (LROR) and follows the exposition in [6].

The Output Regulation Theory addresses the problem of rendering the output $y(\cdot)$ of a linear control system, possibly with plant uncertainties w, to asymptotically track any reference $y_R(\cdot)$ belonging to a given family and, at the same time, reject asymptotically any disturbance $d(\cdot)$ that may be found in a certain set, while maintaining the internal stability of the closed loop system.

The formulation of the problem to be solved takes advantage of the following fact: if one thinks in nullifying the output error $e = y - y_R$, there is no need to separate the roles of y_R and d, because both may be seen as components of an exogenous input that has to be rejected.

Therefore, consider the system

$$\begin{aligned} \dot{x} &= A_w x + B_w u + P_w v, \\ e &= C_w x + Q_w v, \end{aligned} \tag{2.1}$$

with $x \in \mathbb{R}^n$ and $u, e \in \mathbb{R}^m$. The exogenous input $v \in \mathbb{R}^p$ is assumed to satisfy the exosystem

$$\dot{v} = Sv, \tag{2.2}$$

while $w \in \mathbb{R}^q$ stands for the plant uncertainty vector. A_w, B_w, C_w, P_w, and Q_w are real matrices with appropriate dimensions, whose coefficients depend on the plant uncertainties, their nominal values matrices being A_0, B_0, C_0, P_0, and Q_0. Notice, finally, that in (2.1) it is assumed that $d = P_w v$, $y = C_w x$, and $y_R = -Q_w v$.

Let the dynamic state feedback law

$$\begin{aligned} \dot{z} &= \Phi z + Ne, \\ u &= H_1 x + H_2 z, \end{aligned} \tag{2.3}$$

with z, Φ, N, H_1, and H_2 being real vector and matrices of appropriate dimensions. Then, the forced closed-loop system consisting of the plant (2.1), the exosystem (2.2) and the control law (2.3) is

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A_w + B_w H_1 & B_w H_2 \\ NC_w & \Phi \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} P_w \\ NQ_w \end{pmatrix} v, \\ \dot{v} &= Sv, \\ e &= C_w x + Q_w v. \end{aligned} \tag{2.4}$$

Denote also by

$$A_{cw} = \begin{pmatrix} A_w + B_w H_1 & B_w H_2 \\ NC_w & \Phi \end{pmatrix} \tag{2.5}$$

the matrix of the unforced closed-loop system, A_{c0} being its nominal value matrix.

Within this framework, the LROR problem may be posed as follows.

Definition 2.1. Let W be an open subset of $\mathbb{R}^q$ that contains the origin $w = 0$. The LROR problem of (2.1) with exosystem (2.2) in W, by means of dynamic state feedback, consists of designing a control law of the form (2.3) such that one has the following:

(i) The matrix A_{c0} defined from (2.5) is Hurwitz, that is, $\sigma(A_{c0}) \subset \mathbb{C}^-$, where $\sigma(A_{c0})$ denotes the spectrum of A_{c0}.

(ii) The matrix A_{cw} in (2.5) is such that $\sigma(A_{cw}) \subset \mathbb{C}^-$, for all $w \in W$; furthermore, for all $x(0), z(0), v(0)$ and for all $w \in W$, the trajectories of (2.4) satisfy

$$\lim_{t \to \infty} e(t) = \lim_{t \to \infty} [C_w x(t) + Q_w v(t)] = 0. \tag{2.6}$$

A necessary and sufficient condition for the solvability of the LROR problem using dynamic state feedback is established in the next result.

Theorem 2.2 (see [6]). *Consider the plant* (2.1) *with exosystem* (2.2). *Assume that the pair* (A_0, B_0) *is stabilizable and also that* $\sigma(S) \subset \overline{\mathbb{C}^+} = \{\lambda \in \mathbb{C};\ \mathrm{Re}(\lambda) \geq 0\}$. *Then, the LROR problem is solvable* *by means of a dynamic state feedback controller* (2.3) *if and only if*

$$\operatorname{rank}\begin{pmatrix} A_0 - \lambda \mathbb{I} & B_0 \\ C_0 & 0 \end{pmatrix} = n + m, \quad \forall \lambda \in \sigma(S). \tag{2.7}$$

Assume that system (2.1)-(2.2) satisfies condition (2.7). Then, the construction of a linear robust regulator with a dynamic state feedback control law of the form (2.3) may be carried out as follows [6]. Let

$$mp_S(\lambda) = \lambda^r + a_{r-1}\lambda^{r-1} + \cdots + a_1\lambda + a_0 \tag{2.8}$$

be the minimal polynomial of S. Then, consider the matrices

$$\Phi_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{r-1} \end{pmatrix}, \qquad N_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{2.9}$$

with $\Phi_i \in M_{r\times r}(\mathbb{R})$, $N_i \in M_{r\times 1}(\mathbb{R})$, and define

$$\begin{aligned} \Phi &= \operatorname{diag}(\Phi_1, \ldots, \Phi_m) \in M_{rm\times rm}(\mathbb{R}), \\ N &= \operatorname{diag}(N_1, \ldots, N_m) \in M_{rm\times m}(\mathbb{R}). \end{aligned} \tag{2.10}$$

This selection of Φ and N, which ensures the controllability of (Φ, N), together with the assumed stabilizability of (A_0, B_0) and the fulfillment of (2.7), yields the stabilizability of the following pair:

$$\overline{A}_0 = \begin{pmatrix} A_0 & 0 \\ NC_0 & \Phi \end{pmatrix}, \qquad \overline{B}_0 = \begin{pmatrix} B_0 \\ 0 \end{pmatrix}, \tag{2.11}$$

where $\overline{A}_0$ and $\overline{B}_0$ are, respectively, $(n + rm) \times (n + rm)$ and $(n + rm) \times m$ matrices. Thus, there exists $H \in M_{m\times(n+rm)}(\mathbb{R})$ such that

$$\sigma\left(\overline{A}_0 + \overline{B}_0 H\right) = \sigma\left[\begin{pmatrix} A_0 + B_0 H_1 & B_0 H_2 \\ NC_0 & \Phi \end{pmatrix}\right] \subset \mathbb{C}^-. \tag{2.12}$$

Selecting Φ, N, H_1, and H_2 as indicated above, the dynamic state feedback regulator (2.3) solves the LROR problem for system (2.1)-(2.2).

Remark 2.3. It is proved in [6] that, under the assumption $\sigma(S) \in \overline{\mathbb{C}^+}$, if a controller (2.3) solves the LROR problem for (2.1) and (2.2), then, for all $w \in W$, there exist unique matrices (Π_w, Σ_w) that satisfy the following matrix equations:

$$\begin{pmatrix} \Pi_w \\ \Sigma_w \end{pmatrix} S = \begin{pmatrix} A_w + B_w H_1 & B_w H_2 \\ N C_w & \Phi \end{pmatrix} \begin{pmatrix} \Pi_w \\ \Sigma_w \end{pmatrix} + \begin{pmatrix} P_w \\ N Q_w \end{pmatrix}, \tag{2.13}$$

$$0 = C_w \Pi_w + Q_w. \tag{2.14}$$

For every $w \in W$, (2.13) indicates that $M_{cw} = \{(x, z, v);\ x = \Pi_w v,\ z = \Sigma_w v\}$ is an invariant manifold for the closed-loop system (2.4), while (2.14) means that the error is zero on the invariant manifold M_{cw}. Furthermore, let $v(0) = v^*$ be any initial condition for the exosystem; then, the corresponding exogenous input is $v^*(t) = \exp(St)v^*$. If the initial state of the plant (x, z) in (2.4) is set to $x(0) = \Pi_w v^*$, $z(0) = \Sigma_w v^*$, it is immediate that $x(t) = \Pi_w v^*(t)$, $z(t) = \Sigma_w v^*(t)$, and, subsequently, $e = 0$, for all $t \geq 0$.

3. Output Regulation in a Class of Nonlinear Switched Power Converters

The basic nonlinear switched power converters NonInverting Buck-Boost, Full-Bridge NonInverting Buck-Boost, Watkins-Johnson and Inverse of Watkins-Johnson have a general state-space representation in terms of an averaged model consisting of a two-dimensional system with the inductor current i_L and the capacitor voltage v_C as state variables, and a control variable $\hat{u} = (\hat{u}_1, \hat{u}_2)^\top$. It is worth mentioning that the control action in the physical system is carried out by means of switches; hence, $\hat{u}_1$ and $\hat{u}_2$ are to be actually implemented through an appropriate PWM signal.

For a systematic analysis it is advisable to minimize the number of parameters of the system. This goal may be achieved with the change of variables and parameters:

$$x_1 = \frac{1}{V_g}\sqrt{\frac{L}{C}}\, i_L, \qquad x_2 = \frac{1}{V_g} v_C, \qquad t = \frac{1}{\sqrt{LC}}\tau, \qquad \mu = \frac{1}{R}\sqrt{\frac{L}{C}}, \tag{3.1}$$

which make the system dimensionless:

$$\begin{aligned} \dot{x}_1 &= \hat{u}_1 - x_2 \hat{u}_2 + k_1(\hat{u}_2 - 1) + k_2 x_2 (1 - \hat{u}_1), \\ \dot{x}_2 &= -\mu x_2 + x_1 \hat{u}_2 - k_2 x_1 (1 - \hat{u}_1). \end{aligned} \tag{3.2}$$

The control gains $\hat{u}_1, \hat{u}_2$ take values in $[\hat{u}^-, \hat{u}^+] \times [\hat{u}^-, \hat{u}^+]$, with $\hat{u}_- < \hat{u}^+$. The values of the parameters k_1, k_2 and of the lower and upper bounds of the control gains for the different converters are summarized in Table 1. Moreover, assume an unknown value R for the load resistance, due to the addition of a constant disturbance term R_w to its nominal value R_N;

Table 1: State space descriptors of a class of switched power converters.

Converters	k_1	k_2	$\hat{u}^-$	$\hat{u}^+$
NonInverting Buck-Boost	0	0	0	1
Full-Bridge NonInverting Buck-Boost	0	0	−1	1
Watkins-Johnson	1	0	0	1
Inverse of Watkins-Johnson	0	1	0	1

that is, $R = R_N + R_w$, with $R_N > 0$ and $R_w \in (-R_N, +\infty)$. Consequently, the parameter μ may be written as $\mu = \mu_N - w$, with

$$\mu_N = \frac{1}{R_N}\sqrt{\frac{L}{C}} > 0, \qquad w = \frac{\mu_N R_w}{R_N + R_w} \in (-\infty, \mu_N), \tag{3.3}$$

w being the only uncertain parameter of the system.

Assigning

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \hat{u} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}, \qquad \delta = \begin{pmatrix} -k_1 \\ 0 \end{pmatrix}, \qquad A_N = \begin{pmatrix} 0 & k_2 \\ -k_2 & -\mu_N \end{pmatrix}, \tag{3.4}$$

$$A_w = \begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}, \qquad B(x) = \begin{pmatrix} -k_2 x_2 + 1 & -x_2 + k_1 \\ k_2 x_1 & x_1 \end{pmatrix}, \tag{3.5}$$

the dynamical system (3.2) may be written as

$$\dot{x} = (A_N + A_w)x + \delta + B(x)\hat{u}. \tag{3.6}$$

Furthermore, notice that $\det B(x) = x_1(1 - k_1 k_2) = x_1$ according to the admissible values for k_1, k_2 indicated in Table 1. Hence, assuming $x_1 \neq 0$, the state feedback control law

$$\hat{u} = B^{-1}(x)[u - A_N x - \delta] \tag{3.7}$$

transforms system (3.6) into

$$\dot{x} = A_w x + u. \tag{3.8}$$

Let us now consider the problem of rendering the state x of system (3.8) to asymptotically track a certain reference profile $x = x_R(t)$, which can be expressed as a linear combination of the solutions of a time-invariant, linear exosystem; that is, there exist real matrices S and Q, of appropriate dimensions, such that

$$\begin{aligned} \dot{v} &= Sv, \\ v(0) &= v_0, \\ x_R &= Qv. \end{aligned} \tag{3.9}$$

Notice that an exosystem as (3.9) with $\sigma(S) \subset \overline{\mathbb{C}^+}$ can generate a large class of functions, including combinations of step functions with arbitrary amplitude, ramps with arbitrary slope or sinusoidal signals with arbitraries amplitude and initial phase. These are the type of references/disturbances usually faced by power converters.

Therefore, the problem may be posed as that of finding a linear robust output regulator for

$$\begin{aligned} \dot{x} &= A_w x + u, \\ \dot{v} &= Sv, \\ e &= x - Qv. \end{aligned} \tag{3.10}$$

Let $\mathbb{I}_2$ and 0_2 denote, respectively, the identity matrix and the null matrix in the set of 2×2 matrices. Then, identifying the elements of the original system (2.1) with those of the particular case (3.10) one gets

$$B_w = B_0 = \mathbb{I}_2, \qquad P_w = P_0 = 0_2, \qquad C_w = C_0 = \mathbb{I}_2, \qquad Q_w = Q_0 = -Q, \tag{3.11}$$

while A_w is defined in (3.5) and, subsequently, $A_0 = 0_2$.

Proposition 3.1. *Let us consider system* (3.10) *and the equivalences* (3.5)–(3.11). *Then,*

(i) *The pair* $(A_0, B_0) = (0_2, \mathbb{I}_2)$ *is controllable.*

(ii) *The following matrix is nonsingular for all* $\lambda \in \mathbb{C}$*:*

$$\begin{pmatrix} A_0 - \lambda\mathbb{I}_2 & B_0 \\ C_0 & 0_2 \end{pmatrix} = \begin{pmatrix} -\lambda\mathbb{I}_2 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0_2 \end{pmatrix}. \tag{3.12}$$

Proof. The proof is immediate. □

Theorem 3.2. *Consider the plant* (3.10), *and assume that* $\sigma(S) \subset \overline{\mathbb{C}^+}$. *Then, the LROR problem is solvable by means of a dynamic state feedback controller* (2.3).

Proof. The result follows using Proposition 3.1 and Theorem 2.2. □

Let (2.3) be a dynamic state feedback controller that solves the LROR problem for the system (3.10). Turning back now to the original system (3.6), the corresponding feedback control law is to be obtained using (2.3) in (3.7). However, recalling that the original control vector $\hat{u}$ has fixed gain values and taking into account (3.7), it is easily realizable that (2.3) will be actually useful for output regulation situations in which the trajectories $x(\cdot)$ of (3.6) remain entirely inside the state-space region X defined as

$$X = \left\{ x \in \mathbb{R}^2 ; \det\, B(x) \neq 0 \wedge B^{-1}(x)[u - A_N x - \delta] \in \widehat{U} \right\}, \tag{3.13}$$

with $\widehat{U} = [\hat{u}^-, \hat{u}^+] \times [\hat{u}^-, \hat{u}^+]$. Notice that the restriction $\det B(x) \neq 0$ is necessary and sufficient for guaranteeing the diffeomorphic character of the transformation (3.7), while the requirement $\hat{u}$ in $\hat{U}$ ensures nonsaturation of the controller performance. Eventually, this fact entails restrictions on the set of initial values $x(0)$, $z(0)$, $v(0)$ and parametric uncertainties W (see issue (ii) in Definition 2.1) from which robust output regulation is attainable.

At this point, specific conditions to be accomplished by candidate reference profiles x_R are especially interesting. They follow immediately from the assumption that x_R lies in X. Hence, assume that system (3.10) has achieved a steady-state $x = x_R$ under the action of the dynamic state feedback controller (2.3), and denote z_R, u_R, the corresponding stationary behavior for u and z; it follows from Remark 2.3 that $x_R = \Pi_w v$, $z_R = \Sigma_w v$, (Π_w, Σ_w) being the solution of (2.13) and (2.14), and $u_R = H_1 x_R + H_2 z_R$ by construction. Moreover, it is straightforward from the assignment (3.5)–(3.11) that $\Pi_w = Q$, while Σ_w is such that

$$\begin{aligned} QS &= (A_w + H_1)Q + H_2\Sigma_w, \\ \Sigma_w S &= \Phi S. \end{aligned} \tag{3.14}$$

Hence, using (3.9),

$$x_R = Qv = Q\exp(St)v_0, \tag{3.15}$$

$$z_R = \Sigma_w v = \Sigma_w \exp(St)v_0, \tag{3.16}$$

$$u_R = (H_1 Q + H_2\Sigma_w)\exp(St)v_0, \tag{3.17}$$

where Σ_w satisfies (3.14). Furthermore, from (3.8) we obtain an alternative expression for u_R:

$$u_R = \dot{x}_R - A_w x_R = QSv - A_w Qv = (QS - A_w Q)\exp(St)v_0. \tag{3.18}$$

Proposition 3.3. *Let $x_R = (x_{1R}, x_{2R})^\top$ satisfying* (3.15) *be a reference profile for system* (3.8), (3.9), *and let X be the set defined in* (3.13). *Then, $x_R \in$ X, for all $t \geq 0$, if and only if the following relations are fulfilled:*

$$\begin{aligned} x_{1R} = q_1 v = q_1\exp(St)v_0 \neq 0, &\quad \forall t \geq 0, \\ B^{-1}(x_R)\left[(QS - A_N Q - A_w Q)\exp(St)v_0 - \delta\right] \in \hat{U}, &\quad \forall t \geq 0, \end{aligned} \tag{3.19}$$

where q_1 denotes the first row of matrix Q, that is, $Q = \mathrm{col}(q_1, q_2)$.

Proof. The proof is immediate using (3.4)-(3.5), Table 1, (3.15), and (3.18) in (3.13). □

The next result establishes sufficient conditions for a command profile x_R in such a way that the dynamic state feedback regulator (2.3) that solves the LROR for system (3.10) also yields robust tracking of x_R by the original system (3.6).

Theorem 3.4. *Let $x_R = (x_{1R}, x_{2R})^\top$ satisfying* (3.15) *be a reference profile for system* (3.6) *in such a way that*

$$x_{1R} = q_1 v = q_1 \exp(St) v_0 \neq 0, \quad \forall t \geq 0,$$
$$B^{-1}(x_R)\left[(QS - A_N Q)\exp(St)v_0 - \delta\right] \in \widehat{U}, \quad \forall t \geq 0, \tag{3.20}$$

where q_1 denotes the first row of matrix Q. Let also (2.3) *be a dynamic state feedback controller that solves the LROR problem for system* (3.10)*. Then, there exist open subsets $X_0 \subset \mathbb{R}^2$, $Z_0 \subset \mathbb{R}^{2r}$ and $W_0 \in \mathbb{R}$, with $0 \in W_0$, such that, for all $(x(0), z(0)) \in X_0 \times Z_0$ and for all $w \in W_0$, the controller* (2.3) *and* (3.7)*, produces*

$$\lim_{t\to\infty} x(t) = x_R(t) \tag{3.21}$$

in system (3.6)*, with $x(t) \in X$, for all $t \geq 0$, X being the region defined in* (3.13).

Proof. As, by hypothesis, (2.3) solves the LROR problem for (3.10), Definition 2.1 ensures the existence of a neighborhood $W \subset \mathbb{R}$ of the origin $w = 0$ where

(i) $\sigma(A_{cw}) \subset \mathbb{C}^-$, for all $w \in W$, A_{cw} being the matrix defined in (2.5), which now is (see (3.5)–(3.11))

$$A_{cw} = \begin{pmatrix} A_w + H_1 & H_2 \\ N & \Phi \end{pmatrix}; \tag{3.22}$$

(ii) for all $x(0), z(0)$ and for all $w \in W$, the trajectories of (3.10) satisfy $\lim_{t\to\infty} x(t) = x_R(t)$.

Assume that x_R satisfies (3.20). It then follows by continuity that there exists an open subset $\widehat{W} \subset \mathbb{R}$, containing the origin, such that x_R also satisfies (3.19), for all $w \in \widehat{W}$; moreover, $W \cap \widehat{W}$ is trivially nonempty and open. Continuity also guarantees the existence of three open subsets, $X_0 \in \mathbb{R}^2$, with $Qv_0 \in X_0$, $Z_0 \in \mathbb{R}^{2r}$, with $\Sigma_0 v_0 \in Z_0$, Σ_0 satisfying (3.16) for $w = 0$, and $W_0 \subseteq W \cap \widehat{W}$, such that, for all $(x(0), z(0)) \in X_0 \times Z_0$ and for all $w \in W_0$, it results that $x(t) \in X$, for all $t \geq 0$. □

The size of the open set $W_0 \subseteq W \cap \widehat{W} \subseteq (-\infty, \mu_N)$ (recall from (3.3) that $w \in (-\infty, \mu_N)$) of possible parametric uncertainties that can be accommodated by the control system (3.6), (3.7), (2.3) during the tracking task of a certain reference x_R satisfying (3.9), is studied below.

On the one hand, W depends on the features of the regulator (2.3). Indeed, W coincides with the set where the matrix A_{cw} is exponentially stable [6], that is,

$$W = \left\{w \in (-\infty, \mu_N); \sigma(A_{cw}) \subset \mathbb{C}^-\right\}, \tag{3.23}$$

with A_{cw} defined in (3.22). The next section contains a design procedure for (2.3) in such a way that a necessary condition for having $W = (-\infty, \mu_N)$ is fulfilled; furthermore, this condition is also sufficient for regulation (x_R constant, i.e., $S = 0$.) tasks.

On the other hand, the size of $\widehat{W}$ can be tuned at will under certain restrictions. Indeed, assuming that the perturbed parameter w belongs to a known, closed interval $[w_m, w_M]$, with $w_m \leq 0 \leq w_M$, and also that x_{2R} satisfies mild hypotheses that include periodicity, a technique based on semi-infinite programming methods developed in [9] allows the obtention of an also periodic reference x_{1R} for x_1, with minimum Root Mean Square, in such a way that (3.19) are verified for all $w \in [w_m, w_M]$, that is, $\widehat{W} = [w_m, w_M]$. Then, if the dynamic state feedback regulator (2.3) is also designed in order to have $W = (-\infty, \mu_N)$, then $W \cap \widehat{W} = \widehat{W}$.

Finally, assume that x_R is such that (3.19) are satisfied for all $w \in W \cap \widehat{W}$. The set $W_0 \subseteq W \cap \widehat{W}$ is strongly dependent on the initial conditions $x(0)$, $z(0)$ and the distance between the actual value of w and $w = 0$. A good selection for $x(0)$ is $x(0) \sim x_R(0) = Qv_0$. However, $z_R(t)$ depends on w (see (3.16)); thus, the setting $z(0) \sim \Sigma_0 v_0$ makes W_0 contain the values w for which the distance $\|(\Sigma_w - \Sigma_0)v_0\|$ is small enough. Otherwise, assuming that $w \in [w_m, w_M]$, alternative assignments such as $z(0) \sim \Sigma_{\overline{w}} v_0$, $\overline{w}$ being a certain value of the interval $[w_m, w_M]$, should be considered (see Remark 2.3).

4. Construction of a Dynamic State Feedback Regulator

The construction of a dynamic state feedback regulator (2.3) for system (3.10) is carried out using the results of Section 3.

Assume that $\sigma(S) \subset \overline{\mathbb{C}^+}$. Let $mp_S(\lambda)$ be the minimal polynomial of S written as in (2.8), with $\deg mp_S(\lambda) = r$, and consider the matrices Φ_i and N_i defined in (2.9). Therefore, using (2.10), let

$$\begin{aligned} \Phi &= \operatorname{diag}(\Phi_1, \Phi_2) \in M_{2r\times 2r}(\mathbb{R}), \\ N &= \operatorname{diag}(N_1, N_2) \in M_{2r\times 2}(\mathbb{R}). \end{aligned} \tag{4.1}$$

It was already commented in Section 2 that, with this selection of Φ and N, the pair $(\overline{A}_0, \overline{B}_0)$ defined in (2.11) is stabilizable. Hence, $H = (H_1 \ H_2) \in M_{2\times(2+2r)}(\mathbb{R})$ can be selected in such a way that $\sigma(\overline{A}_0 + \overline{B}_0 H) \subset \mathbb{C}^-$. However, the situation for system (3.10) is even better, because the corresponding pair is controllable, as stated in the next result.

Proposition 4.1. *Consider the matrices* Φ*,* N*, defined in* (4.1)*. Then, the following pair is controllable:*

$$\overline{A}_0 = \begin{pmatrix} 0_2 & 0_{2\times 2r} \\ N & \Phi \end{pmatrix}, \qquad \overline{B}_0 = \begin{pmatrix} \mathbb{I}_2 \\ 0_{2r\times 2} \end{pmatrix}. \tag{4.2}$$

Proof. Notice that

$$\overline{A}_0^k \overline{B}_0 = \begin{pmatrix} 0_2 \\ \Phi^{k-1} N \end{pmatrix}, \quad \forall k \geq 1. \tag{4.3}$$

Therefore, the controllability matrix $\mathcal{C}(\overline{A}_0, \overline{B}_0) = (\overline{B}_0, \overline{A}_0\overline{B}_0, \dots, \overline{A}_0^{2r+1}\overline{B}_0)$ is such that

$$\operatorname{rank} \mathcal{C}\left(\overline{A}_0, \overline{B}_0\right) = \operatorname{rank}\begin{pmatrix} \mathbb{I}_2 & 0_2 & 0_2 & \cdots & 0_2 \\ 0_{2r\times 2} & N & \Phi N & \cdots & \Phi^{2r}N \end{pmatrix} = \operatorname{rank} \mathbb{I}_2 + \operatorname{rank}\left[\mathcal{C}(\Phi, N), \Phi^{2r}N\right] = 2 + 2r, \tag{4.4}$$

because of the fact that (Φ, N) is controllable by construction. □

Then, by Proposition 4.1, there exists $H = (H_1 \ H_2)$ that allows an arbitrary placement of the poles of the closed-loop system $\overline{A} + \overline{B}H$. The regulator is therefore ensured to be robust for all $w \in W$, W being the set defined in (3.23).

As discussed in Section 3, it is of obvious interest to place the poles of the unperturbed system in such a way that $W = (-\infty, \mu_N)$. The design procedure suggested below, besides guaranteeing robustness in an open neighborhood of $w = 0$, gives a general necessary condition for having $W = (-\infty, \mu_N)$. However, arbitrary pole-placement must be replaced by stable pole-placement. This condition appears to be sufficient in the lowest dimensional case $S = 0$ (i.e., $r = 1$), that is, for regulation purposes. Other cases may demand further analysis of the resulting A_{cw} in order to establish the region W where robustness is preserved.

Hence, consider the perturbed system associated to (4.2):

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \overline{A}_w \begin{pmatrix} x \\ z \end{pmatrix} + \overline{B}_w u, \tag{4.5}$$

with

$$\overline{A}_w = \begin{pmatrix} A_w & 0_{2\times 2r} \\ N & \Phi \end{pmatrix}, \qquad \overline{B}_w = \overline{B}_0 = \begin{pmatrix} \mathbb{I}_2 \\ 0_{2r\times 2} \end{pmatrix}, \tag{4.6}$$

and $\overline{A}_w(w = 0) = \overline{A}_0$. The change of variables

$$\overline{x}_1 = \begin{pmatrix} \overline{x}_{11} \\ \overline{x}_{12} \\ \vdots \\ \overline{x}_{1,r+1} \end{pmatrix} = \begin{pmatrix} x_1 \\ z_1 \\ \vdots \\ z_R \end{pmatrix}, \qquad \overline{x}_2 = \begin{pmatrix} \overline{x}_{21} \\ \overline{x}_{22} \\ \vdots \\ \overline{x}_{2,r+1} \end{pmatrix} = \begin{pmatrix} x_2 \\ z_{r+1} \\ \vdots \\ z_{2R} \end{pmatrix} \tag{4.7}$$

transforms system (4.5) into the block-diagonal form:

$$\begin{pmatrix} \dot{\overline{x}}_1 \\ \dot{\overline{x}}_2 \end{pmatrix} = \begin{pmatrix} \overline{A}_1 & 0_{r+1} \\ 0_{r+1} & \overline{A}_2 \end{pmatrix} \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} + \begin{pmatrix} \overline{B}_1 & 0_{(r+1)\times 1} \\ 0_{(r+1)\times 1} & \overline{B}_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{4.8}$$

where

$$\overline{A}_1 = \begin{pmatrix} 0 & 0_{1\times r} \\ N_1 & \Phi_1 \end{pmatrix}, \qquad \overline{A}_2 = \begin{pmatrix} w & 0_{1\times r} \\ N_2 & \Phi_2 \end{pmatrix}, \qquad \overline{B}_1 = \overline{B}_2 = \begin{pmatrix} 1 \\ 0_{r\times 1} \end{pmatrix}. \tag{4.9}$$

Since the disturbance free pair $(\overline{A}_1, \overline{B}_1)$ is trivially controllable, its poles can be arbitrarily placed by means of appropriate feedback and are not affected by the perturbation. Thence, let us denoted by

$$p_{\overline{A}_1}(\lambda) = -\lambda p_{\Phi_1}(\lambda) = -\lambda mp_S(\lambda) = -\left(\lambda^{r+1} + a_{r-1}\lambda^r + \cdots + a_0\lambda\right) \tag{4.10}$$

the characteristic polynomial of $\overline{A}_1$. It is well-known that $(\overline{A}_1, \overline{B}_1)$ achieves the controllable canonical form:

$$\widetilde{A}_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -a_0 & -a_1 & \cdots & -a_{r-1} \end{pmatrix}, \qquad \widetilde{B}_1 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{4.11}$$

on the base of $\mathbb{R}^{n+1}$ defined by the column vectors of the matrix

$$T = \operatorname{col}\left(\overline{A}_1^r \overline{B}_1 + a_{r-1}\overline{A}_1^{r-1}\overline{B}_1 + \cdots + a_0\overline{B}_1, \ldots, \overline{A}_1\overline{B}_1 + a_{r-1}\overline{B}_1, \overline{B}_1\right). \tag{4.12}$$

Let us now assume that the feedback subsystem is in canonical form

$$\begin{aligned} \dot{\widetilde{x}}_1 &= \widetilde{A}_1\widetilde{x}_1 + \widetilde{B}_1 u_1, \\ u_1 &= \widetilde{H}_1\widetilde{x}_1, \end{aligned} \tag{4.13}$$

where $\widetilde{x}_1 = T^{-1}\overline{x}_1$, is expected to possess a spectrum such as $\sigma(\widetilde{A}_1 + \widetilde{B}_1\widetilde{H}_1) = \{\lambda_{11}, \ldots, \lambda_{1,r+1}\} \subset \mathbb{C}^-$, and let $\{\alpha_{10}, \ldots, \alpha_{1r}\} \subset \mathbb{R}^+$ be the coefficients of the corresponding characteristic polynomial:

$$(\lambda - \lambda_{11})(\lambda - \lambda_{12})\cdots(\lambda - \lambda_{1,r+1}) = \lambda^{r+1} + \alpha_{1r}\lambda^r + \cdots + \alpha_{10}. \tag{4.14}$$

Proposition 4.2. *Let $\widetilde{H}_1 = (\widetilde{h}_{11} \cdots \widetilde{h}_{1,r+1})$, $h_{1i} \in \mathbb{R}$, for all $i \in \{1, \ldots, r+1\}$, be a feedback matrix for system* (4.13)*. If the gains are selected as*

$$\begin{aligned} \widetilde{h}_{11} &= -\alpha_{10}, \\ \widetilde{h}_{1k} &= -\alpha_{1,k-1} + a_{k-2}, \quad k = 2, \ldots, r+1, \end{aligned} \tag{4.15}$$

then the characteristic polynomial of $\widetilde{A}_1 + \widetilde{B}_1\widetilde{H}_1$ coincides with (4.14)*, which makes the system robust for all $w \in (-\infty, \mu_N)$.*

Proof. It is immediate from (4.11) that the characteristic polynomial of $\widetilde{A}_1 + \widetilde{B}_1\widetilde{H}_1$ is

$$p_1(\lambda) = \lambda^{r+1} + \left(a_{r-1} - \widetilde{h}_{1,r+1}\right)\lambda^r + \cdots + \left(a_0 - \widetilde{h}_{12}\right)\lambda - \widetilde{h}_{11}. \tag{4.16}$$

The substitution of (4.15) in (4.16) yields the result. □

The final step should be the transformation of $\widetilde{H}_1$ into the original $\overline{x}_1$-base:

$$H_{\overline{x}_1} = \widetilde{H}_1 T^{-1}. \tag{4.17}$$

Proposition 4.3. *Let* $(\overline{A}_2, \overline{B}_2)$ *be defined from* (4.9). *The base transformation with associated matrix* T *introduced in* (4.12) *reduces the pair to the canonical form* $(\widetilde{A}_2, \widetilde{B}_2)$, *with*

$$\widetilde{A}_2 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ wa_0 & (wa_1 - a_0) & \cdots & (w - a_{r-1}) \end{pmatrix}, \qquad \widetilde{B}_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{4.18}$$

Proof. Taking into account (4.9), the linear system associated to the pair $(\overline{A}_2, \overline{B}_2)$ may be written as

$$\dot{\overline{x}}_2 = \overline{A}_2\overline{x}_2 + \overline{B}_2\overline{u}_2 = \overline{A}_1\overline{x}_2 + \begin{pmatrix} w\overline{x}_{21} \\ 0_{r\times 1} \end{pmatrix} + \begin{pmatrix} 1 \\ 0_{r\times 1} \end{pmatrix} u_2. \tag{4.19}$$

Assigning

$$\overline{u}_2 = w\overline{x}_{21} + u_2, \tag{4.20}$$

Equation (4.19) takes the form: $\dot{\overline{x}}_2 = \overline{A}_1\overline{x}_2 + \overline{B}_2\overline{u}_2$. Performing the base transformation $\widetilde{x}_2 = T^{-1}\overline{x}_2$ and recalling from (4.9) that $\overline{B}_2 = \overline{B}_1$, one gets that

$$\dot{\widetilde{x}}_2 = \widetilde{A}_1\widetilde{x}_2 + \widetilde{B}_2\overline{u}_2, \tag{4.21}$$

with $\widetilde{x}_2 = T^{-1}\overline{x}_2$, $\widetilde{B}_2 = \widetilde{B}_1$ and $\widetilde{A}_1$, $\widetilde{B}_1$ being described in (4.11). Observe now that the state vector component $\overline{x}_{21}$ may be expressed as $\overline{x}_{21} = T_1\widetilde{x}_2 = T_{11}\widetilde{x}_{21} + \cdots + T_{1,r+1}\widetilde{x}_{2,r+1}$, where $T_1 = (T_{11}, \ldots, T_{1,r+1})$ stands for the first row of T; denoting the Kronecker product by $\otimes$, the reversion of the change (4.20) in (4.21) results in

$$\dot{\widetilde{x}}_2 = \widetilde{A}_2\widetilde{x}_2 + \widetilde{B}_2 u_2, \quad \text{with } \widetilde{A}_2 = T^{-1}\overline{A}_2 T = \widetilde{A}_1 + w\widetilde{B}_2 \otimes T_1. \tag{4.22}$$

Therefore, $\widetilde{A}_2$ is a matrix in controllable canonical form, and its characteristic polynomial coincides with that of $\overline{A}_2$ due to the invariance property under base transformations:

$$
\begin{aligned}
p_{\widetilde{A}_2}(\lambda) &= p_{\overline{A}_2}(\lambda) = (w-\lambda)p_{\Phi_2}(\lambda) \\
&= -\Big[\lambda^{r+1} + (a_{r-1}-w)\lambda^r + +(a_{r-2}-wa_{r-1})\lambda^{r-1} + \cdots + (a_0 - wa_1)\lambda - wa_0\Big].
\end{aligned}
\tag{4.23}
$$

Hence, the result follows. □

Consider the canonical feedback subsystem:

$$
\begin{aligned}
\dot{\tilde{x}}_2 &= \widetilde{A}_2\tilde{x}_2 + \widetilde{B}_2 u_2, \\
u_2 &= \widetilde{H}_2\tilde{x}_2.
\end{aligned}
\tag{4.24}
$$

On the one hand notice that since $\widetilde{A}_2|_{w=0} = \widetilde{A}_1$, an assignment of feedback gains equivalent to (4.15) guarantees robustness at least in a certain neighborhood of $w = 0$. On the other hand, it is well-known that a necessary condition for the stability of a polynomial is that all its coefficients have the same sign, which is also sufficient for polynomials of degree 2.

Therefore, assume that (4.24) is expected to possess the following spectrum for $w = 0$: $\sigma(\widetilde{A}_2|_{w=0} + \widetilde{B}_2\widetilde{H}_2) = \{\lambda_{21}, \ldots, \lambda_{2,r+1}\} \subset \mathbb{C}^-$, and let $\{\alpha_{20}, \ldots, \alpha_{2r}\} \subset \mathbb{R}^+$ be the coefficients of the corresponding characteristic polynomial:

$$
(\lambda-\lambda_{21})(\lambda-\lambda_{22})\cdots(\lambda-\lambda_{2,r+1}) = \lambda^{r+1} + \alpha_{2r}\lambda^r + \cdots + \alpha_{20}. \tag{4.25}
$$

Proposition 4.4. *Let* $\widetilde{H}_2 = (\tilde{h}_{21} \cdots \tilde{h}_{2,r+1})$, $\tilde{h}_{2i} \in \mathbb{R}$, *for all* $i \in \{1, \ldots, r+1\}$, *be a feedback matrix for system* (4.24).

(i) *If the gains are selected as*

$$
\begin{aligned}
\tilde{h}_{21} &= -\alpha_{20}, \\
\tilde{h}_{2k} &= -\alpha_{2,k-1} + a_{k-2}, \quad k = 2, \ldots, r+1,
\end{aligned}
\tag{4.26}
$$

then the characteristic polynomial of $\widetilde{A}_2 + \widetilde{B}_2\widetilde{H}_2$ *coincides with* (4.25) *for* $w = 0$, *which makes the subsystem robust in a neighbourhood of* $w = 0$.

(ii) *Moreover, if*

$$
\begin{aligned}
\alpha_{2k} &> \epsilon_k + \mu_N|a_k|, \quad k = 0, \ldots, r-1, \\
\alpha_{2r} &> \mu_N,
\end{aligned}
\tag{4.27}
$$

with $\epsilon_k > 0$, *for all* $k = 0, \ldots, r-1$, *then all the coefficients of the characteristic polynomial of the feedback system* $\widetilde{A}_2 + \widetilde{B}_2\widetilde{H}_2$ *are positive, for all* $w \in (-\infty, \mu_N)$; *furthermore, if* $r = 1$ *then the system is stable for all* $w \in (-\infty, \mu_N)$.

Proof. It is straightforward from (4.18) that the characteristic polynomial of $\widetilde{A}_2 + \widetilde{B}_2\widetilde{H}_2$ is

$$p_2(\lambda) = \lambda^{r+1} + \left(a_{r-1} - w - \widetilde{h}_{2,r+1}\right)\lambda^r + \cdots + \left(a_0 - wa_1 - \widetilde{h}_{22}\right)\lambda + \left(wa_0 + \widetilde{h}_{21}\right). \tag{4.28}$$

(i) The replacement (4.26) in (4.28) yields

$$p_2(\lambda) = \lambda^{r+1} + (\alpha_{2r} - w)\lambda^r + \cdots + (\alpha_{21} - wa_1)\lambda + (\alpha_{20} - wa_0), \tag{4.29}$$

from which issue (i) follows immediately.

(ii) The assumption $\sigma(S) \subset \overline{\mathbb{C}^+}$ entails $a_k \in \mathbb{R}$, for all $k = 0, \dots, r-1$ and, since $\epsilon_k > 0$, for all $k = 0, \dots, r-1$ by hypothesis, (4.27) results in the coefficients of (4.29) satisfying

$$\begin{gathered} \alpha_{2k} - wa_k > \epsilon_k + (\mu_N - w)|a_i| > 0, \quad k = 0, \dots, r-1, \\ \alpha_{2r} - w > \mu_N - w > 0, \end{gathered} \tag{4.30}$$

for all $w \in (-\infty, \mu_N)$. The statement for the case $r = 1$ is therefore trivial. □

Remark 4.5. Propositions 4.2 and 4.4 allow to conclude the following.

(i) The assignments (4.15) and (4.26) yield robustness in a neighborhood of $w = 0$. Indeed, the actual set W is given by $W = \{w < \mu_N; \sigma[p_2(\lambda)] \subset \mathbb{C}^-\}$, with $p_2(\lambda)$ defined in (4.29).

(ii) The assignments (4.15) and (4.26) and the restriction (4.27) constitute a necessary condition for having $W = (-\infty, \mu_N)$. In case that $S = 0$, that is, $r = 1$, the condition becomes necessary and sufficient.

Once the feedback matrix $\widetilde{H}_2$ has been constructed following either (4.26) or (4.26)-(4.27), the transformation into the $\overline{x}_2$-base is to be carried out:

$$H_{\overline{x}_2} = \widetilde{H}_2 T^{-1}. \tag{4.31}$$

Eventually, the dynamic state feedback control law (2.3) is now completely determined, with Φ, N selected as indicated in (4.1) and

$$\begin{pmatrix} H_1 & H_2 \end{pmatrix} = \begin{pmatrix} H_{\overline{x}_1} & 0_{1\times(r+1)} \\ 0_{1\times(r+1)} & H_{\overline{x}_2} \end{pmatrix} M^{-1}, \tag{4.32}$$

M^{-1} being the matrix associated to the change of variables defined in (4.7) for system (4.5):

$$\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = M^{-1} \begin{pmatrix} x \\ z \end{pmatrix}. \tag{4.33}$$

5. Example: Output Voltage Regulation in a Noninverting Buck-Boost Converter

In this section we address the robust regulation of the output voltage x_2 of a NIBB, described by system (3.6) and (3.7) and Table 1, to a constant level $x_{2R} \in \mathbb{R}$ is addressed. Firstly, dynamic state feedback regulators are constructed following Section 4. Later, restrictions arising from control saturation as discussed in Section 3 are considered and, for a certain command profile x_{2R}, a reference x_{1R} for the input current is selected in such a way that (3.19) are fulfilled for all $t \geq 0$ and for all $w \in \widehat{W} = [w_m, w_M] \subset (-\infty, \mu_N)$, where w_m, w_M are, respectively, lower and upper bounds for the uncertain parameter, with $w_m \leq 0 \leq w_M$.

Hence, assume that the control goal is the robust regulation of the state variable x to a constant level $x = x_R$. Then, $S = 0$ and, being $r = \deg mp_S(\lambda) = 1$, (4.1) indicates that $\Phi = 0_2$, $N = \mathbb{I}_2$, which results in the dynamic state feedback regulator (2.3) appearing as

$$\begin{aligned} \dot{z} &= e, \\ u &= H_1 x + H_2 z. \end{aligned} \tag{5.1}$$

The construction of matrices H_1 and H_2 is made from the pole-placement design method provided by Propositions 4.2 and 4.4. Notice that now the closed-loop system (3.22) is

$$A_{cw} = \begin{pmatrix} A_w + H_1 & H_2 \\ \mathbb{I}_2 & 0_2 \end{pmatrix}, \tag{5.2}$$

where A_w is defined in (3.5). Therefore, let us assign complex conjugated, stable poles:

$$\lambda_{11,12} = \lambda_{21,22} = \lambda_{1,2}; \qquad \overline{\lambda}_2 = \lambda_1 \wedge \operatorname{Re}(\lambda_1) < 0, \tag{5.3}$$

for subsystems (4.13) and (4.24). Then, (4.15) and (4.26) yield

$$\widetilde{H}_1 = \widetilde{H}_2 = \left(-|\lambda_1|^2 \; -2|\operatorname{Re}(\lambda_1)| \right), \tag{5.4}$$

and the base transformation matrices T, M^{-1} being (see (4.12) and (4.33), resp.)

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.5}$$

equations (4.17), (4.31), and (4.32) provide

$$\left(H_1 \; \vdots \; H_2 \right) = \begin{pmatrix} -2|\operatorname{Re}(\lambda_1)| & 0 & \vdots & -|\lambda_1|^2 & 0 \\ 0 & -2|\operatorname{Re}\,(\lambda_1)| & \vdots & 0 & -|\lambda_1|^2 \end{pmatrix}. \tag{5.6}$$

Remark 5.1. According to Remark 4.5(i)

$$W = \left\{ w < \mu_N ; \sigma\left[\lambda^2 + (2|\mathrm{Re}(\lambda_1)| - w)\lambda + |\lambda_1|^2\right] \subset \mathbb{C}^- \right\}, \tag{5.7}$$

with $W = (-\infty, 2|\mathrm{Re}(\lambda_1)|) \subset (-\infty, \mu_N)$ in case that $2|\mathrm{Re}(\lambda_1)| < \mu_N$. Otherwise, the assumption $2|\mathrm{Re}\,(\lambda_1)| \geq \mu_N$ results in the fulfillment of (4.27), this yielding $W = (-\infty, \mu_N)$ in accordance to Remark 4.5(ii).

On the other hand, let $v_0 = 1$, $Q = (q_1, q_2)^\top \in \mathbb{R}^2$ in (3.9), which yield $x_{1R} = q_1$, $x_{2R} = q_2$. Next result provides a selection criteria for x_{1R} which is shown to be sufficient for the fulfillment of (3.19).

Proposition 5.2. *Let the NIBB converter, described by* (3.6), (3.7) *and Table 1, be regulated by* (5.1), *with* $(H_1 \mid H_2)$ *defined in* (5.6). *Let also* W *be the set defined in* (5.7). *Assume that the output voltage* x_2 *is expected to attain a certain reference level* $x_{2R} = q_2 \in \mathbb{R} \setminus \{0\}$, *while the uncertain parameter* w *belongs to the set with known bounds* $[w_m, w_M]$, $w_m \leq 0 \leq w_M < \mu_N$. *If the reference* $x_{1R} = q_1 \in \mathbb{R}^+$ *is such that*

$$q_1 > (\mu_N + |w_m|) \max\left\{ |q_2|, q_2^2 \right\}, \tag{5.8}$$

then one has the following:

(i) *Restrictions* (3.19) *are fulfilled for all* $t \geq 0$, *for all* $w \in \widehat{W} = [w_m, w_M]$.

(ii) *If* $2|\mathrm{Re}(\lambda_1)| > w_M$, *then* $W \cap \widehat{W} = \widehat{W}$; *furthermore, there exist open sets* X_0, Z_0 *and* W_0, *with* $W_0 \subseteq \widehat{W}$ *and* $0 \in W_0$, *in such a way that, for all* $(x(0), z(0)) \in X_0 \times Z_0$, *and for all* $w \in W_0$, *the regulated system* (3.6), (3.7), (5.1), (5.6) *is able to accommodate any disturbance* $w \in [w_m, w_M]$ *and, at the same time, maintain the system trajectories evolving inside the state-space region* X *defined from* (3.13), *for all* $t \geq 0$.

Proof. (i) The statement follows from the fact that, in this case, (3.19) answers to

$$q_1 \neq 0, \quad 0 < \frac{(\mu_N - w)q_2^2}{q_1} < 1, \quad 0 < \frac{(\mu_N - w)q_2}{q_1} < 1. \tag{5.9}$$

(ii) As $2|\mathrm{Re}(\lambda_1)| > w_M$ by hypothesis, it follows from Remark 5.1 and item (i) that $\widehat{W} \subset W$, this yielding $W \cap \widehat{W} = \widehat{W}$. Then, the result follows from Theorem 3.4. □

Remark 5.3. It is worth mentioning that the procedure described in [9] for the obtention of an input current reference x_{1R} when both x_{1R}, x_{2R} are assumed to be constant, yields the same result as that of Proposition 5.2.

Finally, recall that the stationary values x_R, z_R, obtained following (3.15), (3.16), are:

$$x_R = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad z_R = \Sigma_w = -H_2^{-1}(A_w + H_1)\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{5.10}$$

where $S = 0$, $v_0 = 1$ and $\Phi = 0_2$ has been taken into account.

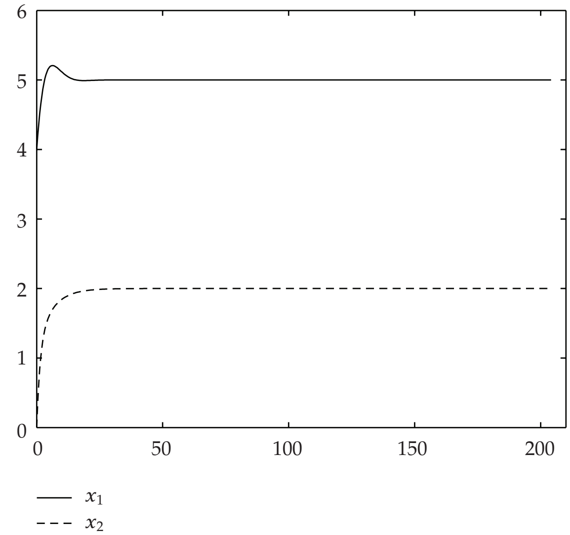

Figure 1: $R_w = -5\Omega$: inductor current (x_1) and output voltage (x_2).

6. Simulation Results

The parameters selected for the NIBB converter are $V_g = 40\,\text{V}$, $L = 0.001\,\text{H}$, $C = 0.00006\,\text{F}$, $R_N = 10\Omega$, and it is expected to suffer an additive load disturbance at $t = 0$ that may vary the nominal value R_N in the range -50% to $+100\%$, that is, admissible values for R_w and R are: $R_w \in [R_{wm}, R_{wM}] = [-5\Omega, 10\Omega]$ and $R \in [R_m, R_M] = [5\Omega, 20\Omega]$. These settings, translated to normalized variables, result in (see (3.3)) $\mu_N = 0.4082$, $w \in [w_m, w_M] = [-0.4082, 0.2041]$ and $\mu \in [0.1361, 0.8165]$. Using $\lambda_{1,2} = -1/4 \pm (1/4)i$ in (5.3), the regulator obtained from (5.6) is

$$\left(H_1 \ \vdots \ H_2\right) = \begin{pmatrix} -\dfrac{1}{2} & 0 & \vdots & -\dfrac{1}{8} & 0 \\ 0 & -\dfrac{1}{2} & \vdots & 0 & -\dfrac{1}{8} \end{pmatrix}. \tag{6.1}$$

Then, as $1/2 = 2|\operatorname{Re}(\lambda_1)| > \mu_N = 0.4082$, it follows from (5.7) and Remark 4.5(ii) that $W = (-\infty, \mu_N) = (-\infty, 0.4082) \supset [-0.4082, 0.2041] = [w_m, w_M]$.

Let us now assign references for the state variables: $x_{1R} = 5$, $x_{2R} = 2$, corresponding to $i_{\mathrm{L}} = 48.9898A$, $v_{\mathrm{C}} = 80\,\text{V}$, respectively. It can be immediately checked that this selection guarantees the fulfillment of restriction (5.8) in Proposition 5.2: $5 = q_1 > (\mu_N + |w_m|)\max\{|q_2|, q_2^2\} = 3.2660$ and, therefore, $\widehat{W} = [-0.4082, 0.2041] \subset W$. Thus, considering that $\widehat{W} = W \cap \widehat{W} = [-0.4082, 0.2041] = [w_m, w_M]$, Proposition 5.2(ii), ensures the existence of open sets $W_0 \subset W \cap \widehat{W}$, X_0, Z_0, with $0 \in W_0$, in such a way that, for all $(x(0), z(0)) \in X_0 \times Z_0$, the regulated system (3.6), (3.7), (5.1) is able to accommodate any disturbance $w \in W_0$ and, at the same time, maintain the system trajectories evolving in the unsaturated region of the state-space region X defined from (3.13), for all $t \geq 0$.

Finally, notice that the steady-state values for the state variables x, z are (see (5.10)): $x_R = (5, 2)^\top$, $z_R = \Sigma_w = -(20, 8 - 16w)^\top$.

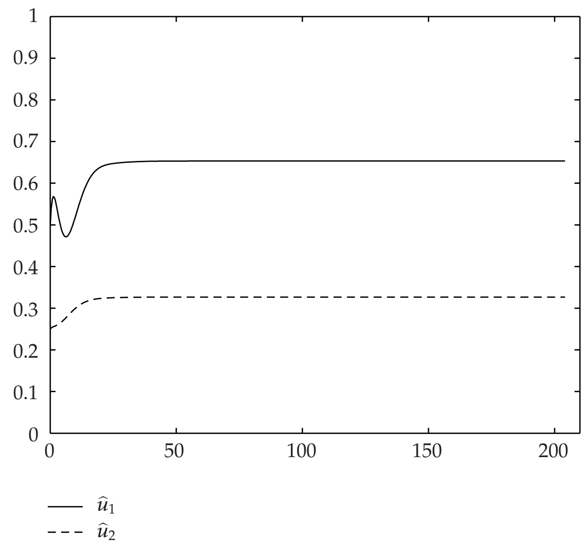

Figure 2: $R_w = -5\Omega$: equivalent controls $\hat{u}_1$ and $\hat{u}_2$.

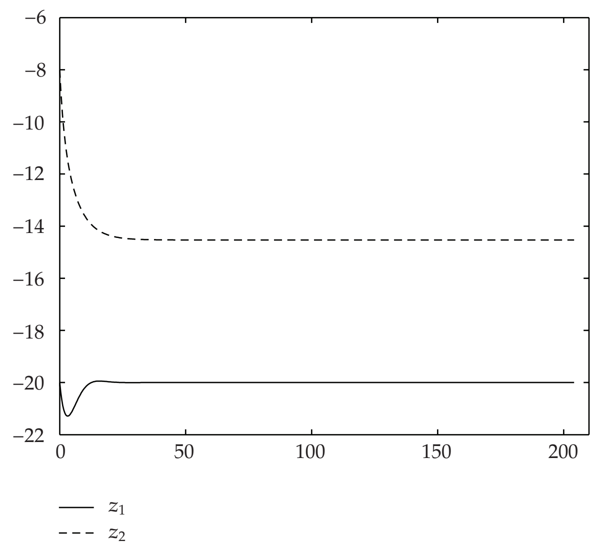

Figure 3: $R_w = -5\Omega$: auxiliary variables z_1 and z_2.

Ideal simulations have been carried out with a SIMULINK model of the system for the cases $R_w = -5\Omega$ and $R_w = 10\Omega$. The control signal $\hat{u}$ is considered continuous. The duration of the simulations is 204.1241 time units, corresponding to 0.05 s. In both cases the selected initial conditions have been: $x(0) = (4, 2)^\top$, $z(0) = \Sigma_0 = -(20, 8)^\top$.

Figures 1, 2, and 3 depict, respectively the state variables x_1 and x_2, the control actions $\hat{u}_1$, $\hat{u}_2$, and the auxiliary variables z_1 and z_2 for the case $R_w = -5\Omega$: the results show an excellent agreement with the theory developed in Section 5, which is confirmed by plots in Figures 4, 5, and 6, corresponding to $R_w = 10\Omega$.

The section closes with realistic simulations that use the software package PSIM. Besides the converter's main parameters indicated above, the model incorporates internal

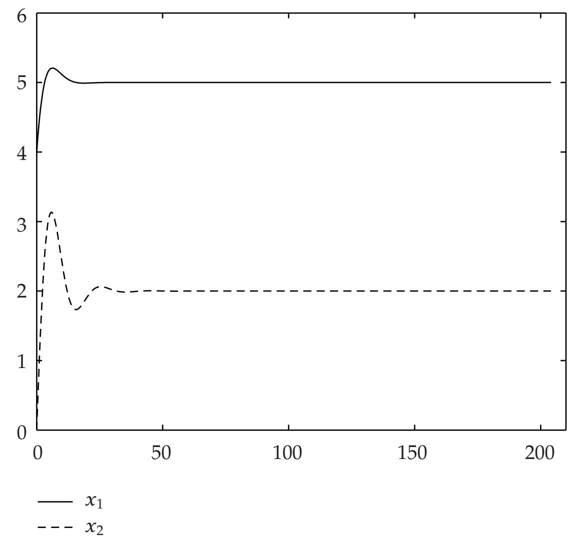

Figure 4: $R_w = 10\Omega$: inductor current (x_1) and output voltage (x_2).

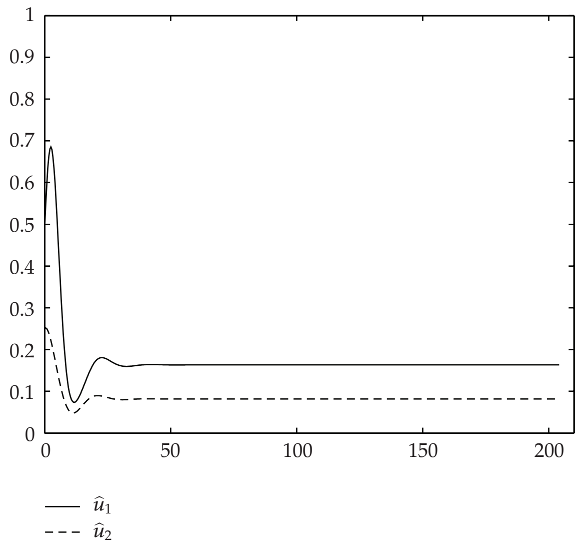

Figure 5: $R_w = 10\Omega$: equivalent controls $\hat{u}_1$ and $\hat{u}_2$.

resistances of 10 mΩ for the inductor and the capacitor. Each switch is implemented by means of an IGBT with a saturation voltage of 2 V and a power diode with a voltage drop of 0.5 V. The implementation of the control law uses PWM with a frequency of 50 kHz.

The current and voltage references have been set to 49 A and 80 V, respectively, and null initial conditions have been selected for both the state and the auxiliary variables. The total duration of the simulations is 9 ms. Figure 7 depicts the inductor current and the output voltage reaching their reference levels under an actual output load of $R = 5\Omega$ instead of the nominal $R_N = 10\Omega$, that is, with a disturbance $R_w = -5\Omega$. In turn, Figure 8 plots an equivalent situation for $R = 20\Omega$, that is, for $R_w = 10\Omega$. The results confirm the validity of the proposed approach.

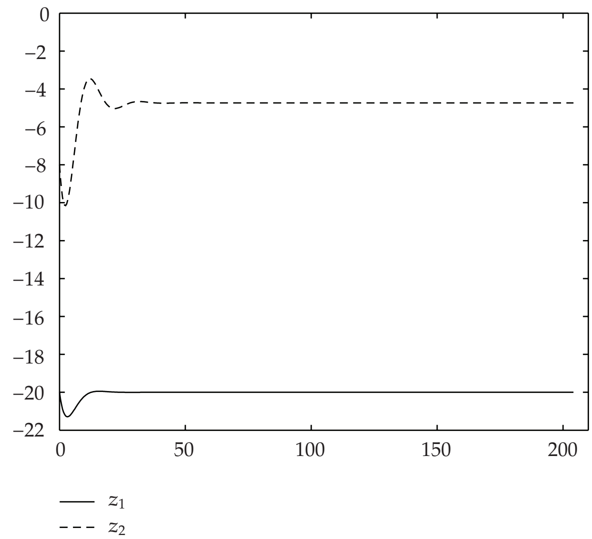

Figure 6: $R_w = 10\Omega$: auxiliary variables z_1 and z_2.

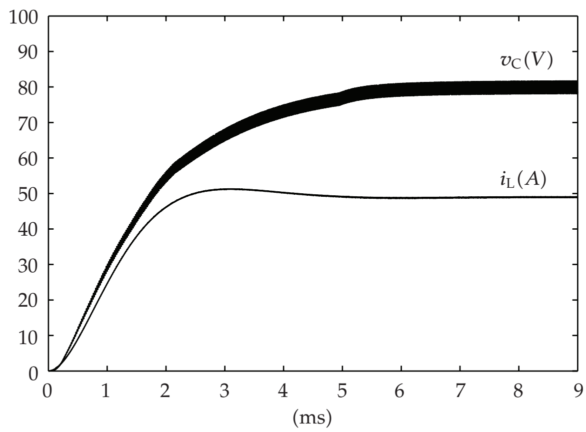

Figure 7: $R_w = -5\Omega$: output voltage v_C (in Volts) and inductor current i_L (in Ampères).

7. Conclusions

The robust output regulation problem for a family of nonlinear switched power converters that includes the NonInverting Buck-Boost, the Full-Bridge NonInverting Buck-Boost, the Watkins-Johnson and the Inverse of Watkins-Johnson has been addressed. Linear techniques, available after a transformation of the control variable, render an efficient solution of the problem. The methodology employs a dynamic state feedback control law and considers resistive loads with load resistance uncertainty. Restrictions due to fixed values of the control gains are considered. The proposed technique is successfully tested via realistic numerical simulations of the robust output voltage regulation in a NonInverting Buck-Boost converter.

Further research should explore the possibility of using state feedback linearization plus linear robust output regulation techniques in converters with a single control switch, such as the boost or the buck-boost.

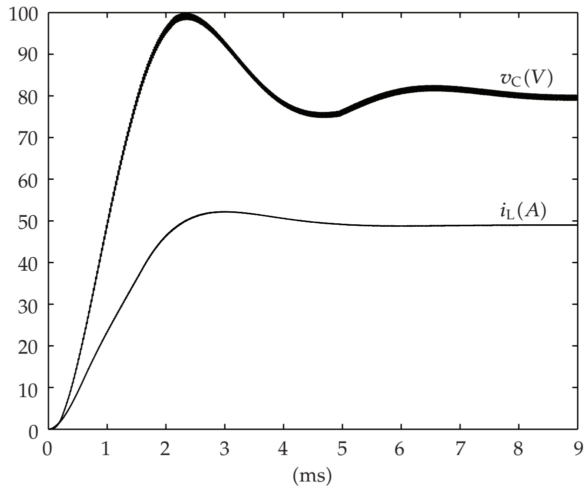

Figure 8: $R_w = 10\,\Omega$: output voltage v_C (in Volts) and inductor current i_L (in Ampères).

Acknowledgments

The authors thank the anonymous reviewers for their valuable suggestions. This work is partially supported by the spanish Ministerio de Educación under project DPI2007-62582 (D. Biel and J. M. Olm) and the spanish Ministerio de Ciencia e Innovación, through the Programa Nacional de Movilidad de Recursos Humanos of the Plan Nacional de I-D+i 2008-2011 (J. M. Olm).

References

[1] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica*, vol. 12, no. 5, pp. 457–465, 1976.

[2] B. A. Francis, "The linear multivariable regulator problem," *SIAM Journal on Control and Optimization*, vol. 15, no. 3, pp. 486–505, 1977.

[3] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 35, no. 2, pp. 131–140, 1990.

[4] C. I. Byrnes and A. Isidori, "Output regulation for nonlinear systems: an overview," *International Journal of Robust and Nonlinear Control*, vol. 10, no. 5, pp. 323–337, 2000.

[5] C. I. Byrnes, F. Delli Priscoli, and A. Isidori, *Output Regulation of Uncertain Nonlinear Systems*, Systems & Control: Foundations & Applications, Birkhäuser, Boston, Mass, USA, 1997.

[6] J. Huang, *Nonlinear Output Regulation: Theory and Application*, vol. 8 of *Advances in Design and Control*, SIAM, Philadelphia, Pa, USA, 2004.

[7] R. Erickson and D. Maksimović, *Fundamentals of Power Electronics*, Kluwer Academic Publishers, Norwell, Mass, USA, 2nd edition, 2001.

[8] D. Biel, E. Fossas, and J. M. Olm, "Robust step-up DC/AC conversion with a full-bridge non-inverting buck-boost," in *Proceedings of the 16th IEEE International Conference on Control Applications*, pp. 593–599, 2007.

[9] J. M. Olm, D. Biel, E. Fossas, A. S. I. Zinober, and L. M. Sanz, "Sliding motion, robust control and power loss minimization in a class of non-linear switched converters," *International Journal of Control*, vol. 80, no. 6, pp. 968–981, 2007.