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Research Article

Adomian Decomposition Method for a Nonlinear Heat Equation with Temperature Dependent Thermal Properties

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The solutions of nonlinear heat equation with temperature dependent diffusivity are investigated using the modified Adomian decomposition method. Analysis of the method and examples are given to show that the Adomian series solution gives an excellent approximation to the exact solution. This accuracy can be increased by increasing the number of terms in the series expansion. The Adomian solutions are presented in some situations of interest.

1. Introduction

In the classical model of the heat equation, the thermal diffusivity and thermal conductivity of the medium are assumed to be constant. In some media such as gases, these parameters are proportional to the temperature of the medium giving rise to a nonlinear heat equation of the following form [1]:

$$C(x)\frac{\partial u}{\partial t} = \lambda\frac{\partial}{\partial x}\left(ku\frac{\partial u}{\partial x}\right), \tag{1.1}$$

where C is the conductivity, k is diffusivity, and λ is a constant.

However, in some situations the diffusivity is proportional to u^{α}, which gives rise to a more general nonlinear heat equation

$$C(x)\frac{\partial u}{\partial t} = \lambda \frac{\partial}{\partial x}\left(u^{\alpha}\frac{\partial u}{\partial x}\right). \tag{1.2}$$

In this paper we investigate the nonlinear heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial u}{\partial x}\right), \tag{1.3}$$

with $f(u) = u^m$, using the Adomian decomposition method. This method was presented by Adomian to solve algebraic, differential, integrodifferential equations and stochastic problems [2–5]. In these papers Adomian presented the so-called decomposition method in which the problem is split into linear (solvable) and nonlinear part. By assuming that the solution admits a power series representation, the nonlinear contribution to the solution is obtained in the form of "Adomian polynomials" [6]. Alternative methods of calculating Adomian polynomials have been discussed by Babolian and Javadi [7] and Wazwaz [8–11]. For the convergence of the Adomian method, see [12–14]. For a detailed treatment and applications of the Adomian decomposition method one may refer to [6]. Chiu and Chen [15] have applied the Adomian method to study fin problem with variable conductivity. Wazwaz in [10] established an algorithm for calculating Adomian polynomials that depend mainly on algebraic and trigonometric identities and on Taylor's expansion. A feature of this method is that it involves less formulas and is straightforward to implement. The reader is referred to [10, Section 2] for details of algorithm and its connection with earlier approach of Adomian [6]. We will use the modified Adomian algorithm given by Wazwaz [10] to find the Adomian solutions to our models of nonlinear heat equation with temperature dependent diffusivity.

2. Method of Solution

Introducing the operator $L_t = \partial/\partial t$, (1.3) takes the form

$$L_t u(x,t) = \left[f'(u)u_x^2 + f(u)u_{xx}\right]. \tag{2.1}$$

We solve (2.1) subject to the initial condition

$$u(x,0) = g(x). \tag{2.2}$$

Applying inverse operator L_t^{-1} to both sides of (2.1) yields

$$u(x,t) = u(x,0) + L_t^{-1}\left[\left(f'(u)u_x^2 + f(u)u_{xx}\right)\right]. \tag{2.3}$$

The desired series solution by Adomian decomposition method is given by (cf. [2–6] for details)

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{2.4}$$

and $u_1, u_2, u_3, \ldots$ are calculated from recursive relation

$$\begin{aligned} u_0 &= u(x,0), \\ u_{n+1} &= L_t^{-1}[(A_n)], \quad n \geq 0, \end{aligned} \tag{2.5}$$

where A_n are the Adomian polynomials for the nonlinear operator

$$F(u(x,t)) = f'(u)u_x^2 + f(u)u_{xx}. \tag{2.6}$$

The formulas that can be used to generate Adomian polynomials are discussed by Adomian in [6]. Here we employ the algorithm of Wazwaz [10] to calculate Adomian polynomials, which seems quite natural and suited for implementation by software.

3. Applications and Results

We consider the nonlinear heat equation

$$\begin{gathered} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial u}{\partial x}\right), \\ u(x,0) = g(x) \end{gathered} \tag{3.1}$$

with power nonlinearity $f(u) = u^m$. We are interested in investigating the case of power nonlinearity due to the fact that this assumption is made in most of the applied nonlinear problems of heat transfer and flows in porous media. For instance, $f(u) = u^{-1/2}$ corresponds to fast diffusion processes of plasma diffusion and thermal expulsion of liquid Helium [16–18]. The diffusivity $f(u) = u^2$ is used to model process of melting and evaporation of metals [17–19]. For the initial temperature profile, we consider typical cases like $g(x)$ a quadratic function or $g(x) = e^{-ax^2}$ or $g(x) = \operatorname{sech}^2 x$ which corresponds to soliton like initial profile.

Case A ($g(x) = ax^2 + bx + c$). The Adomian solution $u(x,t)$ for general a, b, c, and m can be obtained from authors as Mathematica file. Some particular cases for a, b, c, and m are considered as follows.

(i) $a = b = c = 1$ and $m = 2$.

The Mathematica code to obtain Adomina solution in this case consists of the following commands:

$$f[n\_] = \sum_{i=0}^{n-1} u_i[x]\alpha^i + O[\alpha]^n,$$
$$f1d[n\_] = \sum_{i=0}^{n-1} \partial_x u_i[x]\alpha^i + O[\alpha]^n, \tag{3.2}$$
$$f2d[n\_] = \sum_{i=0}^{n-1} \partial_{x,x} u_i[x]\alpha^i + O[\alpha]^n,$$

maximum number of polynomials and solution terms:

$$k = 5 \tag{3.3}$$

Finding Adomian polynomials:

$$\begin{aligned} &\text{apoly} = m f[k]^{m-1} f1d[k]^2 //\text{Simplify} \\ &\text{bpolynomial} = f[k]^m\, f2d[k]. \end{aligned} \tag{3.4}$$

Making vector of admian polynomials:

$$v = \text{CoefficientList}\left[\text{coeffpoly}, \alpha\right]. \tag{3.5}$$

Finding solution *u(x,t)*:

$$\begin{gathered} u_0[x\_] = a * x^2 + b * x + c \\ \text{Do}\left[u_i[x\_] = \int_0^t v[[i]]dt, \{i, 1, k\}\right] \\ u[x\_, t\_] = u_0[x] \\ \text{Do}[u[x\_, t\_] = u[x, t] + u_i[x], \{i, 1, k\}] \\ u[x, t] \\ a = 1; b = 1; c = 1 \\ m = 2 \\ u[x, t]. \end{gathered} \tag{3.6}$$

The Adomian solution obtained is

$$
\begin{aligned}
&u(x,t)\\
&\quad = 1 + x + x^2 + 2t^2(1 + x(1 + x))\\
&\quad \times \left(24 + 45x + 185x^2 + 4x\left(25 + 70x^2\right) + 4\left(2 + 25x^2 + 35x^4\right)\right) + \frac{1}{3}t^3(1 + x(1 + x))\\
&\quad \times \left(60 + 1860x + 8x\left(1860 + 7995x^2\right) + 2\left(570 + 8370x^2\right)\right.\\
&\qquad + 8x\left(1665 + 10770x^2 + 12825x^4\right) + 4\left(720 + 14490x^2 + 29370x^4\right)\\
&\qquad \left.+8\left(75 + 1665x^2 + 5385x^4 + 4275x^6\right)\right) + \frac{1}{3}t^4(1 + x(1 + x))\\
&\quad \times \left(8160 + 11310x + 180390x^2 + 4x\left(50610 + 318390x^2\right)\right.\\
&\qquad + 4\left(13380 + 414000x^2 + 1172820x^4\right) + 4x\left(160950 + 1507380x^2 + 2417550x^4\right)\\
&\qquad + 16x\left(20370 + 222060x^2 + 585450x^4 + 429000x^6\right)\\
&\qquad + 4\left(17970 + 605070x^2 + 2705190x^4 + 2807850x^6\right)\\
&\qquad \left.+16\left(600 + 20370x^2 + 111030x^4 + 195150x^6 + 107250x^8\right)\right) + \frac{1}{60}t^5(1 + x(1 + x))\\
&\quad \times \left(15120 + 1275120x + 2\left(486360 + 16082040x^2\right)\right.\\
&\qquad + 8x\left(5172000 + 45672540x^2\right) + 4\left(3226560 + 145378920x^2 + 558401640x^4\right)\\
&\qquad + 8x\left(36485460 + 477387720x^2 + 1007253300x^4\right)\\
&\qquad + 64x\left(8126460 + 123096840x^2 + 424022700x^4 + 390053400x^6\right)\\
&\qquad + 8\left(5708700 + 279944820x^2 + 1685856660x^4 + 2244119100x^6\right)\\
&\qquad + 32x\left(5402520 + 86621760x^2 + 373392000x^4 + 591280800x^6 + 309309000x^8\right)\\
&\qquad + 16\left(2417760 + 119127600x^2 + 868513680x^4 + 1945018800x^6 + 1317980400x^8\right)\\
&\qquad \left.+32\left(113400 + 5402520x^2 + 43310880x^4 + 124464000x^6 + 147820200x^8 + 61861800x^{10}\right)\right)\\
&\quad + t(1 + x(1 + x))(2 + 2(1 + 5x(1 + x))).
\end{aligned}
\tag{3.7}
$$

The solutions in Figure 1 increase algebraically as is expected from algebraic behavior of initial condition and the form of $f(u)$.

(ii) $a = b = c = 1$ and $m = -2$ (Figure 2).

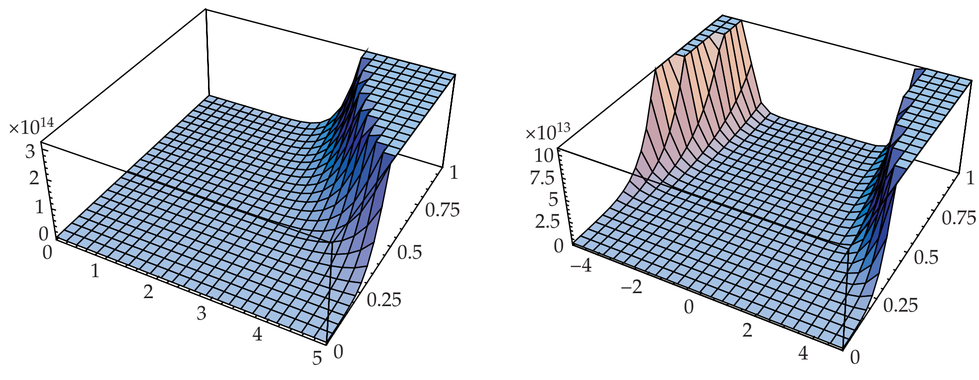


Figure 1: (a) Graph of Adomian solution for $\{x, 0, 5\}, \{t, 0, 1\}$. (b) Graph Adomian solution for $\{x, -5, 5\}, \{t, 0, 1\}$.

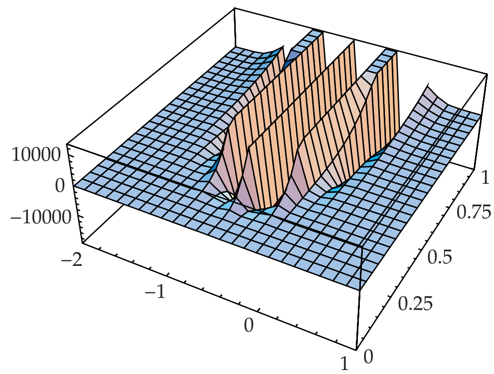


Figure 2: Graph of Adomian solution for the range $\{x, -2, 1\}, \{t, 0, 1\}$.

As the diffusivity in this case is decreasing function of u, the solution exhibits the change in the quadratically increasing initial temperature.

(iii) $a = b = c = 1$ and $m = 1/2$ (Figure 3).

Case B ($g(x) = e^{-ax^2}$). The Adomian solution for general a, m can be obtained from authors as Mathematica file. Some particular cases are considered as follows.

(i) $a = 2$ and $m = 2$

The Adomian solution is

$$\begin{aligned} u(x,t) = {} & \mathrm{e}^{-2x^2} + 4\mathrm{e}^{-6x^2} t\left(-1 + 12x^2\right) + 8\mathrm{e}^{-10x^2} t^2\left(11 - 400x^2 + 1200x^4\right) \\ & + \frac{32}{3}\mathrm{e}^{-14x^2} t^3\left(-315 + 22692x^2 - 181552x^4 + 291648x^6\right) \\ & + \frac{32}{3}\mathrm{e}^{-18x^2} t^4\left(16425 - 1947360x^2 + 28962720x^4 - 115402752x^6 + 123607296x^8\right) \\ & + \frac{128}{15}\mathrm{e}^{-22x^2} t^5\left(-1326840 + 233242200x^2 - 5491343520x^4 + 38961513344x^6\right. \\ & \left. -99063148800x^8 + 78562446336x^{10} + 945\left(-1 + 20x^2\right)\right). \end{aligned} \tag{3.8}$$

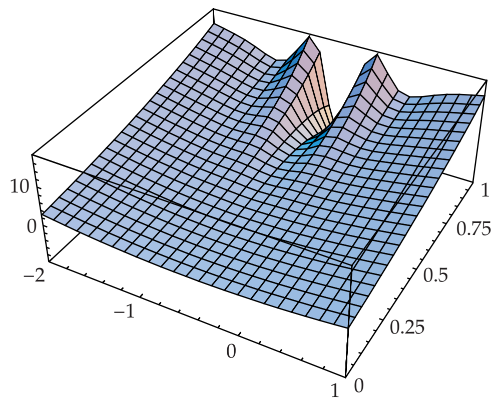


Figure 3: Graph of Adomian solution for the range $\{x, -2, 1\}, \{t, 0, 1\}$.

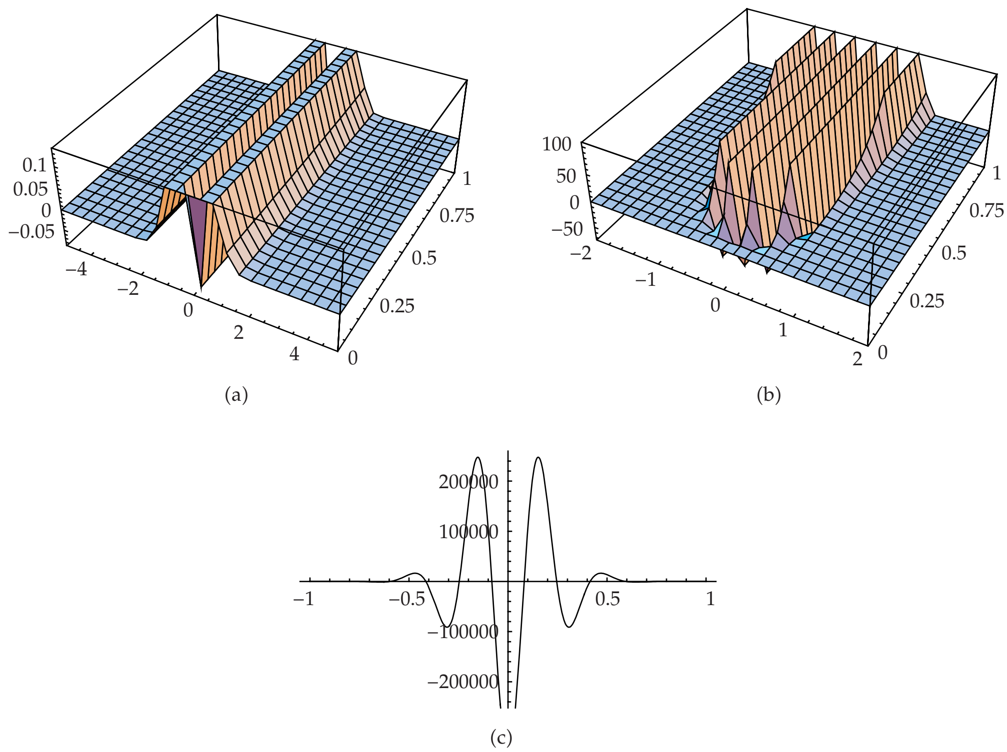


Figure 4: (a) Graph of solution for the range $\{x, -5, 5\}, \{t, 0, 1\}$. (b) Graph of solution for the range $\{x, -2, 2\}, \{t, 0, 1\}$. (c) Graph for fixed $t = 0.5$ for the range $\{x, -1, 1\}$.

Figure 4 displays how the bell-shaped initial temperature interacts with quadratic dependence of diffusivity.

(ii) $a = 2$ and $m = -2$ (Figure 5).

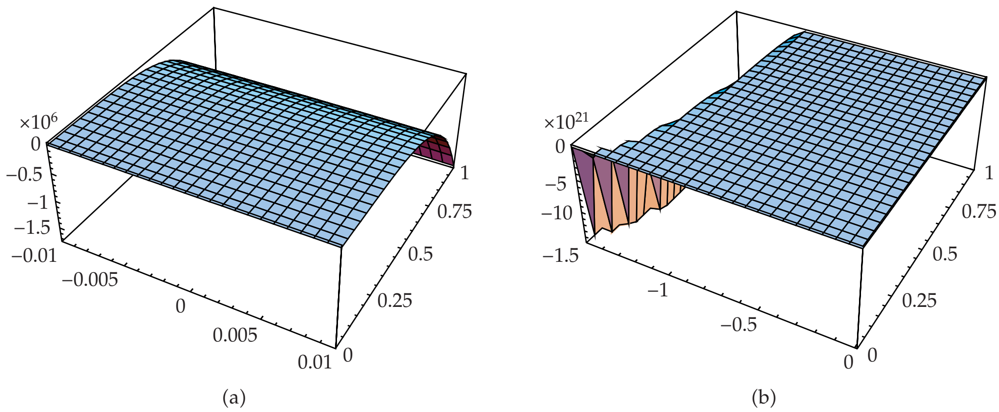


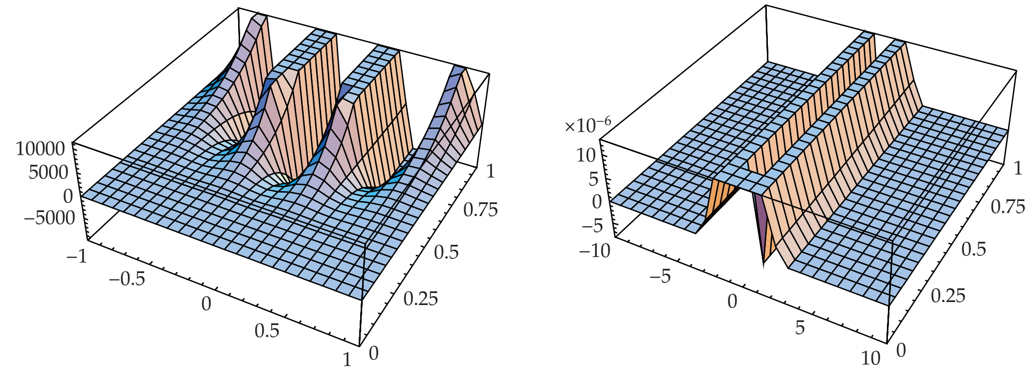
Figure 5: (a) Graph for the range $\{x, -.01, .01\}, \{t, 0, 1\}$. (b) Graph for the range $\{x, -1.5, .01\}, \{t, 0, 1\}$.

The Adomian solution is

$$\begin{aligned}
u(x,t) = {} & e^{-2x^2} + 4e^{2x^2}t\left(-1-4x^2\right) + 8e^{6x^2}t^2\left(-5-96x^2-144x^4\right) \\
& + \frac{32}{3}e^{10x^2}t^3\left(-91-4028x^2-19120x^4-17600x^6\right) \\
& + \frac{32}{3}e^{14x^2}t^4\left(-3287-260480x^2-2523104x^4-6375936x^6-4202240x^8\right) \\
& + \frac{128}{15}e^{18x^2}t^5\left(-191704-23954712x^2-390296736x^4-1877037696x^6\right. \\
& \qquad \left. -3158528256x^8-1613177856x^{10}+945\left(-1+20x^2\right)\right).
\end{aligned} \tag{3.9}$$

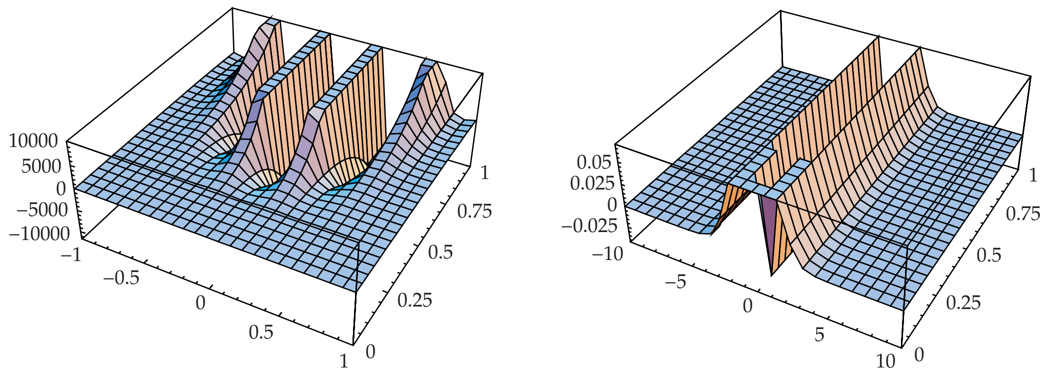
(iii) $a = 2$ and $m = 1/2$ (Figure 6).
The Adomian solution is

$$\begin{aligned}
u(x,t) = {} & e^{-2x^2} + 4\left(e^{-2x^2}\right)^{3/2}t\left(-1+6x^2\right) + 8e^{-4x^2}t^2\left(5-76x^2+96x^4\right) \\
& + \frac{32}{3}\left(e^{-2x^2}\right)^{5/2}t^3\left(-\frac{219}{4}+\frac{3009x^2}{2}-4615x^4+2850x^6\right) \\
& + \frac{32}{3}e^{-6x^2}t^4\left(\frac{2031}{2}-43365x^2+233406x^4-337716x^6+131760x^8\right) \\
& + \frac{128}{15}\left(e^{-2x^2}\right)^{7/2}t^5\left(-\frac{433389}{16}+\frac{13476579x^2}{8}-\frac{27882147x^4}{2}+34527269x^6\right. \\
& \qquad \left. -30761241x^8+8579214x^{10}+945\left(-1+20x^2\right)\right).
\end{aligned} \tag{3.10}$$



(a) (b)

Figure 6: (a) Graph for the range $\{x, -1, 1\}, \{t, 0, 1\}$. (b) Graph for the range $\{x, -10, 10\}, \{t, 0, 1\}$.



(a) (b)

Figure 7: (a) Graph for the range $\{x, -1, 1\}, \{t, 0, 1\}$. (b) Graph for the range $\{x, -10, 10\}, \{t, 0, 1\}$.

Case C ($g(x) = \operatorname{sech}^2 x$). The Adomian solution for general m can be obtained from authors as Mathematica file. Some particular cases are considered as follows.

(i) $m = 2$.

The Adomian solution is

$$\begin{aligned}
u(x,t) &= \operatorname{sech}(x)^2 + 2t(-4 + 3\cosh(2x))\operatorname{sech}(x)^8 \\
&\quad + 3t^2(161 - 178\cosh(2x) + 25\cosh(4x))\operatorname{sech}(x)^{14} \\
&\quad + t^3(-54900 + 71641\cosh(2x) - 18772\cosh(4x) + 1519\cosh(6x)) \\
&\quad \times \operatorname{sech}(x)^{20} + \frac{1}{4}t^4(35318621 - 50550350\cosh(2x) + 18047504\cosh(4x) \\
&\qquad\qquad -2916178\cosh(6x) + 160947\cosh(8x)) \\
&\quad \times \operatorname{sech}(x)^{26} + \frac{1}{20}t^5(-35893153056 + 54495231330\cosh(2x) - 23506173696\cosh(4x) \\
&\qquad\qquad + 5488700877\cosh(6x) - 621401568\cosh(8x) \\
&\qquad\qquad + 25573713\cosh(10x))\operatorname{sech}(x)^{32}.
\end{aligned} \tag{3.11}$$

Here the initial condition is soliton like. This is reflected in the Figure 7 as the diffusivity varies quadratically.

(ii) $m = -2$ (Figure 8).

The Adomian solution is

$$
\begin{aligned}
u(x,t) = {} & -2t\cosh(2x) - t^2\cosh(x)^2(1 - 2\cosh(2x) + 9\cosh(4x)) \\
& - \frac{1}{3}t^3\cosh(x)^4(-44 + 85\cosh(2x) - 76\cosh(4x) + 275\cosh(6x)) \\
& - \frac{1}{12}t^4\cosh(x)^6(2865 - 5862\cosh(2x) + 5968\cosh(4x) - 5178\cosh(6x) \\
& \qquad +16415\cosh(8x)) - \frac{1}{60}t^5\cosh(x)^8 \\
& \times (-303864 + 606738\cosh(2x) - 616768\cosh(4x) + 638373\cosh(6x) \\
& \quad -544328\cosh(8x) + 1575369\cosh(10x)) + \operatorname{sech}(x)^2.
\end{aligned}
\tag{3.12}
$$

(iii) $m = 1/2$ (Figure 9).

The Adomian solution is

$$
\begin{aligned}
u(x,t) = {} & \operatorname{sech}(x)^2 + \frac{3}{2}t^2(56 - 52\cosh(2x) + 4\cosh(4x))\operatorname{sech}(x)^8 + \frac{1}{8}t^4 \\
& \times \left(\frac{5889415}{8} - \frac{3750383}{4}\cosh(2x) + 232028\cosh(4x)\right. \\
& \qquad \left. -\frac{76417}{4}\cosh(6x) + \frac{2745}{8}\cosh(8x)\right) \\
& \times \operatorname{sech}(x)^{14} + 2t\left(-\frac{5}{2} + \frac{3}{2}\cosh(2x)\right)\operatorname{sech}(x)^4\sqrt{\operatorname{sech}(x)^2} \\
& + \frac{1}{2}t^3\left(-\frac{37917}{8} + \frac{86005}{16}\cosh(2x) - \frac{7163}{8}\cosh(4x) + \frac{475}{16}\cosh(6x)\right) \\
& \times \operatorname{sech}(x)^8\left(\operatorname{sech}(x)^2\right)^{3/2} + \frac{1}{40}t^5 \\
& \times \left(-\frac{22986251157}{128} + \frac{31585649589}{128}\cosh(2x) - \frac{2501116101}{32}\cosh(4x)\right. \\
& \quad \left. +\frac{2695647273}{256}\cosh(6x) - \frac{64605399}{128}\cosh(8x) + \frac{1429869}{256}\cosh(10x)\right) \\
& \times \operatorname{sech}(x)^{12}\left(\operatorname{sech}(x)^2\right)^{5/2}.
\end{aligned}
\tag{3.13}
$$

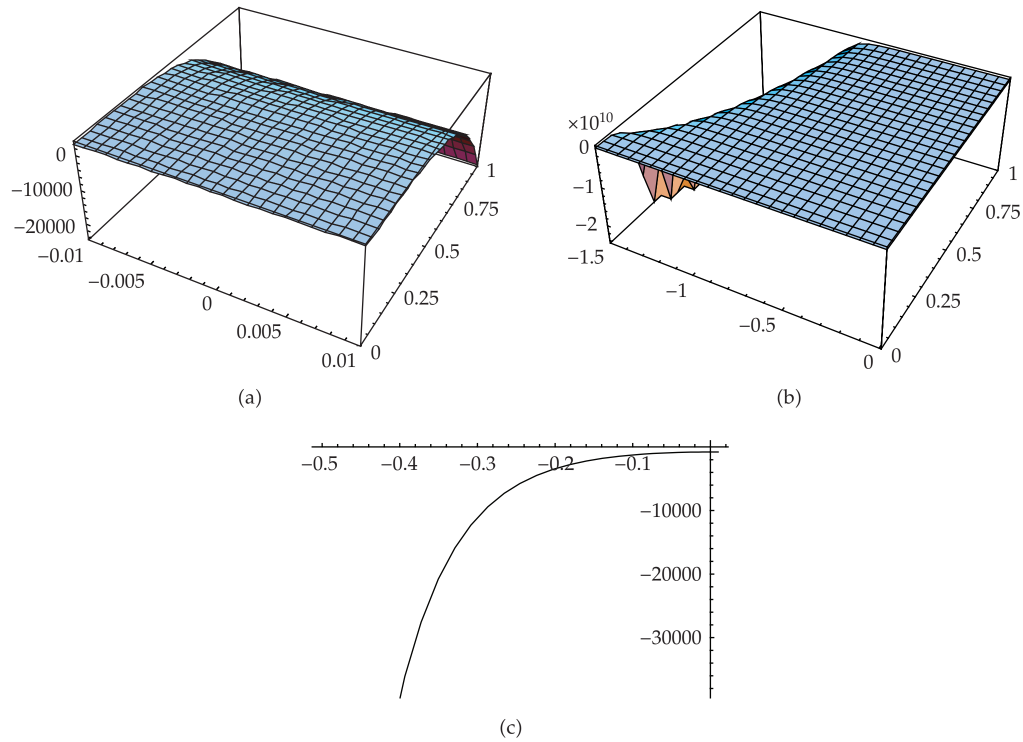


Figure 8: (a) Graph for the range $\{x, -.01, .01\}, \{t, 0, 1\}$. (b) Graph for the range $\{x, -1.5, .01\}, \{t, 0, 1\}$. (c) Graph for fixed $t = 0.5$ for the range $\{x, -0.5, 0.01\}$.

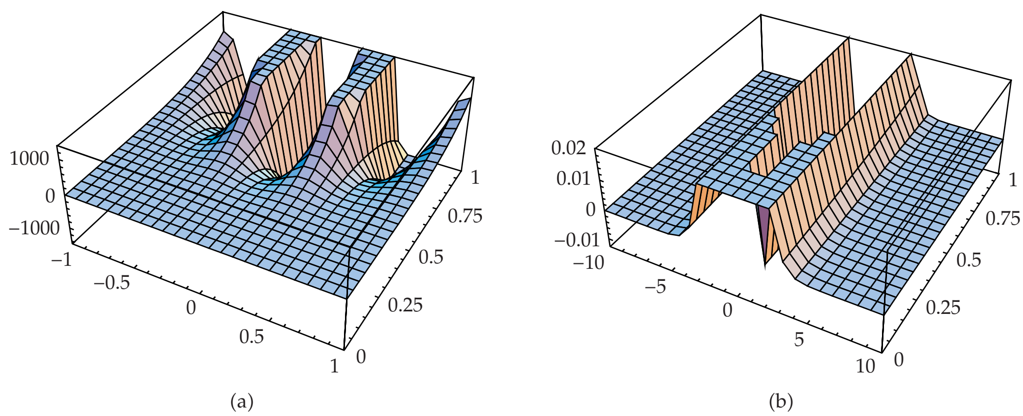


Figure 9: (a) Graph for the range $\{x, -1, 1\}, \{t, 0, 1\}$. (b) Graph for the range $\{x, -10, 10\}, \{t, 0, 1\}$.

4. Conclusion

The Adomian decomposition method has been applied to obtain solutions of the heat equation with power nonlinearity in the diffusivity. The solutions are presented for some typical initial temperature profiles like a quadratic function or or e^{-ax^2} or $\operatorname{sech}^2 x$. The interaction of the initial temperature with diffusivity is also discussed for different cases of solutions investigated here.

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