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Research Article

An Approximation to Solution of Space and Time Fractional Telegraph Equations by He's Variational Iteration Method

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He's variational iteration method (VIM) is used for solving space and time fractional telegraph equations. Numerical examples are presented in this paper. The obtained results show that VIM is effective and convenient.

1. Introduction

In recent years, there has been a great deal of interest in fractional differential equations since there have been a wide variety of applications in physics and engineering. The space and time fractional telegraph equations have been studied by Orsingher and Zhao [1] and Orsingher and Beghin [2]. The telegraph equation is used in signal analysis for transmission and propagation of electrical signals and also used modeling reaction diffusion [3, 4]. In the papers by Momani [5] and Yildirim [6], Adomian decomposition method (ADM) and homotopy perturbation method (HPM) were used for solving the space and time fractional telegraph equations, respectively. Variational iteration method was used for solving linear telegraph equation in [7]. In this paper we will use variational iteration method (VIM) for solving the space and time fractional telegraph equations. The variational iteration method (VIM) which was developed in 1999 by He [8] has been applied to a wide variety of differential equations by many authors. He [9] used the variational iteration method (VIM) for solving seepage flow with fractional derivatives in porous media. Momani and Odibat [10] constructed numerical solutions of the space-time fractional advection dispersion equation by decomposition method and variational iteration method. Momani and Odibat [11] also compared homotopy perturbation method (HPM) and VIM for linear fractional partial differential equations. Drăgănescu [12] used VIM for viscoelastic models with fractional derivatives. Yulita at al. [13] applied the variational iteration method for fractional heat and wave-like equations. Dehghan at al. [14] studied telegraph and space telegraph equations using variational iteration method. In [14], space fractional telegraph equation was considered for $\alpha = 3/2$. However, in this paper space fractional telegraph equation has been considered for $1 < \alpha \leq 2$ and also variational iteration method has been applied for time fractional telegraph equation.

We note that the space and time fractional derivatives are considered in Caputo sense in this paper. The main objective of the present paper is to extend the application of the variational iteration method (VIM) to obtain approximate solution of the space and time fractional telegraph equations.

2. He's Variational Iteration Method

We will give a brief description of He's variational iteration method. The basic concepts of the variational iteration method can be expressed as follows. Consider the differential equation of the form

$$Lu(x,t) + Nu(x,t) = f(x,t), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and $f(x,t)$ is the inhomogeneous term. According to the variational iteration method, a correction functional for (2.1) can be constructed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)\left(Lu_n(x,s) + N\tilde{u}_n(x,s) - f(x,s)\right)ds, \quad n = 0,1,2,\ldots, \tag{2.2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory [15, 16], the subscript n denotes the nth approximations, and $\tilde{u}_n$ is considered as restricted variation [17, 18], that is, $\delta\tilde{u}_n = 0$. The successive approximations $u_{n+1}(x,t)$, $n = 0,1,2\ldots$, of the solution $u(x,t)$ can be obtained after finding the Lagrange multiplier and by using the selective function $u_0(x,t)$ which is usually selected from initial conditions.

3. Space and Time Fractional Derivatives in Caputo Sense

For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u}{\partial \tau^m}(x,\tau)d\tau, & m-1 < \alpha < m, \\ \dfrac{\partial^m u}{\partial t^m}(x,t), & \alpha = m \in \mathbf{N}, \end{cases} \tag{3.1}$$

and for m to be the smallest integer that exceeds β, the Caputo space-fractional derivative operator of order $\beta > 0$ is defined as

$$\frac{\partial^{\beta} u}{\partial x^{\beta}}(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^x (x-\xi)^{m-\beta-1} \frac{\partial^m u}{\partial \tau^m}(\xi,t)\,d\xi, & m-1 < \beta < m, \\[2ex] \dfrac{\partial^m u}{\partial x^m}(x,t), & \beta = m \in \mathbf{N}, \end{cases} \tag{3.2}$$

where Γ is the Gamma function.

Further information about fractional derivatives and its properties can be found in [19, 20].

4. Application to Space-Time Fractional Telegraph Equations

In this section we will obtain iteration formulas for space-time fractional telegraph equations. We first consider the following space fractional telegraph equation for $1 < \alpha \leq 2$:

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) + a\frac{\partial u}{\partial t}(x,t) + bu(x,t) + f(x,t), \quad 0 < x < 1, \; t > 0, \tag{4.1}$$

where a and b are given constants, $f(x,t)$ given function. The correctional functional for (4.1) can be approximately expressed as follows

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(s) \\ \times \left\{ \frac{\partial^m u_n}{\partial s^m}(s,t) - \frac{\partial^2 \tilde{u}_n}{\partial t^2}(s,t) - a\frac{\partial \tilde{u}_n}{\partial t}(s,t) - b\tilde{u}_n(s,t) - f(s,t) \right\} ds, \quad n = 0,1,\ldots. \end{aligned} \tag{4.2}$$

Making the correctional functional in (4.2) stationary, and noticing that $\delta \tilde{u}_n = 0$, we obtain the following stationary conditions for $m = 2$:

$$\begin{gathered} \lambda''(s) = 0, \\ 1 - \lambda'(s)\big|_{s=x} = 0, \qquad \lambda(s)|_{s=x} = 0. \end{gathered} \tag{4.3}$$

Lagrange multiplier can be identified from (4.3) as

$$\lambda(s) = s - x. \tag{4.4}$$

Substituting the above obtained Lagrange multiplier into (4.2), we get the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x (s-x) \times \left\{ \frac{\partial^m u_n}{\partial s^m}(s,t) - \frac{\partial^2 u_n}{\partial t^2}(s,t) - a\frac{\partial u_n}{\partial t}(s,t) - bu_n(s,t) - f(s,t) \right\} ds, \quad n = 0,1,\ldots. \tag{4.5}$$

Now consider the following time fractional telegraph equation for $0 < \alpha \leq 1$:

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}(x,t) + a\frac{\partial^\alpha u}{\partial t^\alpha}(x,t) = b\frac{\partial^2 u}{\partial x^2} + g(x,t), \quad 0 < x < 1,\ t > 0, \tag{4.6}$$

where a and b are given constants, and $g(x,t)$ given function.. The correctional functional for the equation (4.6) can be approximately expressed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s) \times \left\{ \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}(x,s) + a\frac{\partial^\alpha \tilde{u}}{\partial t^\alpha}(x,s) - b\frac{\partial^2 \tilde{u}}{\partial x^2} - g(x,s) \right\} ds, \quad n = 0,1,\ldots. \tag{4.7}$$

Making the correctional functional in (4.7) stationary, and noticing that $\delta\tilde{u}_n = 0$, we obtain the following stationary conditions for $m = 1$:

$$\lambda''(s) = 0, \qquad 1 - \lambda'(s)\big|_{s=t} = 0, \qquad \lambda(s)|_{s=t} = 0. \tag{4.8}$$

Lagrange multiplier can be identified from (4.8) as

$$\lambda(s) = s - t. \tag{4.9}$$

Substituting the above obtained Lagrange multiplier into (4.7), we get the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t) \times \left\{ \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}(x,s) + a\frac{\partial^\alpha u}{\partial t^\alpha}(x,s) - b\frac{\partial^2 u}{\partial x^2} - g(x,s) \right\} ds, \quad n = 0,1,\ldots. \tag{4.10}$$

5. Numerical Examples

We will give the following three examples to illustrate variational iteration method for solving the space and time fractional telegraph equations.

Example 5.1. We first consider the following one-dimensional initial and boundary value problem of space-fractional homogeneous telegraph equation for $1 < \alpha \leq 2$, (see [5, 21])

$$\frac{\partial^\alpha u}{\partial x^\alpha}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial u}{\partial t}(x,t) + u(x,t), \quad 0 < x < 1, \ t > 0, \tag{5.1}$$

$$u(x,0) = e^x, \quad 0 \leq x \leq 1, \tag{5.2}$$

$$u(0,t) = e^{-t}, \quad u_x(0,t) = e^{-t}, \quad t \geq 0. \tag{5.3}$$

It follows from (4.5) for $a = b = 1$ and $f(x,t) = 0$; the iteration formula for (5.1) can be written in the following form:

$$\begin{aligned} u_{n+1}(x,t) = u_n(x,t) + \int_0^x (s-x) \\ \times \left\{ \frac{\partial^\alpha u_n}{\partial s^\alpha}(s,t) - \frac{\partial^2 u_n}{\partial t^2}(s,t) - \frac{\partial u_n}{\partial t}(s,t) - u_n(s,t) \right\} ds, \quad n = 0, 1 \ldots. \end{aligned} \tag{5.4}$$

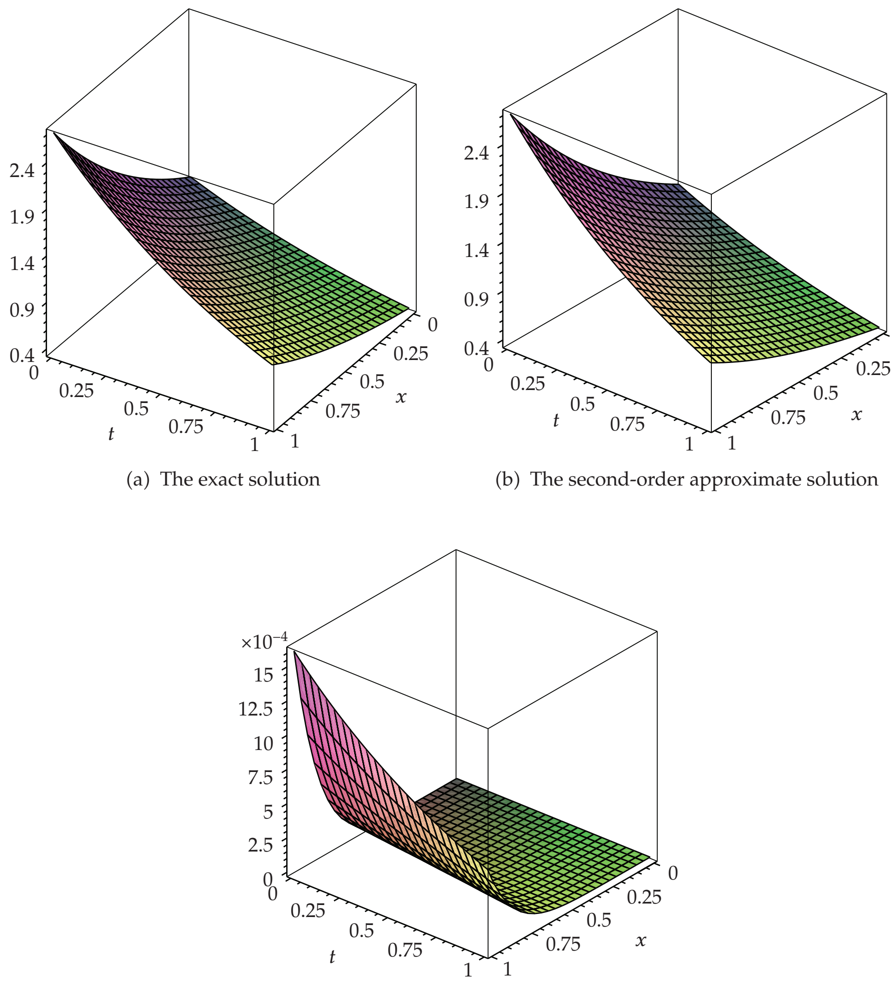
We start with initial approximation:

$$u_0(x,t) = u(0,t) + x u_x(0,t) = (1+x)e^{-t}, \tag{5.5}$$

and by the iteration formula (5.4) we obtain the first two approximations as

$$\begin{aligned} u_1(x,t) &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)e^{-t}, \\ u_2(x,t) &= \left(1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{x^{5-\alpha}}{\Gamma(6-\alpha)}\right)e^{-t}, \end{aligned} \tag{5.6}$$

and so on; in the same manner further approximations of the iteration formula (5.4) can be obtained by Mapple. We observe that, setting $\alpha = 2$ in the nth approximations yields the exact solution $u(x,t) = e^{x-t}$ as $n \to \infty$. In Figures 1(a), 1(b), and 1(c) exact and second-order approximate solutions of (5.1)–(5.3) are given. Figures 2(a) and 2(b) show the evolution



(a) The exact solution (b) The second-order approximate solution

(c) $|u_{ex} - u_{app}|$

Figure 1: The surfaces related with the solution of (5.1)–(5.3) for $\alpha = 2$.

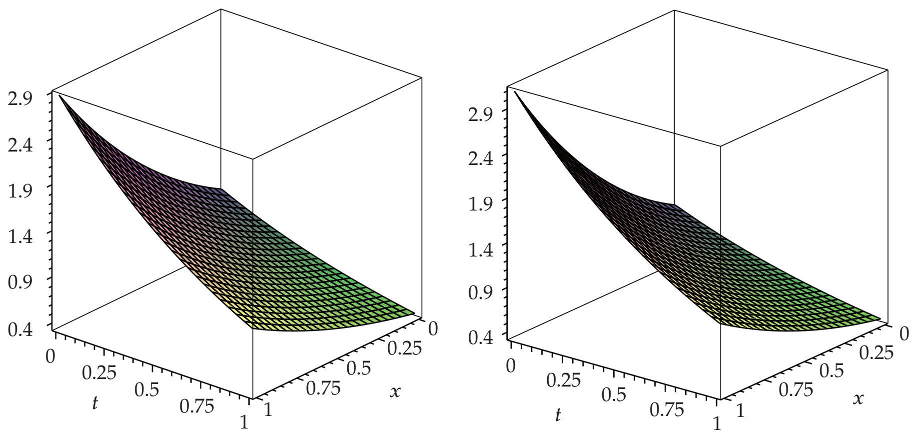
results for the second-order approximate solutions of (5.1)–(5.3) obtained for different values of α using the variational iteration method.

Example 5.2. We now consider the following one-dimensional initial and boundary value problem of space-fractional inhomogeneous telegraph equation for $0 < \alpha \leq 1$, (see, [5, 21]):

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial u}{\partial t}(x,t) + u(x,t) - x^2 - t + 1, \tag{5.7}$$

$$u(x,0) = x^2, \quad 0 \leq x \leq 1, \tag{5.8}$$

$$u(0,t) = t, \quad u_x(0,t) = 0, \quad t \geq 0. \tag{5.9}$$



(a) $\alpha = 1.25$

(b) $\alpha = 1.75$

Figure 2: The surfaces show the second-order approximate solutions of (5.1)–(5.3).

The iteration formula for (5.7) can be written in the following form:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x (s-x) \times \left\{ \frac{\partial^{2\alpha} u_n}{\partial s^{2\alpha}}(s,t) - \frac{\partial^2 u_n}{\partial t^2}(s,t) - \frac{\partial u_n}{\partial t}(s,t) + u_n(s,t) + s^2 + t - 1 \right\} ds, \quad n = 0, 1 \ldots. \tag{5.10}$$

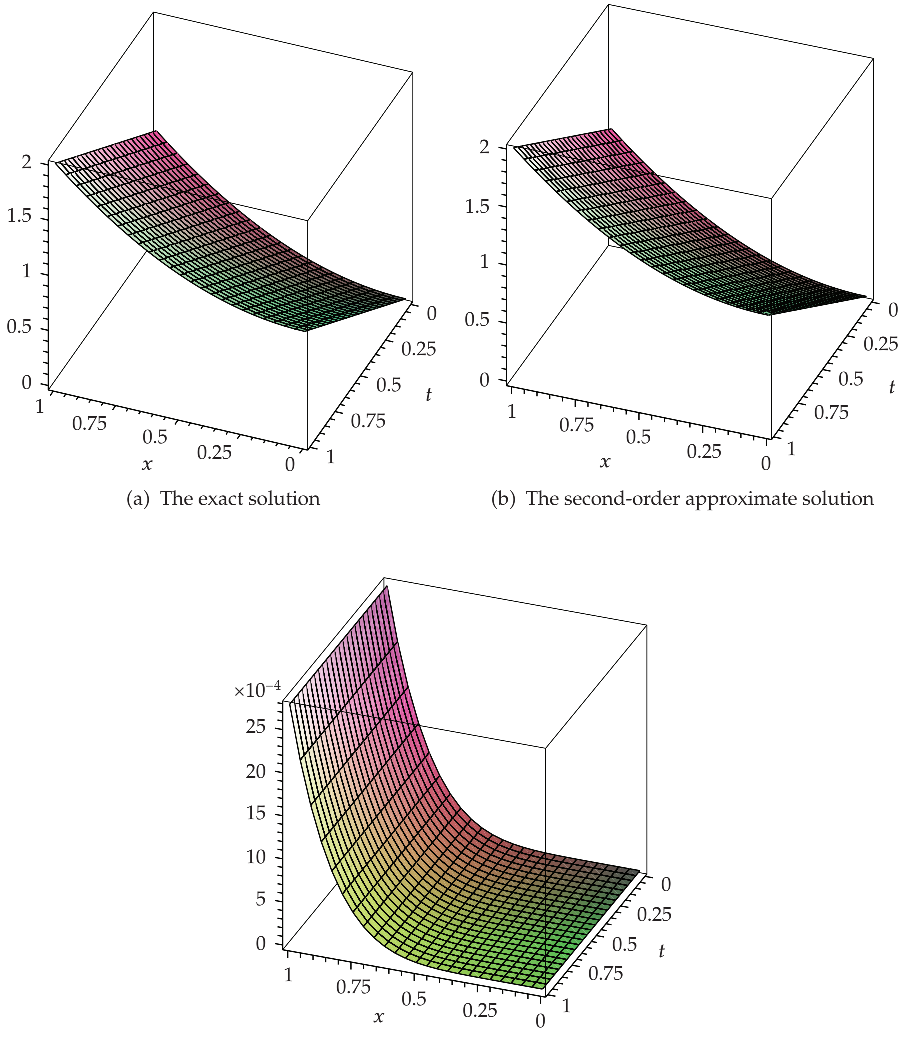
We start with initial approximation:

$$u_0(x,t) = u(0,t) + x u_x(0,t) = t, \tag{5.11}$$

and by iteration formula (5.10) we obtain the first two approximations as

$$\begin{aligned} u_1(x,t) &= \left(t + x^2 - \frac{x^4}{12} \right), \\ u_2(x,t) &= \left(t + 2x^2 - \frac{x^4}{12} - \frac{x^6}{300} - \frac{2x^{4-2\alpha}}{\Gamma(5-2\alpha)} + \frac{2x^{6-2\alpha}}{\Gamma(7-2\alpha)} \right), \end{aligned} \tag{5.12}$$

and so on; in the same manner further approximations of the iteration formula (5.10) can be obtained by Mapple. We observe that, setting $\alpha = 1$ in the nth approximations and canceling noise terms yields the exact solution $u(x,t) = x^2 + t$ as $n \to \infty$. In Figures 3(a), 3(b), and 3(c) exact and second-order approximate solutions of (5.7)–(5.9) are given. Figures 4(a) and 4(b) show the evolution results for the second-order approximate solutions of (5.7)–(5.9) obtained for different values of α using the variational iteration method.



(a) The exact solution

(b) The second-order approximate solution

(c) $|u_{\text{ex}} - u_{\text{app}}|$

Figure 3: The surfaces related with the solution of (5.7)–(5.9) for $\alpha = 1$.

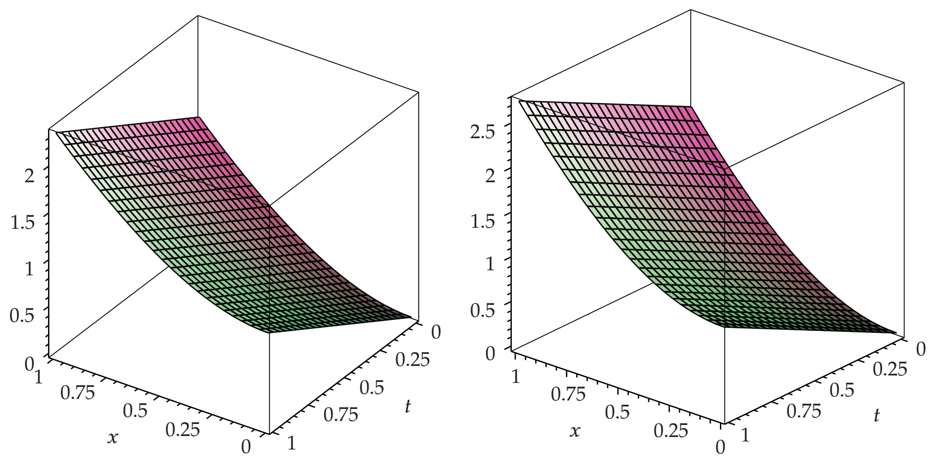
Example 5.3. We last consider the following initial and boundary value problem of time fractional telegraph equation of order $0 < \alpha \leq 1$, (see, [5, 21]):

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}(x,t) + \lambda \frac{\partial^{\alpha} u}{\partial t^{\alpha}}(x,t) = \nu \frac{\partial^2 u}{\partial x^2}(x,t), \quad t > 0, \tag{5.13}$$

$$u(x,0) = h_1(x), \tag{5.14}$$

$$u_t(x,0) = h_2(x), \tag{5.15}$$

$$u_x(0,t) = s(t). \tag{5.16}$$



(a) $\alpha = 0.25$ (b) $\alpha = 0.75$

Figure 4: The surfaces show the second-order approximate solutions of (5.7)–(5.9).

It follows from (4.10) for $a = \lambda,\ b = \nu$, and $g(x,t) = 0$; the iteration formula for (5.13) can be written in the following form:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t) \times \left\{ \frac{\partial^{2\alpha} u_n}{\partial s^{2\alpha}}(x,s) + \lambda \frac{\partial^{\alpha} u_n}{\partial s^{\alpha}}(x,s) - \nu \frac{\partial^2 u_n}{\partial x^2}(x,s) \right\} ds, \quad n = 0,1\ldots. \tag{5.17}$$

We start with the following initial approximation:

$$u_0(x,t) = u(x,0) + tu_t(x,0) = h_1(x) + th_2(x), \tag{5.18}$$

and by the iteration formula (5.17), we get

$$u_1(x,t) = h_1(x) + th_2(x) + \nu h_1''(x)\frac{t^2}{2!} + \nu h_2''(x)\frac{t^3}{3!} - \lambda h_2(x)\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} - h_2(x)\frac{x^{3-2\alpha}}{\Gamma(4-2\alpha)}, \tag{5.19}$$

and so on; in the same manner further approximations of the iteration formula (5.17) can be obtained by Mapple.

6. Conclusion

The variational iteration method has been successfully applied for finding the solution of space and time fractional telegraph equations. The space and time fractional derivatives are considered in the Caputo sense. We have achieved a very good agreement between the approximate solution obtained by He's VIM and the exact solution. The results of the examples show that He's variational iteration method is reliable and efficient method for solving space and time fractional telegraph equations and also other equations.

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