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Research Article

On the Operator $\oplus_B^k$ Related to Bessel Heat Equation

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We study the equation $(\partial/\partial t)u(x,t) = c^2\oplus_B^k u(x,t)$ with the initial condition $u(x,0) = f(x)$ for $x \in R_n^+$. The operator $\oplus_B^k$ is the operator iterated k-times and is defined by $\oplus_B^k = ((\sum_{i=1}^{p} B_{x_i})^4 - (\sum_{j=p+1}^{p+q} B_{x_i})^4)^k$, where $p+q=n$ is the dimension of the $\mathbb{R}_n^+$, $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -1/2$, $i = 1,2,3,\ldots,n$, and k is a nonnegative integer, $u(x,t)$ is an unknown function for $(x,t) = (x_1,x_2,\ldots,x_n,t) \in R_n^+ \times (0,\infty)$, $f(x)$ is a given generalized function, and c is a positive constant. We obtain the solution of such equation, which is related to the spectrum and the kernel, which is so called Bessel heat kernel. Moreover, such Bessel heat kernel has interesting properties and also related to the kernel of an extension of the heat equation.

1. Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\Delta u(x,t) \tag{1.1}$$

with the initial condition

$$u(x,0) = f(x), \tag{1.2}$$

where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplace operator and $(x,t) = (x_1,x_2,\ldots,x_n,t) \in \mathbb{R}^n \times (0,\infty)$, we obtain

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}}\int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4c^2 t}\right) f(y)\,dy \tag{1.3}$$

as the solution of (1.1). Equation (1.3) can be written as

$$u(x,t) = E(x,t) * f(x), \tag{1.4}$$

where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2 t}\right). \tag{1.5}$$

$E(x,t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$ (see [1, pages 208-209]).

In 2004, Yildirim et al. [2, 3] first introduced the Bessel diamond operator $\Diamond_B^k$ iterated k-times, defined by

$$\Diamond_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right)^k, \tag{1.6}$$

where $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -1/2$, $x_i > 0$. The operator $\Diamond_B^k$ can be expressed by $\Diamond_B^k = \Delta_{\mathrm{B}}^k \Box_B^k = \Box_B^k \Delta_B^k$, where

$$\Delta_B^k = \left(\sum_{i=1}^{p} B_{x_i}\right)^k, \tag{1.7}$$

$$\Box_B^k = \left(\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j}\right)^k. \tag{1.8}$$

And, Yildirim et al. [2, 3] have shown that the solution of the convolution form

$$u(x) = (-1)^k S_{2k}(x) * R_{2k}(x) \tag{1.9}$$

is a unique elementary solution of $\Diamond_B^k$ that is

$$\Diamond_B^k\left((-1)^k S_{2k}(x) * R_{2k}(x)\right) = \delta. \tag{1.10}$$

Now, the purpose of this work is to study the following equation:

$$\frac{\partial}{\partial t} u(x,t) = c^2 \bigoplus_B^k u(x,t) \tag{1.11}$$

with the initial condition

$$u(x,0) = f(x), \quad \text{for } x \in R_n^+, \tag{1.12}$$

where the operator $\oplus_B^k$ is first introduced by Satsanit and Kananthai [4] and is defined by

$$\bigoplus_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i}\right)^4 - \left(\sum_{j=p+1}^{p+q} B_{x_i}\right)^4\right)^k$$
$$= \left(\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_i}\right)^2\right)^k \left(\left(\sum_{i=1}^{p} B_{x_i}\right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_i}\right)^2\right)^k. \tag{1.13}$$

Let us denote the operator

$$\circledcirc_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i}\right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j}\right)^2\right)^k. \tag{1.14}$$

By (1.7) we obtain

$$\circledcirc_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i^2}\right)^2 + \left(\sum_{j=p+1}^{p+q} B_{x_j^2}\right)^2\right]^k$$
$$= \left[\left(\frac{\Delta_B + \Box_B}{2}\right)^2 + \left(\frac{\Delta_B - \Box_B}{2}\right)^2\right]^k$$
$$= \left(\frac{\Delta_B^2 + \Box_B^2}{2}\right)^k. \tag{1.15}$$

Thus, (1.13) can be written as

$$\bigoplus_B^k = \Diamond_B^k \circledcirc_B^k, \tag{1.16}$$

where $\Diamond_B^k$ and $\circledcirc_B^k$ are defined by (1.6) and (1.15), respectively, $p + q = n$ is the dimension of the $R_n^+ = x : x = (x_1, x_2, \ldots, x_n, t),\ x_i > 0, i = 1, 2, 3, \ldots, n$, $u(x,t)$ is an unknown function, $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -1/2$, $f(x)$ is the given generalized function, k is a positive integer, and c is a constant.

Moreover, Bessel heat kernel has interesting properties and also related to the kernel of an extension of the heat equation. We obtain the solution in the classical convolution form

$$u(x,t) = E(x,t) * f(x), \tag{1.17}$$

where the symbol $*$ is the B-convolution in (2.3), as a solution of (1.11), which satisfies (1.12), and

$$E(x,t) = C_v \int_{\Omega^+} e^{c^2 t[(y_1^2+\cdots+y_p^2)^4-(y_{p+1}^2+\cdots+y_{p+q}^2)^4]^k} \prod_{i=1}^{n} J_{v_i-1/2}(x_i,y_i) y_i^{2v_i} dy, \tag{1.18}$$

and $\Omega^+ \subset R_n^+$ is the spectrum of $E(x,t)$ for any fixed $t > 0$ and $J_{v_i-1/2}(x_i,y_i)$ is the normalized Bessel function.

Before going into details, the following definitions and some important concepts are needed.

2. Preliminaries

The shift operator according to the law remarks that this shift operator connected to the Bessel differential operator (see [2, 3, 5]):

$$\begin{aligned} T_x^y \varphi(x) = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi\left(\sqrt{x_1^2+y_1^2-2x_1y_1\cos\theta_1}, \ldots, \sqrt{x_n^2+y_n^2-2x_ny_n\cos\theta_n}\right) \\ \times \left(\prod_{i=1}^{n} \sin^{2v_i-1}\theta_i\right) d\theta_1 \cdots d\theta_n, \end{aligned} \tag{2.1}$$

where $x, y \in R_n^+$, $C_v^* = \prod_{i=1}^n \Gamma(v_i+1)/\Gamma(1/2)\Gamma(v_i)$. We remark that this shift operator is closely connected to the Bessel differential operator (see [4]):

$$\begin{gathered} \frac{d^2U}{dx^2} + \frac{2v}{x}\frac{dU}{dx} = \frac{d^2U}{dy^2} + \frac{2v}{y}\frac{dU}{dy}, \\ U(x,0) = f(x), \qquad U_y(x,0) = 0. \end{gathered} \tag{2.2}$$

The convolution operator determined by the T_x^y is as follows:

$$(f * \varphi)(y) = \int_{R_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy. \tag{2.3}$$

Convolution (2.3) is known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator.

(1) $T_x^y \cdot 1 = 1$.

(2) $T_x^0 \cdot f(x) = f(x)$.

(3) If $f(x), g(x) \in C(R_n^+)$, $g(x)$ is a bounded function for all $x > 0$, and $\int_0^\infty |f(x)| (\prod_{i=1}^n x_i^{2v_i}) dx < \infty$, then $\int_{R_n^+} T_x^y f(x) g(y) (\prod_{i=1}^n y_i^{2v_i}) dy = \int_{R_n^+} f(y) T_x^y g(x) (\prod_{i=1}^n y_i^{2v_i}) dy$.

(4) From (3),we have the following equality for $g(x) = 1$: $\int_{R_n^+} T_x^y f(x) (\prod_{i=1}^n y_i^{2v_i}) dy = \int_{R_n^+} f(y) (\prod_{i=1}^n y_i^{2v_i}) dy$.

(5) $(f * g)(x) = (g * f)(x)$.

The Fourier-Bessel transformation and its inverse transformation are defined as follows:

$$(\digamma_B f)(x) = C_v \int_{R_n^+} f(y) \left(\prod_{i=1}^{n} J_{v_i - 1/2}(x_i, y_i) y_i^{2v_i} \right) dy, \tag{2.4}$$

$$\left(\digamma_B^{-1} f \right)(x) = (\digamma_B f)(-x), \qquad C_v = \left(\prod_{i=1}^{n} 2^{v_i - 1/2} \Gamma\left(v_i + \frac{1}{2} \right) \right)^{-1}, \tag{2.5}$$

where $J_{v_i-1/2}(x_i, y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. The following equalities for Fourier-Bessel transformation are true (see [5–7]):

$$\digamma_B \delta(x) = 1, \tag{2.6}$$

$$\digamma_B (f * g)(x) = \digamma_B f(x) \cdot \digamma_B g(x). \tag{2.7}$$

Definition 2.1. The spectrum of the kernel $E(x,t)$ of (1.18) is the bounded support of the Fourier Bessel transform $\digamma_B E(y,t)$ for any fixed $t > 0$.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be a point in $\mathbb{R}_n^+$ and denote by

$$\Gamma_+ = \left\{ x \in \mathbb{R}_n^+ : x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 > 0,\ \xi_1 > 0 \right\} \tag{2.8}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω^+ be spectrum of $E(x,t)$ defined by (1.18) for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $F_B E(y,t)$ be the Fourier Bessel transform of $E(x,t)$, which is defined by

$$\digamma_B E(y,t) = \begin{cases} e^{c^2 t \left[(y_1^2 + \cdots + y_p^2)^4 - (y_{p+1}^2 + \cdots + y_{p+q}^2)^4 \right]^k} & \text{for } \xi \in \Omega_+, \\ 0 & \text{for } \xi \notin \Omega_+. \end{cases} \tag{2.9}$$

Lemma 2.3 (Fourier Bessel transform of $\square_B^k$ operator). *One has*

$$\digamma_B \square_B^k u(x) = (-1)^k V_1^k(x) \digamma_B u(x), \tag{2.10}$$

where

$$V_1^k(x) = \left(\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \right)^k. \tag{2.11}$$

Proof. See [8]. □

Lemma 2.4 (Fourier Bessel transform of Δ_B^k operator). *One has*

$$\mathcal{F}_B \Delta_B^k u(x) = (-1)^k |x|^{2k} \mathcal{F}_B u(x), \tag{2.12}$$

where

$$|x|^{2k} = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^k. \tag{2.13}$$

Proof. See [8]. □

Lemma 2.5 (Fourier Bessel transform of $\oplus_B^k$ operator). *One has*

$$\mathcal{F}_B \bigoplus_B^k u(x) = V^k(x) \mathcal{F}_B u(x), \tag{2.14}$$

where

$$V^k(x) = \left(\left(\sum_{i=1}^{p} x_i^2\right)^4 - \left(\sum_{j=p+1}^{p+q} x_j^2\right)^4\right)^k. \tag{2.15}$$

Proof. We can use the mathematical induction method; for $k = 1$, we have

$$\begin{aligned}
\mathcal{F}_B\left(\bigoplus_B u\right)(x) &= C_v \int_{R_n^+} \left(\bigoplus_B u(y)\right)\left(\prod_{i=1}^{n} J_{v_i-1/2}(x_i, y_i) y_i^{2v_i}\right) dy \\
&= C_v \int_{R_n^+} (\Diamond_B \circledcirc_B u(y)) \left(\prod_{i=1}^{n} J_{v_i-1/2}(x_i, y_i) y_i^{2v_i}\right) dy \\
&= C_v \int_{R_n^+} \frac{(\Delta_B^2 + \square_B^2)}{2} g(y) \left(\prod_{i=1}^{n} J_{v_i-1/2}(x_i, y_i) y_i^{2v_i}\right) dy, \quad g(y) = \Diamond_B u(y) \\
&= \mathcal{F}_B \frac{(\Delta_B^2 g + \square_B^2 g)}{2}(x) \\
&= \frac{\left((-1)^2 (x_1^2 + \cdots + x_n^2)^2 + (-1)^2 \left(x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2\right)^2\right)}{2} \mathcal{F}_B g(x) \\
&= \left(\left(x_1^2 + x_2^2 + \cdots + x_p^2\right)^2 + \left(x_{p+1}^2 + \cdots + x_{p+q}^2\right)^2\right) \mathcal{F}_B \Diamond_B u(x)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\left(\left(\sum_{i=1}^{p} x_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} x_j^2 \right)^2 \right) \cdot \left(\left(\sum_{i=1}^{p} x_i^2 \right)^2 - \left(\sum_{j=p+1}^{p+q} x_j^2 \right)^2 \right) \mathcal{F}_B u(x) \\
&= \left(\left(x_1^2 + x_2^2 + \cdots + x_p^2 \right)^4 - \left(x_{p+1}^2 + \cdots + x_{p+q}^2 \right)^4 \right) \mathcal{F}_B u(x) \\
&= V(x) \mathcal{F}_B u(x),
\end{aligned}
\tag{2.16}
$$

where $V(x) = (x_1^2 + x_2^2 + \cdots + x_p^2)^4 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^4$. Then, from inverse Fourier transform we obtain

$$\bigoplus_B u(x) = \mathcal{F}_B^{-1} V(x) \mathcal{F}_B u(x). \tag{2.17}$$

Assume that the statement is true for $k-1$, that is,

$$\bigoplus_B^{k-1} u(x) = \mathcal{F}_B^{-1} V^{k-1}(x) \mathcal{F}_B u(x). \tag{2.18}$$

Then, we must prove that is also true for $k \in N$. So we have

$$
\begin{aligned}
\bigoplus_B^{k} u(x) &= \bigoplus_B \left(\bigoplus_B^{k-1} u(x) \right) \\
&= \mathcal{F}_B^{-1} V(x) \mathcal{F}_B \mathcal{F}_B^{-1} V^{k-1}(x) \mathcal{F}_B u(x) \\
&= \mathcal{F}_B^{-1} V^{k}(x) \mathcal{F}_B u(x).
\end{aligned}
\tag{2.19}
$$

This completes the proof. □

Lemma 2.6. *For $t, v > 0$ and $x, y \in \mathbb{R}^n$, one has*

$$
\begin{aligned}
\int_0^\infty e^{-c^2 x^2 t} x^{2v} dx &= \frac{\Gamma(v)}{2c^{2v+1} t^{v+1/2}}, \\
\int_0^\infty e^{-c^2 x^2 t} J_{v-1/2}(xy) x^{2v} dx &= \frac{\Gamma(v+1/2)}{2(c^2 t)^{v+1/2}} e^{-y^2/4c^2 t},
\end{aligned}
\tag{2.20}
$$

where c is a positive constant.

Proof. See [9]. □

Lemma 2.7. *Let the operator L be defined by*

$$L = \frac{\partial}{\partial t} - c^2 \bigoplus_B^{k}, \tag{2.21}$$

where $\oplus_B^k$ *is the operator iterated* k*-times and is given by*

$$\bigoplus_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i} \right)^4 - \left(\sum_{j=p+1}^{p+q} B_{x_i} \right)^4 \right)^k,$$
$$B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}, \tag{2.22}$$

$p + q = n$ *is the dimension* $\mathbb{R}_n^+$, k *is a positive integer,* $(x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+$, *and* c *is a positive constant. Then*

$$E(x,t) = C_v \int_{\Omega^+} e^{c^2 t[(y_1^2+\cdots+y_p^2)^4 - (y_{p+1}^2+\cdots+y_{p+q}^2)^4]^k} \prod_{i=1}^{n} J_{v_i - 1/2}(x_i, y_i) y_i^{2v_i} dy \tag{2.23}$$

is the elementary solution of (2.21) *in the spectrum* $\Omega^+ \subset R_n^+$ *for* $t > 0$.

Proof. Let $LE(x,t) = \delta(x,t)$, where $E(x,t)$ is the elementary solution of L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t} E(x,t) - c^2 \bigoplus_B^k E(x,t) = \delta(x)\delta(t). \tag{2.24}$$

Applying the Fourier Bessel transform, which is defined by (2.4) to the both sides of the above equation and using Lemma 2.5 by considering $\mathcal{F}_B \delta(x) = 1$, we obtain

$$\frac{\partial}{\partial t} \mathcal{F}_B E(x,t) - c^2 \left[\left(x_1^2 + x_2^2 + \cdots + x_p^2 \right)^4 - \left(x_{p+1}^2 + \cdots + x_{p+q}^2 \right)^4 \right]^k \mathcal{F}_B E(x,t) = \delta(t). \tag{2.25}$$

Thus, we get

$$\mathcal{F}_B E(x,t) = H(t) e^{c^2 t[(x_1^2+x_2^2+\cdots+x_p^2)^4 - (x_{p+1}^2+\cdots+x_{p+q}^2)^4]^k}, \tag{2.26}$$

where $H(t)$ is the Heaviside function, because $H(t) = 1$ holds for $t \geq 0$.

Therefore,

$$\mathcal{F}_B E(x,t) = e^{c^2 t[(x_1^2+x_2^2+\cdots+x_p^2)^4 - (x_{p+1}^2+\cdots+x_{p+q}^2)^4]^k}, \tag{2.27}$$

which has been already defined by (2.7). Thus from (2.5), we have

$$E(x,t) = C_v \int_{\mathbb{R}_n^+} e^{c^2 t[(y_1^2+\cdots+y_p^2)^4 - (y_{p+1}^2+\cdots+y_{p+q}^2)^4]^k} \prod_{i=1}^{n} J_{v_i - 1/2}(x_i, y_i) y_i^{2v_i} dy, \tag{2.28}$$

where Ω^+ is the spectrum of $E(x,t)$. Thus, we obtain

$$E(x,t) = C_v \int_{\Omega^+} e^{c^2 t[(y_1^2+\cdots+y_p^2)^4-(y_{p+1}^2+\cdots+y_{p+q}^2)^4]^k} \prod_{i=1}^{n} J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy \tag{2.29}$$

as an elementary solution of (2.21) in the spectrum $\Omega^+ \subset R_n^+$ for $t > 0$. □

3. Main Results

Theorem 3.1. *Let us consider the equation*

$$\frac{\partial}{\partial t} u(x,t) - c^2 \bigoplus_B^k u(x,t) = 0 \tag{3.1}$$

with the initial condition

$$u(x,0) = f(x), \tag{3.2}$$

where $\oplus_B^k$ is the operator iterated k-times and is defined by

$$\bigoplus_B^k = \left(\left(\sum_{i=1}^{p} B_{x_i} \right)^4 - \left(\sum_{j=p+1}^{p+q} B_{x_i} \right)^4 \right)^k, \qquad B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i}, \tag{3.3}$$

$p+q=n$ is the dimension $\mathbb{R}_n^+, k$ is a positive integer, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}_n^+ \times (0,\infty)$, $f(x)$ is the given generalized function, and c is a positive constant. Then

$$u(x,t) = E(x,t) * f(x) \tag{3.4}$$

is a solution of (3.1)*, which satisfies* (3.2)*, where $E(x,t)$ is given by* (2.23)*. In particular, if one puts $k=1$ and $q=0$ in* (3.1)*, then* (3.1) *reduces to the equation*

$$\frac{\partial}{\partial t} u(x,t) - c^2 \Delta_B^4 u(x,t) = 0, \tag{3.5}$$

which is related to the Bessel heat equation.

Proof. Taking the Fourier Bessel transform, which is defined by (2.4), of both sides of (3.1) for $x \in \mathbb{R}_n^+$ and using Lemma 2.5, we obtain

$$\frac{\partial}{\partial t}\mathcal{F}_B u(x,t) = c^2\left(\left(x_1^2+\cdots+x_p^2\right)^4 - \left(x_{p+1}^2+\cdots+x_{p+q}^2\right)^4\right)^k \mathcal{F}_B u(x,t). \tag{3.6}$$

Thus, we consider the initial condition (3.2); then we have the following equality for (3.6):

$$u(x,t) = f(x) * \mathcal{F}_B^{-1} e^{c^2t[(x_1^2+\cdots+x_p^2)^4-(x_{p+1}^2+\cdots+x_{p+q}^2)^4]^k}. \tag{3.7}$$

Here, if we use (2.4) and (2.5), then we have

$$\begin{aligned} u(x,t) &= f(x) * \mathcal{F}_B^{-1} e^{c^2t[(y_1^2+\cdots+y_p^2)^4-(y_{p+1}^2+\cdots+y_{p+q}^2)^4]^k} \\ &= \int_{\mathbb{R}_n^+} \mathcal{F}_B^{-1} e^{c^2t[(y_1^2+\cdots+y_p^2)^4-(y_{p+1}^2+\cdots+y_{p+q}^2)^4]^k} T_x^y f(x)\left(\prod_{i=1}^n y_i^{2v_i}\right)dy \\ &= \int_{\mathbb{R}_n^+}\left(C_v\int_{\mathbb{R}_n^+} e^{c^2tV^k(z)}\prod_{i=1}^n J_{v_i-1/2}(y_i,z_i)z_i^{2v_i}dz\right)T_x^y f(x)\left(\prod_{i=1}^n y_i^{2v_i}\right)dy, \end{aligned} \tag{3.8}$$

where $V(z) = (z_1^2+z_2^2+\cdots+z_p^2)^4 - (z_{p+1}^2+z_{p+2}^2+\cdots+z_{p+q}^2)^4$. Set

$$E(x,t) = C_v\int_{\mathbb{R}_n^+} e^{c^2t[(y_1^2+\cdots+y_p^2)^4-(y_{p+1}^2+\cdots+y_{p+q}^2)^4]^k}\prod_{i=1}^n J_{v_i-1/2}(x_i,y_i)y_i^{2v_i}dy. \tag{3.9}$$

Since the integral in (3.9) is divergent, therefore we choose $\Omega^+ \subset \mathbb{R}_n^+$ to be the spectrum of $E(x,t)$, and by (2.21), we have

$$\begin{aligned} E(x,t) &= C_v\int_{\mathbb{R}_n^+} e^{c^2t[(y_1^2+\cdots+y_p^2)^4-(y_{p+1}^2+\cdots+y_{p+q}^2)^4]^k}\prod_{i=1}^n J_{v_i-1/2}(x_i,y_i)y_i^{2v_i}dy \\ &= C_v\int_{\Omega^+} e^{c^2t[(y_1^2+\cdots+y_p^2)^4-(y_{p+1}^2+\cdots+y_{p+q}^2)^4]^k}\prod_{i=1}^n J_{v_i-1/2}(x_i,y_i)y_i^{2v_i}dy. \end{aligned} \tag{3.10}$$

Thus (3.8) can be written in the convolution form

$$u(x,t) = E(x,t) * f(x). \tag{3.11}$$

Moreover, since $E(x,t)$ exists, we can see that

$$\begin{aligned}\lim_{t\to 0} E(x,t) &= C_v \int_{\Omega^+} \prod_{i=1}^{n} J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy \\ &= C_v \int_{\mathbb{R}_n^+} \prod_{i=1}^{n} J_{v_i-1/2}(x_i, y_i) y_i^{2v_i} dy \\ &= \delta(x), \quad \text{for } x \in \mathbb{R}_n^+,\end{aligned} \tag{3.12}$$

hold (see [8]). Thus for the solution $u(x,t) = E(x,t) * f(x)$ of (3.1), then we have

$$\lim_{t\to 0} u(x,t) = u(x,0) = \delta * f(x) = f(x), \tag{3.13}$$

which satisfies (3.2). This completes the proof. □

Theorem 3.2. *The kernel $E(x,t)$ defined by* (3.10) *has the following properties.*

(1) $E(x,t) \in C^\infty$*-the space of continuous function for $x \in \mathbb{R}^n$, $t > 0$ with infinitely differentiable.*

(2) $(\partial/\partial t - c^2 \oplus_B^k) E(x,t) = 0$ *for all* $x \in \mathbb{R}_n^+$, $t > 0$.

(3) $\lim_{t\to 0} E(x,t) = \delta$ *for all* $x \in \mathbb{R}_n^+$.

Proof. (1) From (3.10) and

$$\frac{\partial^n}{\partial t^n} E(x,t) = C_v \int_{\mathbb{R}_n^+} \frac{\partial^n}{\partial t^n} e^{c^2 t [(y_1^2+\cdots+y_p^2)^4 - (y_{p+1}^2+\cdots+y_{p+q}^2)^4]^k} \prod_{i=1}^{n} J_{v_i-(1/2)}(x_i, y_i) y_i^{2v_i} dy. \tag{3.14}$$

we have $E(x,t) \in C^\infty$ for $x \in \mathbb{R}_n^+, t > 0$.

(2) We have $u(x,t) = E(x,t)$ since $u(x,t) = E(x,t) * f(x)$ holds. Note here that we use the fact $f(x) = \delta(x)$ by the Fourier Bessel transformation. Then, we obtain

$$\left(\frac{\partial}{\partial t} - c^2 \bigoplus_B^k\right) E(x,t) = 0 \tag{3.15}$$

by direct computation.

(3) This case is obvious by (3.12).

In particular, if we put $k = 1$ and $q = 0$ in (3.1), then (3.1) reduces to the equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \Delta_B^4 u(x,t) = 0, \tag{3.16}$$

and we obtain the solution of (3.16) in the convolution form

$$u(x,t) = E(x,t) * f(x), \tag{3.17}$$

where $E(x,t)$ is defined by (2.23) with $k = 1$ which is related to Bessel heat equation. This completes the proof. □

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