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Research Article

Hopf Bifurcation with the Spatial Average of an Activator in a Radially Symmetric Free Boundary Problem

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An interface problem derived by a bistable reaction-diffusion system with the spatial average of an activator is studied on an n-dimensional ball. We analyze the existence of the radially symmetric solutions and the occurrence of Hopf bifurcation as a parameter varies in two and three-dimensional spaces.

1. Introduction

The study of interfacial patterns is important in several areas of biology, chemistry, physics, and other fields [1–4]. Internal layers (or free boundary), which separate two stable bulk states by a sharp transition near interfaces, are often observed in bistable reaction-diffusion equations when the reaction rate is faster than the diffusion effect. We consider a reaction-diffusion system with a sufficiently small positive constant ε [5, 6]

$$\begin{gathered} \sigma\varepsilon u_t = \varepsilon^2\ \nabla^2 u + f(u,v), \\ v_t = D\nabla^2 v + g(u,v) \quad t>0,\ \mathbf{x}\in\Omega, \\ 0 = \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu}, \end{gathered} \tag{1.1}$$

where ε, σ, and D are positive constants, $\Omega = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < R\}$ is the ball in n-dimensional space, and ν stands for the unit outward normal on the boundary $\partial\Omega$.

The nonlinear functions are

$$f(u,v) = -u + H(u - a_0) - v + (1 - \mu)\langle u \rangle, \qquad g(u,v) = \mu u - v, \tag{1.2}$$

where $0 < \mu < 1$ and $\langle u \rangle$ denotes the spatial average, describing a global feedback effect, namely,

$$\langle u \rangle = \frac{1}{|\Omega|}\int_{\Omega} u\, d\Omega = \frac{1}{|\Omega|}\int_{\Omega} u(\mathbf{x},t)d\mathbf{x}. \tag{1.3}$$

The system (1.1) with (1.2) is a model for flow discharges proposed by et al. [7], in which u is interpreted by the current density and v by the voltage drop across the gas gap [7, 8] in a gas-discharge system. This system also exhibits a codimension-two Turing Hopf bifurcation [9], where the conditions of a spatial Turing instability [10] with a certain wavelength ε and a temporal Hopf bifurcation with a certain frequency $1/\sqrt{\varepsilon\sigma}$ are met simultaneously. Equation (1.1) determines the dynamics of an internal layer, and equation (1.1) together with (1.2) represents a basic model of globally coupled bistable medium which is relevant for current density dynamics in large area bistable semiconductor systems [11–14]. The internal layer has a physical reason as the current filament has a sharp profile with a narrow transition layer connecting flat on- and off-states.

When ε in (1.1) is sufficiently small for the case of without the spatial average, the singular limit analysis is applied to show the existence and the stability of localized radially symmetric equilibrium solutions [15, 16]. In one-dimensional space for the case of without the spatial average, such equilibrium solutions should undergo certain instabilities, and the loss of stability resulting from a Hopf bifurcation produces a kind of periodic oscillation in the location of the internal layers [2, 17–19]. As the parameter D varies, the stability of the spherically symmetric solutions and their symmetry-breaking bifurcations into layer solutions for the case of without the spatial average have been examined in [5, 6].

In this paper, the free boundary problem of (1.1) with (1.2) for the case when $\varepsilon = 0$ in two- and three-dimensional space will be studied. Suppose that there is only one $(n-1)$-dimensional hypersurface $\eta(t)$ which is a single closed curve given in the domain Ω in such a way that $\Omega \times (0,\infty) = \Omega^+(t) \cup \eta(t) \cup \Omega^-(t)$, where $\Omega^-(t) = \{(\mathbf{x},t) \in \Omega \times (0,\infty) : u(\mathbf{x},t) > a_0\}$ and $\Omega^+(t) = \{(\mathbf{x},t) \in \Omega \times (0,\infty) : u(\mathbf{x},t) < a_0\}$. When $D = 1$ and $\varepsilon = 0$ in (1.1), the spatial average $\langle u \rangle$ satisfies

$$\mu\langle u \rangle\langle H(x-\eta)\rangle - \langle v \rangle, \quad \langle H(x-\eta)\rangle = \frac{1}{|\Omega_n|}\int_{\Omega_n} H(x-\eta)dx = 1 - \left(\frac{n}{R}\right)^n, \tag{1.4}$$

where $|\Omega_2| = \pi R^2$ and $|\Omega_3| = (4/3)\pi R^3$. The spatial average of v is a solution of $\langle v \rangle'(t) = \mu\langle u \rangle - \langle v \rangle = 1 - (\eta/R)^n - 2\langle v \rangle$. Equation of $\eta(t)$ is given by (see [20, 21])

$$\frac{d\eta(t)}{dt} \cdot \nu = C(v_i), \qquad x \in \eta(t), \tag{1.5}$$

where ν is the outward normal vector on $\eta(t)$, v_i is the value of v on the interface $\eta(t)$, and $C(v)$ is the velocity of the interface. The reaction terms (1.2) satisfy the bistable condition, that is, the nullclines of $f(u,v) = 0$ and $g(u,v) = 0$ must have three intersection points and the nullcline $f(u,v) = 0$ is the triple-valued function of u which is called h^+, h^-, and h^0. From [2, 20, 22], the trajectory with a unique value of $C = C(v)$ exists which is given by $C(v) = h^+ - 2h^0 + h^-$. Furthermore, the velocity of the interface $C(v)$ is a continuously differentiable function defined on an interval $I := (-a_0, 1 - a_0)$, and thus the velocity of the interface can be normalized by

$$C((v(\eta), \eta, \langle v\rangle), t) = -\frac{1 - 2a_0 - 2V}{\sigma\sqrt{(V + a_0)(1 - a_0 - V)}}, \tag{1.6}$$

where $V = v(\eta) - ((1-\mu)/\mu)(1 - (\eta/R)^n - \langle v\rangle)$.

An analysis of the dynamics of this process has been shown (see, e.g., [2, 5, 6, 15]) to lead a free boundary problem consisting of the initial-boundary value problem

$$\begin{gathered} v_t = \nabla^2 v + g(h^{\pm}, v), \quad (\mathbf{x}, t) \in \Omega^{\pm}(t), \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}), \\ v(\eta(t) - 0, t) = v(\eta(t) + 0, t), \\ \frac{d}{d\nu} v(\eta(t) - 0, t) = \frac{d}{d\nu} v(\eta(t) + 0, t), \\ \eta'(t) = C((v(\eta), \eta, w), t), \\ w'(t) = 1 - \left(\frac{\eta}{R}\right)^n - 2w, \qquad w(0) = w_0, \end{gathered} \tag{1.7}$$

where $w = \langle v\rangle$.

The organization of the paper is as follows. In Section 2, a change of variables is given which regularizes problem (1.7) in such a way that results from the theory of nonlinear evolution equations can be applied. In this way, we obtain enough regularity of the solution for an analysis of the bifurcation. In Section 3, we show the existence of radially symmetric localized equilibrium solutions for (1.7) and obtain the linearization of problem (1.7). In the last section we show the existence of the periodic solutions and the bifurcation of the interface problem as a parameter σ varies in two and three dimensions.

2. Regularized System

We look for an existence problem of radially symmetric equilibrium solutions of (1.7) with $|\mathbf{x}| = r$, where the center and the interface are located at the origin and $r = \eta$, respectively.

The problem is given by

$$
\begin{aligned}
v_t = \frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r}\frac{\partial v}{\partial r} - (\mu+1)v + \mu H(r-\eta(t)) + (1-\mu)\left(1-\left(\frac{\eta(t)}{R}\right)^{\mathrm{n}} - w(t)\right),& \\
\eta'(t) = C\left((v(\eta),\eta,w),t\right), \quad t>0,\ \eta(0)=\eta_0,& \\
w'(t) = -2w(t) + 1 - \left(\frac{\eta(t)}{R}\right)^{n}, \quad w(0)=w_0,& \\
\frac{\partial v}{\partial r}v(0,t) = 0 = \frac{\partial v}{\partial r}v(R,t), \qquad 0<r<R,\ t>0.&
\end{aligned}
\tag{2.1}
$$

As a first step we obtain more regularity for the solution by semigroup methods, considering $A := -(\partial^2/\partial r^2) + ((n-1)/r)(\partial/\partial r) + \mu + 1$ as a densely defined operator $A : D(A) \subset X \to X$, where $D(A) = \{v \in H^{2,2}((0,R)) : (\partial v/\partial r)(0,t) = 0 = (\partial v/\partial r)(R,t)\}$ and $X := L_2((0,R))$ with norm $\|\cdot\|_2$.

We define $g : [0,R] \times [0,R] \times \mathbb{C} \to \mathbb{C}$,

$$
\begin{aligned}
g(r,\eta,w) &:= A^{-1}\left(\mu(H(\cdot-\eta))(r) + (1-\mu)\left(1-\left(\frac{\eta}{R}\right)^{n} - w\right)\right) \\
&= \mu\int_{\eta}^{R} G(r,y)\,dy + \frac{1-\mu}{1+\mu}\left(1-\left(\frac{\eta}{R}\right)^{n} - w\right),
\end{aligned}
\tag{2.2}
$$

and $\gamma : [0,R] \times \mathbb{C} \to \mathbb{C}$,

$$
\gamma(\eta,w) := g(\eta,\eta,w). \tag{2.3}
$$

Here $G : [0,R]^2 \to \mathbb{R}$ is a Green's function of A satisfying the Neumann boundary conditions: for $n=2$,

$$
G(r,z) = \begin{cases}
zI_0\left(r\sqrt{1+\mu}\right)\left(K_0\left(z\sqrt{1+\mu}\right) + \dfrac{K_1\left(R\sqrt{1+\mu}\right)}{I_1\left(R\sqrt{1+\mu}\right)}I_0\left(z\sqrt{1+\mu}\right)\right), & 0<r<z, \\
zI_0\left(z\sqrt{1+\mu}\right)\left(K_0\left(r\sqrt{1+\mu}\right) + \dfrac{K_1\left(R\sqrt{1+\mu}\right)}{I_1\left(R\sqrt{1+\mu}\right)}I_0\left(r\sqrt{1+\mu}\right)\right), & z<r<R,
\end{cases}
\tag{2.4}
$$

and for $n = 3$,

$$G(r,z) = \begin{cases} \dfrac{1}{Y_R} z \dfrac{\sinh(r\sqrt{1+\mu})}{r\sqrt{1+\mu}} \left(R\sqrt{1+\mu}\cosh((R-z)\sqrt{1+\mu}) - \sinh((R-z)\sqrt{1+\mu})\right), \\ \hfill 0 < r < z \\ \dfrac{1}{Y_R} z \dfrac{\sinh(z\sqrt{1+\mu})}{r\sqrt{1+\mu}} \left(R\sqrt{1+\mu}\cosh((R-r)\sqrt{1+\mu}) - \sinh((R-r)\sqrt{1+\mu})\right), \\ \hfill z < r < R, \end{cases} \tag{2.5}$$

where I_i and K_i are modified Bessel functions ($i = 0, 1$) and Y_R is given by

$$Y_R = \sqrt{1+\mu}\left(R\sqrt{1+\mu}\cosh\left(R\sqrt{1+\mu}\right) - \sinh\left(R\sqrt{1+\mu}\right)\right). \tag{2.6}$$

Moreover, $g(\cdot,\eta,w) \in D(A)$ for all η and w, $(\partial g/\partial\eta)(r,\eta,w) \in H^{1,\infty}((0,R)^2 \times \mathbb{C})$ and $\gamma \in C^{\infty}([0,R] \times \mathbb{C})$.

Applying the transformation $u(t)(r) = v(r,t) - g(r,\eta(t),w(t))$, then we obtain an equivalent abstract evolution equation of(2.1)

$$\begin{gathered} \frac{d}{dt}(u,\eta,w) + \widetilde{A}(u,\eta,w) = F(u,\eta,w), \\ (u,\eta,w)(0) = (u_0,\eta_0,w_0), \end{gathered} \tag{2.7}$$

where $\widetilde{A}$ is a 3×3 matrix defined on $D(\widetilde{A}) = D(A) \times (0,R) \times \mathbb{C}$ and given by

$$\begin{pmatrix} A & 0 & \dfrac{2(1-\mu)}{1+\mu} \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{2.8}$$

The nonlinear forcing term F defined on the set $W := \{(u,\eta,w) \in C^1([0,R]) \times (0,R) \times \mathbb{C} : u(\eta) + \gamma(\eta,w) - ((1-\mu)/\mu)(1-(\eta/R)^n - w) \in \mathbf{I}\} \subset_{\text{open}} C^1([0,R]) \times (0,R) \times \mathbb{C}$ as

$$F(u,\eta,w) = \begin{pmatrix} f_2(u,\eta,w) \cdot \left(f_1(\eta) + \dfrac{1-\mu}{1+\mu}\dfrac{n}{R^n}\eta^{n-1}\right) + \dfrac{1-\mu}{1+\mu}\left(1 - \left(\dfrac{\eta}{R}\right)^n\right) \\ f_2(u,\eta,w) \\ 1 - \left(\dfrac{\eta}{R}\right)^n \end{pmatrix}, \tag{2.9}$$

where $f_1 : (0,R) \to X$, $f_1(\eta)(r) := \mu G(r,\eta)$, and $f_2 : W \to \mathbb{C}$, $f_2(u,\eta,w) := C(u(\eta),\eta,w)$. We define $\xi(\eta) := \mu\int_\eta^R G(\eta,y)dy$ and then $\gamma(\eta,w) = \xi(\eta) + ((1-\mu)/(1+\mu))(1-(\eta/R)^n - w)$. Let $\chi(u,\eta) := u(\eta) + \xi(\eta) - ((1-\mu)/(1+\mu))(\eta/R)^n + ((1-\mu)/\mu)(\eta/R)^n$, then the velocity of η is written by

$$\begin{aligned} &C(u(\eta),\eta,w)\\ &= C(\chi(u,\eta),w)\\ &= -\frac{1}{\sigma}\frac{1-2a_0-2(\chi(u,\eta)-((1-\mu)/\mu(1+\mu))(1-w))}{\sqrt{(a_0+\chi(u,\eta)-((1-\mu)/\mu(1+\mu))(1-w))(1-a_0-(\chi(u,\eta)-((1-\mu)/\mu(1+\mu))(1-w)))}}. \end{aligned} \tag{2.10}$$

Lemma 2.1. *The functions $f_1 : (0,R) \to X$, $f_2 : W \to \mathbb{C}$ and $F : W \to X\times\mathbb{C}\times\mathbb{R}$ are continuously differentiable with derivatives given by*

$$\begin{aligned} f_1'(\eta) &= \mu\frac{\partial G}{\partial z}(\cdot,\eta),\\ Df_2(u,\eta,w)(\hat{u},\hat{\eta},\hat{w}) &= C_\chi(\chi(u,\eta),w)\left(u'(\eta)\hat{\eta} + \hat{u}(\eta) + \xi'(\eta)\hat{\eta} + \frac{1-\mu}{\mu(1+\mu)}\frac{n\eta^{n-1}}{R^n}\hat{\eta}\right)\\ &\quad + C_w(\chi(u,\eta),w)\hat{w},\\ DF(u,\eta,w)(\hat{u},\hat{\eta},\hat{w}) &= f_2(u,\eta,w)\cdot\left(f_1'(\eta) + \frac{1-\mu}{1+\mu}\frac{n(n-1)}{R^n}\eta^{n-2},0,0\right)\hat{\eta}\\ &\quad + Df_2(u,\eta,w)(\hat{u},\hat{\eta},\hat{w})\cdot\left(f_1(\eta) + \frac{1-\mu}{1+\mu}\frac{n}{R^n}\eta^{n-1},1,0\right)\\ &\quad + \left(-\frac{1-\mu}{1+\mu}\frac{n}{R^n}\eta^{n-1},0,-\frac{n}{R^n}\eta^{n-1}\right)\hat{\eta}, \end{aligned} \tag{2.11}$$

where $C_\chi = \partial C/\partial\chi$ and $C_w = \partial C/\partial w$.

The well posedness of solutions was shown in [23] applying the semigroup theory using domains of fractional powers $\alpha \in (3/4,1]$ of A and $\widetilde{A}$ [24]. Moreover, they obtained that $F : W \cap D(\widetilde{A}^\alpha) \to X\times\mathbb{C}\times\mathbb{R}$ is a continuously differentiable function, where $D(\widetilde{A}^\alpha) = D(A^\alpha)\times(0,R)\times\mathbb{C}$.

3. Radially Symmetric Equilibrium Solutions and Linearization

The steady states are solutions of the following problem:

$$
\begin{gathered}
Au^* = \left(\mu\, G(\cdot,\eta^*) + \frac{1-\mu}{1+\mu}\frac{n}{R^n}(\eta^*)^{n-1}\right)C(u^*(\eta^*),\eta^*,w^*) + \frac{1-\mu}{1+\mu}\left(-2w^* + 1 - \left(\frac{\eta^*}{R}\right)^n\right), \\
0 = C(u^*(\eta^*) + \gamma(\eta^*, w^*)), \\
0 = -2w^* + 1 - \left(\frac{\eta^*}{R}\right)^n, \\
u^{*\prime}(0) = 0 = u^{*\prime}(R)
\end{gathered}
\tag{3.1}
$$

for $(u^*, \eta^*, w^*) \in D(\widetilde{A}) \cap W$.

Lemma 3.1. *Define $\xi(\eta) := \mu\int_\eta^R G(\eta, y)dy < 0$. Then $\xi'(\eta) < 0$, and $\xi'(\eta) + \mu G(\eta,\eta) > 0$ for $0 < \eta < R$.*

Proof. For $n = 2$, the derivative of ξ is given by

$$
\begin{aligned}
\xi'(\eta) = \mu\eta\Big(&-I_0\left(\eta\sqrt{1+\mu}\right)K_0\left(\eta\sqrt{1+\mu}\right) + I_1\left(\eta\sqrt{1+\mu}\right)K_1\left(\eta\sqrt{1+\mu}\right)\Big) \\
&-\mu\eta\left(\frac{K_1(R\sqrt{1+\mu})}{I_1(R\sqrt{1+\mu})}\left(I_0^2\left(\eta\sqrt{1+\mu}\right) + I_1^2\left(\eta\sqrt{1+\mu}\right)\right)\right).
\end{aligned}
\tag{3.2}
$$

Let $h(\eta) = I_0(\eta\sqrt{1+\mu})K_0(\eta\sqrt{1+\mu}) - I_1(\eta\sqrt{1+\mu})K_1(\eta\sqrt{1+\mu})$. Then $\lim_{\eta\to 0}h(\eta) = \infty$, and $h(R) > 0$. The derivative of h is

$$
\begin{aligned}
h'(\eta) &= -\sqrt{1+\mu}\Big(\left(I_0\left(\eta\sqrt{1+\mu}\right) + I_1\left(\eta\sqrt{1+\mu}\right)\right)K_1\left(\eta\sqrt{1+\mu}\right) \\
&\qquad +2I_1\left(\eta\sqrt{1+\mu}\right)K_0\left(\eta\sqrt{1+\mu}\right)\Big) \\
&\le -2\sqrt{1+\mu}\left(I_1\left(\eta\sqrt{1+\mu}\right)\right)\left(K_1\left(\eta\sqrt{1+\mu}\right) - K_0\left(\eta\sqrt{1+\mu}\right)\right) \le 0.
\end{aligned}
\tag{3.3}
$$

Thus $h(\eta) > 0$, and thus $\xi'(\eta) < 0$ for $0 < \eta < R$. Moreover,

$$
\begin{aligned}
&\xi'(\eta) + \mu G(\eta,\eta) \\
&\quad = \eta\frac{I_1(\eta\sqrt{1+\mu})}{I_1(R\sqrt{1+\mu})}\left(I_1\left(R\sqrt{1+\mu}\right)K_1\left(\eta\sqrt{1+\mu}\right) - I_1\left(\eta\sqrt{1+\mu}\right)K_1\left(R\sqrt{1+\mu}\right)\right).
\end{aligned}
\tag{3.4}
$$

Since I_i are increasing functions and K_i are decreasing functions $(i = 0, 1)$,

$$\begin{aligned} & I_1\left(R\sqrt{1+\mu}\right)K_1\left(\eta\sqrt{1+\mu}\right) - I_1\left(\eta\sqrt{1+\mu}\right)K_1\left(R\sqrt{1+\mu}\right) \\ &\quad = \left(I_1\left(R\sqrt{1+\mu}\right) - I_1\left(\eta\sqrt{1+\mu}\right)\right)K_1\left(\eta\sqrt{1+\mu}\right) \\ &\qquad + \left(K_1\left(\eta\sqrt{1+\mu}\right) - K_1\left(R\sqrt{1+\mu}\right)\right)I_1\left(\eta\sqrt{1+\mu}\right) > 0, \end{aligned} \tag{3.5}$$

and thus $\xi'(\eta) + \mu G(\eta, \eta) > 0$ for $0 < \eta < R$. For $n = 3$, the derivative of ξ is

$$\begin{aligned} \xi'(\eta) = {}& \frac{\mu}{M_2}\left(R\sqrt{1+\mu}+1\right)\left[1 - \left(1 - 2\eta\sqrt{1+\mu} + 2\eta^2(1+\mu)\right)e^{2\eta\sqrt{1+\mu}}\right] \\ & - \frac{\mu}{M_2}\left(R\sqrt{1+\mu}-1\right)\left[e^{2R\sqrt{1+\mu}} - \left(1 + 2\eta\sqrt{1+\mu} + 2\eta^2(1+\mu)\right)e^{2(R-\eta)\sqrt{1+\mu}}\right], \end{aligned} \tag{3.6}$$

where $M_2 = 2\eta^2(1+\mu)\sqrt{1+\mu}[R\sqrt{1+\mu} + 1 + (R\sqrt{1+\mu} - 1)e^{2R\sqrt{1+\mu}}]$. Since $e^{2\eta\sqrt{1+\mu}} \geq (1 + 2\eta\sqrt{1+\mu} + 2\eta^2(1+\mu))$, we have $\xi'(\eta) \leq -(\mu/M_2)(1 + R\sqrt{1+\mu})4\eta^4(1+\mu^2) < 0$ for $0 < \eta < R$. Moreover, $\xi'(\eta) + \mu G(\eta,\eta) > 2(\mu/M_2)(R-\eta)\sqrt{1+\mu}[(\eta\sqrt{1+\mu}+1) + (\eta\sqrt{1+\mu}-1)e^{2\eta\sqrt{1+\mu}}] > 0$ for $0 < \eta < R$. □

Theorem 3.2. *Suppose that (i)* $0 < (1/2) - a_0 < (2\mu - 1)/2\mu$ *and* $\xi'(\eta) + (1-\mu)/2\mu(1+\mu)(n/R^n)\eta^{n-1} < 0$ *or (ii)* $0 < (2\mu-1)/2\mu < (1/2) - a_0$ *and* $\xi'(\eta) + ((1-\mu)/2\mu(1+\mu))(n/R^n)\eta^{n-1} > 0$. *Then the stationary problem of* (2.7) *has the only stationary solution* (u^*, η^*, w^*) *for all* $\sigma \neq 0$ *with* $u^* = 0$ *and* $2w^* = 1 - (\eta^*/R)^n$, $\eta^* \in (0, R)$. *The linearization of* F *at* $(0, \eta^*, w^*)$ *is*

$$DF(0,\eta^*,w^*)(\hat{u},\hat{\eta},\hat{w}) = \begin{pmatrix} \dfrac{4}{\sigma}\left(\hat{u}(\eta^*) + \gamma_\eta(\eta^*,w^*)\hat{\eta} + \gamma_w(\eta^*,w^*)\hat{w} + \dfrac{1-\mu}{\mu}\left(\dfrac{n}{R^n}(\eta^*)^{n-1}\hat{\eta} + \hat{w}\right)\right) \\ \cdot\left(\mu G(\cdot,\eta^*) + \dfrac{1-\mu}{1+\mu}\dfrac{n}{R^n}(\eta^*)^{n-1}\right) - \dfrac{1-\mu}{1+\mu}\left(2\hat{w} + \dfrac{n}{R^n}(\eta^*)^{n-1}\hat{\eta}\right) \\ \dfrac{4}{\sigma}\left(\hat{u}(\eta^*) + \gamma_\eta(\eta^*,w^*)\hat{\eta} + \gamma_w(\eta^*,w^*)\hat{w} + \dfrac{1-\mu}{\mu}\left(\dfrac{n}{R^n}\eta^{*n-1}\hat{\eta} + \hat{w}\right)\right) \\ -2\hat{w} - \dfrac{n}{R^n}\eta^{*n-1}\hat{\eta} \end{pmatrix}. \tag{3.7}$$

The pair $(0,\eta^*,w^*)$ *corresponds to a unique steady state* (v^*,η^*,w^*) *of* (2.1) *for* $\sigma \neq 0$ *with* $v^*(r) = g(r,\eta^*,w^*)$.

Proof. From the system (3.1), η^* and w^* are solutions of the following equations:

$$u^* = 0, \quad C(0,\eta^*,w^*) = 0 \quad 2w^* = 1 - \left(\frac{\eta^*}{R}\right)^n. \tag{3.8}$$

We only check the existence of η^* of (2.10) and (3.8), and thus we let

$$\Gamma(\eta) := \frac{1}{2} - a_0 - \chi(0,\eta) + \frac{1-\mu}{\mu(1+\mu)}\left(1 - \frac{1-(\eta/R)^n}{2}\right). \tag{3.9}$$

Then

$$\Gamma(R) = \frac{1}{2} - a_0, \quad \Gamma(0) = \frac{1}{2} - a_0 - \xi(0) + \frac{1-\mu}{2\mu(1+\mu)}, \quad \Gamma'(\eta) = -\xi'(\eta) - \frac{1-\mu}{2\mu(1+\mu)}\frac{n}{R^n}\eta^{n-1}. \tag{3.10}$$

Since $\xi'(\eta) < 0$ and $\xi(0) = \mu/(1+\mu)$ for $n = 2,3$, there is a unique $\eta^* \in (0,R)$ when $\Gamma'(\eta) < 0$, $\Gamma(0) > 0, \Gamma(R) < 0$, or $\Gamma'(\eta) > 0, \Gamma(0) < 0, \Gamma(R) > 0$.

The formula for $DF(0,\eta^*,w^*)$ follows from Lemma 2.1, the relation $C_\chi(0,\eta^*,w^*) = 4/\sigma$, and $C_w(0,\eta^*,w^*) = 4(1-\mu)/\sigma\mu(1+\mu)$. The corresponding steady state (v^*,η^*,w^*) for (2.1) is obtained using the transformation and Theorem 2.1 in [17]. □

Definition 3.3. Suppose that a_0 and μ satisfy $0 < (1/2) - a_0 < (2\mu-1)/2\mu$ and $\xi'(\eta) + ((1-\mu)/2\mu(1+\mu))(n/R^n)\eta^{n-1} < 0$. One defines (for $1 \geq \alpha > 3/4$) the operator B that is a linear operator from $D(\widetilde{A}^\alpha)$ to $D(\widetilde{A})$ as

$$B := \frac{\sigma}{4}\, DF(0,\eta^*,w^*). \tag{3.11}$$

One then defines $(0,\eta^*,w^*)$ to be a Hopf point for (2.7) if there exists an $\epsilon_0 > 0$ and a C^1-curve

$$(-\epsilon_0 + \tau^*, \tau^* + \epsilon_0) \longmapsto (\lambda(\tau), \phi(\tau)) \in \mathbb{C} \times D\left(\widetilde{A}\right)_{\mathbb{C}} \tag{3.12}$$

($Y_{\mathbb{C}}$ denotes the complexification of the real space Y) of eigendata for $-\widetilde{A} + \tau B$ such that

(i) $(-\widetilde{A} + \tau B)(\phi(\tau)) = \lambda(\tau)\phi(\tau)$, $(-\widetilde{A} + \tau B)(\overline{\phi(\tau)}) = \overline{\lambda(\tau)}\,\overline{\phi(\tau)}$,

(ii) $\lambda(\tau^*) = i\beta$ with $\beta > 0$,

(iii) $\mathrm{Re}(\lambda) \neq 0$ for all λ in the spectrum of $(-\widetilde{A} + \tau^* B) \setminus \{\pm i\beta\}$,

(iv) $\mathrm{Re}\,\lambda'(\tau^*) \neq 0$ (transversality),

where $\tau = 4/\sigma$.

4. Hopf Bifurcation Analysis

We will show that there is a Hopf bifurcation from the curve $\sigma \mapsto (0,\eta^*,w^*)$ of radially symmetric stationary solution. The linearized eigenvalue problem of (2.7) is

$$-\widetilde{A}(u,\eta,w) + \tau B(u,\eta,w) = \lambda I_3(u,\eta,w), \tag{4.1}$$

where I_3 is a 3 by 3 identity matrix. This is equivalent to

$$\begin{aligned}
(A+\lambda)u &= \tau\Big(u(\eta^*)+\gamma_\eta(\eta^*,w^*)\eta+\gamma_w(\eta^*,w^*)w+\frac{1-\mu}{\mu}\Big(w+\frac{n}{R^n}(\eta^*)^{n-1}\eta\Big)\Big)\\
&\quad\cdot\Big(\mu G(\cdot,\eta^*)+\frac{1-\mu}{1+\mu}\frac{n}{R^n}(\eta^*)^{n-1}\Big)+\frac{1-\mu}{1+\mu}\Big(-2w-\frac{n}{R^n}(\eta^*)^{n-1}\eta\Big),\\
\lambda\eta &= \tau\Big(u(\eta^*)+\gamma_\eta(\eta^*,w^*)\eta+\gamma_w(\eta^*,w^*)w+\frac{1-\mu}{\mu}\Big(w+\frac{n}{R^n}(\eta^*)^{n-1}\eta\Big)\Big),\\
\lambda w &= -2w-\frac{n}{R^n}(\eta^*)^{n-1}\,\eta.
\end{aligned}\tag{4.2}$$

Our main theorem is stated as follows.

Theorem 4.1. *Suppose that a_0 and μ satisfy $0<(1/2)-a_0<((2\mu-1)/2\mu)$ and $\xi'(\eta)+((1-\mu)/2\mu(1+\mu))(n/R^n)\eta^{n-1}<0$, the problem (2.7), and (2.1), has a unique stationary solution (u^*,η^*,w^*), where $u^*=0$ and $w^*=(1/2)(1-(n/R^n)(\eta^*)^{n-1})$, and (v^*,η^*,w^*), respectively, for all $\tau>0$. Then there exists a unique τ^* such that the linearization $-\widetilde{A}+\tau^*B$ has a purely imaginary pair of eigenvalues β. The point $(0,\eta^*,w^*,\tau^*)$ is then a Hopf point for (2.7), and there exists a C^0-curve of nontrivial periodic orbits for (2.7) and (2.1), bifurcating from $(0,\eta^*,w^*,\tau^*)$ and (v^*,η^*,w^*,τ^*), respectively.*

We will show the following three theorems that verify the above theorem. The next theorem shows that the steady state is the only Hopf point.

Theorem 4.2. *For $\tau^*\in\mathbb{R}\setminus\{0\}$, the operator $-\widetilde{A}+\tau^*B$ has a unique pair of purely imaginary eigenvalues $\{\pm i\beta\}$. Then the point $(0,\eta^*,w^*,\tau^*)$ satisfies the conditions (i), (ii), and (iii) in Definition 3.3.*

Proof. In the sequel, we denote $b_n=(n/R^n)(\eta^*)^{n-1}$, $n=2,3$. We assume without loss of generality that $\beta>0$ and Φ^* is the (normalized) eigenfunction of $-\widetilde{A}+\tau^*B$ with eigenvalue $i\beta$. We have to show that $(\Phi^*,i\beta)$ can be extended to a C^1-curve $\tau\mapsto(\phi(\tau),\lambda(\tau))$ of eigendata for $-\widetilde{A}+\tau B$ with $\mathrm{Re}(\lambda'(\tau^*))\neq 0$. For this, let $\Phi^*:=(\psi_0,\eta_0,w_0)\in D(\widetilde{A})$. First, we note that if $w_0=0$, then $\eta_0=0$ (vice versa) in the last equation of (4.2). We see that $\eta_0\neq 0$ and $w_0\neq 0$, for otherwise, by (4.2), $(A+i\beta)\psi_0=i\beta\,(\mu G(\cdot,\eta^*)\,\eta_0+((1-\mu)/(1+\mu))b_n+((1-\mu)/(1+\mu))\,w_0)=0$, which is not possible because A is symmetric. So without loss of generality, let $\eta_0=1$. Define

$$\begin{aligned}
&E: D(A)_{\mathbf{C}}\times\mathbb{C}\times\mathbb{C}\times\mathbb{R}\longrightarrow X_{\mathbf{C}}\times\mathbb{C}\times\mathbb{C},\\
&E(u,w,\lambda,\tau)\\
&:=\begin{pmatrix}
(A+\lambda)u-\tau\Big(u(\eta^*)+\gamma_\eta+\gamma_w w+\dfrac{1-\mu}{\mu}(b_n+w)\Big)\Big(\mu G(\cdot,\eta^*)+\dfrac{1-\mu}{1+\mu}\,b_n\Big)\\
+\dfrac{1-\mu}{1+\mu}(2w+b_n)\\
\lambda-\tau\Big(u(\eta^*)+\gamma_\eta+\gamma_w w+\dfrac{1-\mu}{\mu}(b_n+w)\Big)\\
\lambda w+2w+b_n
\end{pmatrix}.
\end{aligned}\tag{4.3}$$

The equation $E(u, w, \lambda, \tau) = 0$ is equivalent to λ being an eigenvalue of $-\widetilde{A} + \tau B$ with eigenfunction $(u, 1, w)$. By (4.2), we have $E(\psi_0, w_0, i\beta, \tau^*) = 0$ which is equivalent to

$$\begin{gathered}(A + i\beta)\psi_0 = i\beta\left(\mu G(\cdot, \eta^*) + \frac{1-\mu}{1+\mu}\,(b_n + w_0)\right),\\ i\beta = \tau^*\left(\psi_0(\eta^*) + \gamma_\eta + \gamma_w w_0 + \frac{1-\mu}{\mu}(b_n + w_0)\right),\\ i\beta w_0 = -2w_0 - b_n.\end{gathered} \tag{4.4}$$

To apply the implicit function theorem to E, we have to check that E is in C^1 and that

$$D_{(u,w,\lambda)}E(\psi_0, w_0, i\beta, \tau^*) \in L(D(A)_{\mathbf{C}} \times \mathbb{C} \times \mathbb{C},\ X_{\mathbf{C}} \times \mathbb{C} \times \mathbb{C}) \text{ is an isomorphism.} \tag{4.5}$$

In addition, the mapping

$$D_{(u,w,\lambda)}E(\psi_0, w_0, i\beta, \tau^*)\left(\widehat{u}, \widehat{w}, \widehat{\lambda}\right) = \begin{pmatrix} (A + i\beta)\widehat{u} + \widehat{\lambda}\psi_0 \\ -\tau^*\left(\widehat{u}(\eta^*) + \gamma_w\widehat{w} + \dfrac{1-\mu}{\mu}\widehat{w}\right)\left(\mu G(\cdot, \eta^*) + \dfrac{1-\mu}{1+\mu}\,b_n\right) + 2\,\dfrac{1-\mu}{1+\mu}\widehat{w} \\ \widehat{\lambda} - \tau^*\left(\widehat{u}(\eta^*) + \gamma_w\widehat{w} + \dfrac{1-\mu}{\mu}\widehat{w}\right) \\ \widehat{\lambda} w_0 + i\beta\widehat{w} + 2\widehat{w} \end{pmatrix} \tag{4.6}$$

is a compact perturbation of the mapping

$$\left(\widehat{u}, \widehat{w}, \widehat{\lambda}\right) \longmapsto \left((A + i\beta)\widehat{u}, \widehat{w}, \widehat{\lambda}\right) \tag{4.7}$$

which is invertible. In order to verify (4.5), it suffices to show that the system

$$D_{(u,w,\lambda)}E(\psi_0, w_0, i\beta, \tau^*)\left(\widehat{u}, \widehat{w}, \widehat{\lambda}\right) = 0, \tag{4.8}$$

which are

$$\begin{gathered}(A + i\beta)\widehat{u} + \widehat{\lambda}\psi_0 = \tau^*\left(\widehat{u}(\eta^*) + \gamma_w\widehat{w} + \frac{1-\mu}{\mu}\widehat{w}\right)\left(\mu G(\cdot, \eta^*) + \frac{1-\mu}{1+\mu}b_n\right) - 2\frac{1-\mu}{1+\mu}\widehat{w},\\ \widehat{\lambda} = \tau^*\left(\widehat{u}(\eta^*) + \gamma_w\widehat{w} + \frac{1-\mu}{\mu}\widehat{w}\right),\\ \widehat{\lambda} w_0 = -i\beta\widehat{w} - 2\widehat{w},\end{gathered} \tag{4.9}$$

necessarily implies that $\widehat{u} = 0$, $\widehat{w} = 0$, and $\widehat{\lambda} = 0$. We define $\phi := \psi_0 - (\mu G(\cdot, \eta^*) + ((1-\mu)/(1+\mu))b_n + ((1-\mu)/(1+\mu))w_0)$, then the first equation of (4.9) is given by

$$(A + i\beta)\widehat{u} + \widehat{\lambda}\phi = i\beta\frac{1-\mu}{1+\mu}\widehat{w}. \tag{4.10}$$

Since $(v, \eta, w, \lambda) = (\psi_0, 1, w, i\beta)$ solves (4.2), ϕ is a solution to the equation

$$(A + i\beta)\phi = -\mu\delta_{\eta^*} - (1-\mu)(b_n + w_0), \tag{4.11}$$

$$\frac{i\beta}{\tau^*} = \phi(\eta^*) + \frac{1-\mu}{\mu}(b_n + w_0), \tag{4.12}$$

$$(2 + i\beta)w_0 = -b_n. \tag{4.13}$$

Multiply (4.11) by r^{n-1} $(n = 2, 3)$ and integrate, then

$$\int_0^R \left(-\phi_{rr} - \frac{n-1}{r}\phi_r + (\mu + 1 + i\beta)\phi\right) r^{n-1}dr = \int_0^R (-\mu\delta_{\eta^*} - (1-\mu)(b_n + w_0))r^{n-1}dr \tag{4.14}$$

which implies that

$$\int_0^R \left(\frac{\partial}{\partial r}\left(-r^{n-1}\phi_r\right) + (\mu + 1 + i\beta)\int r^{n-1}\phi\right)dr = -\mu(\eta^*)^{n-1} - (1-\mu)(b_n + w_0)\frac{R^n}{n}. \tag{4.15}$$

Therefore we obtain

$$(\mu + 1 + i\beta)\int r^{n-1}\phi = -\mu(\eta^*)^{n-1} - (1-\mu)(b_n + w_0)\frac{R^n}{n}. \tag{4.16}$$

Multiply (4.10) by r^{n-1} $(n = 2, 3)$ and (4.11) by $r^{n-1}\phi$. Now we integrate the resulting equation to obtain

$$(\mu + 1 + i\beta)\int r^{n-1}\widehat{u} = -\widehat{\lambda}\int r^{n-1}\phi + i\beta\frac{1-\mu}{1+\mu}\cdot\frac{R^n}{n}\widehat{w}. \tag{4.17}$$

Multiply (4.10) and (4.11) by $r^{n-1}\phi$, and then eliminate the term $\widehat{\lambda}(A + i\beta)r\phi^2$. Integrating the resulting equation, we have

$$\begin{aligned}
0 &= \widehat{\lambda}\mu(\eta^*)^{n-1}\phi(\eta^*) + \widehat{\lambda}(1-\mu)(b_n + w_0)\int r^{n-1}\phi + \mu(\eta^*)^{n-1}\ \widehat{u}(\eta^*)(\mu + 1 + i\beta) \\
&\quad + (1-\mu)(b_n + w_0)(\mu + 1 + i\beta)\int r^{n-1}\ \widehat{u} + i\beta\frac{1-\mu}{1+\mu}\widehat{w}(\mu + 1 + i\beta)\int r^{n-1}\phi \\
&= \mu(\eta^*)^{n-1}\left(\widehat{\lambda}\phi(\eta^*) - i\beta\ \frac{1-\mu}{1+\mu}\widehat{w} + (\mu + 1 + i\beta)\left(\frac{\widehat{\lambda}}{\tau^*} - \frac{1-\mu}{\mu(1+\mu)}\widehat{w}\right)\right)
\end{aligned} \tag{4.18}$$

by (4.16) and (4.17). Using (4.9), (4.12), and (4.13) in the above equation, then we obtain

$$0 = \widehat{\lambda}\left(\frac{\mu + 1 + i2\beta}{\tau^*} - \left(\mu G(\eta^*, \eta^*) + \xi'(\eta^*)\right) - \frac{1-\mu}{\mu}b_n + \frac{1-\mu}{\mu}\frac{b_n}{(2+i\beta)^2}\right). \tag{4.19}$$

Suppose that $\widehat{\lambda} \neq 0$. Then the real and imaginary parts of (4.19) are given by

$$\begin{aligned} 0 &= \frac{\mu+1}{\tau^*} - \left(\mu G(\eta^*, \eta^*) + \xi'(\eta^*)\right) - \frac{(1-\mu)b_n}{\mu} \cdot \frac{\beta^4 + 9\beta^2 + 12}{(4+\beta^2)^2}, \\ 0 &= \frac{1}{\tau^*} - \frac{1-\mu}{\mu} \cdot \frac{2b_n}{(4+\beta^2)^2}. \end{aligned} \tag{4.20}$$

From these equations, we have

$$0 = \mu G(\eta^*, \eta^*) + \xi'(\eta^*) + \frac{(1-\mu)b_n}{\mu} \cdot \frac{\beta^4 + 9\beta^2 + 10 - 2\mu}{(4+\beta^2)^2}. \tag{4.21}$$

This leads to a contradiction that the right hand side is positive for $\beta > 0$ and $0 < \mu < 1$. Therefore, we should have $\widehat{\lambda} = 0$. Thus $\widehat{w} = 0$ and $\widehat{u} = 0$. □

Theorem 4.3. *Under the same condition as in Definition 3.3, $(0, \eta^*, w^*, \tau^*)$ satisfies the transversality condition. Hence this is a Hopf point for (2.7).*

Proof. By implicit differentiation of $E(\psi_0(\tau),\ w(\tau),\ \lambda(\tau), \tau) = 0$,

$$\begin{aligned} &D_{(u,w,\lambda)}E(\psi_0, w_0, i\beta, \tau^*)\left(\psi_0'(\tau^*), w'(\tau^*), \lambda'(\tau^*)\right) \\ &\quad = \begin{pmatrix} \left(\mu\ G(\eta^*, \eta^*) + \dfrac{1-\mu}{1+\mu}b_n\right)\left(\psi_0(\eta^*) + \gamma_\eta + \gamma_w w_0 + \dfrac{1-\mu}{\mu}b_n + \dfrac{1-\mu}{\mu}w'(\tau^*)\right)\Big) \\ \psi_0(\eta^*) + \gamma_\eta + \gamma_w w_0 + \dfrac{1-\mu}{\mu}b_n + \dfrac{1-\mu}{\mu}w'(\tau^*)\Big) \\ 0 \end{pmatrix}. \end{aligned} \tag{4.22}$$

This means that the functions $\widetilde{u} := \psi_0'(\tau^*)$, $\widetilde{w} := w'(\tau^*)$, and $\widetilde{\lambda} := \lambda'(\tau^*)$ satisfy the equations

$$\begin{aligned}
&(A+i\beta)\widetilde{u} + \widetilde{\lambda}\psi_0 - \tau^*\left(\widetilde{u}(\eta^*) + \gamma_w\widetilde{w} + \frac{1-\mu}{\mu}\widetilde{w}\right)\left(\mu\, G(\eta^*,\eta^*) + \frac{1-\mu}{1+\mu}b_n\right) + 2\frac{1-\mu}{1+\mu}\widetilde{w} \\
&\quad = \left(\psi_0(\eta^*) + \gamma_\eta + \gamma_w w_0 + \frac{1-\mu}{\mu}b_n + \frac{1-\mu}{\mu}w_0\right)\left(\mu G(\eta^*,\eta^*) + \frac{1-\mu}{1+\mu}b_n\right), \\
&\widetilde{\lambda} - \tau^*\left(\widetilde{u}(\eta^*) + \gamma_w\widetilde{w} + \frac{1-\mu}{\mu}\widetilde{w}\right) = \psi_0(\eta^*) + \gamma_\eta + \gamma_w w_0 + \frac{1-\mu}{\mu}(b_n + w_0), \\
&\widetilde{\lambda}w_0 + (2+i\beta)\widetilde{w} = 0.
\end{aligned} \tag{4.23}$$

By letting $\phi := \psi_0 - (\mu G(\cdot,\eta) + ((1-\mu)/(1+\mu))b_n + ((1-\mu)/(1+\mu))w_0)$, then

$$(A+i\beta)\widetilde{u} + \widetilde{\lambda}\phi = i\beta\frac{1-\mu}{1+\mu}\widetilde{w}. \tag{4.24}$$

From (4.9) and (4.23), we obtain

$$\frac{\widetilde{\lambda}}{\tau^*} = \widetilde{u}(\eta^*) + \left(\gamma_w + \frac{1-\mu}{\mu}\right)\widetilde{w} + \frac{i\beta}{\tau^{*2}}. \tag{4.25}$$

We now multiply (4.24) by $r^{n-1}\phi$ and (4.11) by $r^{n-1}\widetilde{u}$ and then subtract these two equations, then we get

$$\widetilde{\lambda}r^{n-1}\phi^2 = \mu r^{n-1}\widetilde{u}(r)\delta_{\eta^*} + (1-\mu)(b_n + w_0)r^{n-1}\widetilde{u} + i\beta\frac{1-\mu}{1+\mu}\widetilde{w}r^{n-1}\phi. \tag{4.26}$$

Comparing to (4.11) and then integrating, we have

$$\begin{aligned}
0 = &-\mu(\eta^*)^{n-1}\phi(\eta^*)\widetilde{\lambda} - \widetilde{\lambda}(1-\mu)(b_n + w_0)\int r^{n-1}\phi - \mu(\eta^*)^{n-1}\widetilde{u}(\eta^*)(\mu+1+i\beta) \\
&- (1-\mu)(b_n + w_0)(\mu+1+i\beta)\int r^{n-1}\widetilde{u} - i\beta\frac{1-\mu}{1+\mu}\widetilde{w}(\mu+1+i\beta)\int r^{n-1}\phi.
\end{aligned} \tag{4.27}$$

Using (4.16) and (4.17) in the above equation, then

$$0 = \mu(\eta^*)^{n-1}\left(\widetilde{\lambda}\phi(\eta^*) + (\mu+1+i\beta)\widetilde{u}(\eta^*) - i\beta\frac{1-\mu}{1+\mu}\widetilde{w}\right). \tag{4.28}$$

By substituting (4.12), (4.13), and (4.23), we have

$$(\mu+1+i\beta)\frac{i\beta}{\tau^{*2}} = \widetilde{\lambda}\left(\frac{\mu+1+2i\beta}{\tau^*} - \mu G(\eta^*,\eta^*) - \xi'(\eta^*) - \frac{1-\mu}{\mu}b_n + \frac{1-\mu}{\mu}\frac{b_n}{(2+i\beta)^2}\right). \tag{4.29}$$

The real part of $\tilde{\lambda}$ is given by

$$\frac{\beta}{\tau^*}\left((\mu+1)^2+\beta^2\right)\operatorname{Re}\tilde{\lambda}=\left|\tilde{\lambda}\right|^2((\mu+1)Q-\beta P), \tag{4.30}$$

where

$$\begin{gathered}P=\frac{\mu+1}{\tau^*}-\mu G(\eta^*,\eta^*)-\xi'(\eta^*)-\frac{1-\mu}{\mu}b_n+\frac{(1-\mu)b_n}{\mu}\cdot\frac{4-\beta^2}{(4+\beta^2)^2},\\ Q=\frac{2\beta}{\tau^*}-\frac{1-\mu}{\mu}\cdot\frac{4b_n\beta}{(4+\beta^2)^2},\\ (\mu+1)Q-\beta P=\frac{(\mu+1)\beta}{\tau^*}+\beta(\mu G(\eta^*,\eta^*)+\xi'(\eta^*))+\frac{(1-\mu)b_n\beta}{\mu(4+\beta^2)^2}\left(\beta^4+9\beta^2+8-4\mu\right).\end{gathered} \tag{4.31}$$

Since $(\mu+1)Q-\beta P>0$ for $0<\mu<1$, we have $\operatorname{Re}\tilde{\lambda}>0$. Therefore, $\operatorname{Re}\lambda'(\tau^*)>0$ for $\beta>0$ and for $0<\mu<1$, and thus by the Hopf-bifurcation theorem in [17], there exists a family of periodic solutions which bifurcates from the stationary solution as τ passes τ^*. □

The next theorem shows that a critical Hopf point τ^* exists uniquely.

Theorem 4.4. *Under the same condition as in Definition 3.3, there exists a unique, purely imaginary eigenvalue $\lambda=i\beta$ of* (4.2) *with $\beta>0$ for a unique critical point $\tau^*>0$ in order for $(0,\eta^*,w^*,\tau^*)$ to be a Hopf point.*

Proof. We only need to show that the function $(u,\beta,\tau)\mapsto E(u,w,i\beta,\tau)$ has a unique zero with $\beta>0$ and $\tau>0$. This means solving the system (4.2) with $\lambda=i\beta$, $\eta_0=1$, and $\psi_0=\phi+\mu G(\cdot,\eta^*)+((1-\mu)/(1+\mu))b_n+((1-\mu)/(1+\mu))w_0$,

$$\begin{gathered}(A+i\beta)\phi=-\mu\delta_{\eta^*}-(1-\mu)(b_n+w_0),\\ \frac{i\beta}{\tau^*}=\phi(\eta^*)+\mu G(\eta^*,\eta^*)+\xi'(\eta^*)+\frac{1-\mu}{\mu}(b_n+w_0),\\ (2+i\beta)w_0=-b_n.\end{gathered} \tag{4.32}$$

The second equation becomes

$$\frac{i\beta}{\tau^*}=-\mu G_\beta(\eta^*,\eta^*)+\mu G(\eta^*,\eta^*)+\xi'(\eta^*)-\frac{1}{\mu+1+i\beta}(1-\mu)(b_n+w_0)+\frac{1-\mu}{\mu}(b_n+w_0), \tag{4.33}$$

where G_β is a Green's function of the differential operator $A+i\beta$. The real and imaginary parts of this above equation are given by

$$\begin{aligned}
\frac{\beta}{\tau^*} &= -\mu \operatorname{Im} G_\beta(\eta^*,\eta^*) + \frac{(1-\mu)b_n\beta}{\mu(4+\beta^2)} + \frac{(1-\mu)b_n\beta(1-\mu+\beta^2)}{(4+\beta^2)\left((1+\mu)^2+\beta^2\right)} \\
&= -\mu \operatorname{Im} G_\beta(\eta^*,\eta^*) + \frac{(1-\mu)b_n\beta}{\mu(4+\beta^2)\left((1+\mu)^2+\beta^2\right)}\left((1+\mu)\beta^2+3\mu+1\right), \\
0 &= -\mu \operatorname{Re} G_\beta(\mu^*,\mu^*) + \mu G(\eta^*,\eta^*) + \xi'(\eta^*) + \frac{(1-\mu)b_n\ (2+\beta^2)}{\mu(4+\beta^2)} \\
&\quad - \frac{(1-\mu)b_n((\mu+1)(2+\beta^2)+\beta^2)}{\left((\mu+1)^2+\beta^2\right)(4+\beta^2)} \\
&= -\mu \operatorname{Re} G_\beta(\mu^*,\mu^*) + \mu G(\eta^*,\eta^*) + \xi'(\eta^*) + \frac{(1-\mu)b_n\beta}{\mu(4+\beta^2)\left((1+\mu)^2+\beta^2\right)} \\
&\quad \times \left(\beta^4+3\beta^2+2(1+\mu)\right).
\end{aligned} \tag{4.34}$$

Since $\operatorname{Im} G_\beta(\eta^*,\eta^*) < 0$ in [17, Lemma 12] and $\mu > 0$, there is a unique τ in the first equation if it does guarantee the existence of β. Now, we let

$$\begin{aligned}
T(\beta) &:= -\mu \operatorname{Re} G_\beta(\mu^*,\mu^*) + \mu G(\eta^*,\eta^*) + \xi'(\eta^*) + \frac{(1-\mu)b_n\beta}{\mu(4+\beta^2)\left((1+\mu)^2+\beta^2\right)} \\
&\quad \times \left(\beta^4+3\beta^2+2(1+\mu)\right),
\end{aligned} \tag{4.35}$$

then $T(\infty) = \mu\ G(\eta^*,\eta^*)+\xi'(\eta^*) > 0$ and $T(0) = \xi'(\eta^*)+(1-\mu)b_n/2\mu(1+\mu) < 0$ by assumption. If we show that $T'(\beta) < 0$, then the existence of β is proved:

$$\begin{aligned}
T'(\beta) &= -\mu(\operatorname{Re} G_\beta(\eta^*,\eta^*))' + \frac{2(1-\mu)b_n\beta}{\mu(4+\beta^2)^2\left((1+\mu)^2+\beta^2\right)^2} \\
&\quad \times \left(\left(\mu^2+2\mu+2\right)\beta^4 + 4(1+\mu)(1+2\mu)\beta^2 + 2(1+\mu)\left(1+4\mu-\mu^2\right)\right).
\end{aligned} \tag{4.36}$$

Since $1+4\mu-\mu^2 > 0$ for $0 < \mu < 1$ and $(\operatorname{Re} G_\beta(\eta^*,\eta^*))' < 0$ in [17, Lemma 12], we have $T'(\beta) > 0$ for all $\beta > 0$. □

There is a unique pure imaginary eigenvalue $\beta > 0$ and the critical point τ^* of (2.1) and thus there exists a family of periodic solutions which bifurcates from the stationary solution as τ passes τ^* under the condition of Theorem 4.4. Thus we also found the relationship between μ and a_0 for which Hopf bifurcation occurs for the problem (2.1).

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