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Research Article

On the Numerical Solution of Fractional Hyperbolic Partial Differential Equations

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The stable difference scheme for the numerical solution of the mixed problem for the multidimensional fractional hyperbolic equation is presented. Stability estimates for the solution of this difference scheme and for the first and second orders difference derivatives are obtained. A procedure of modified Gauss elimination method is used for solving this difference scheme in the case of one-dimensional fractional hyperbolic partial differential equations.

1. Introduction

It is known that various problems in fluid mechanics (dynamics, elasticity) and other areas of physics lead to fractional partial differential equations. Methods of solutions of problems for fractional differential equations have been studied extensively by many researchers (see, e.g., [1–11] and the references given therein).

The role played by stability inequalities (well posedness) in the study of boundary-value problems for hyperbolic partial differential equations is well known (see, e.g., [12–25]). In the present paper, the mixed boundary value problem for the multidimensional fractional hyperbolic equation

$$\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{m} \left(a_r(x) u_{x_r}\right)_{x_r} + D_t^{1/2} u(t,x) = f(t,x), \\
x = (x_1, \dots, x_m) \in \Omega, \quad 0 < t < 1, \\
u(0,x) = 0, \quad u_t(0,x) = 0, \quad x \in \overline{\Omega}, \\
u(t,x) = 0, \quad x \in S
\end{gathered} \tag{1.1}$$

is considered. Here $D_t^{1/2} = D_{0+}^{1/2}$ is the standard Riemann-Lioville's derivative of order 1/2 and Ω is the unit open cube in the m-dimensional Euclidean space $\mathbb{R}^m : \{\Omega = x = (x_1, \ldots, x_m) : 0 < x_j < 1,\ 1 \leq j \leq m\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$, $a_r(x)$, $(x \in \Omega)$ and $f(t,x)$ $(t \in (0,1),\ x \in \Omega)$ are given smooth functions and $a_r(x) \geq a > 0$.

The first order of accuracy in t and the second order of accuracy in space variables for the approximate solution of problem (1.1) are presented. The stability estimates for the solution of this difference scheme and its first and second ordes difference derivatives are established. A procedure of modified Gauss elimination method is used for solving this difference scheme in the case of one-dimensional fractional hyperbolic partial differential equations.

2. The Difference Scheme and Stability Estimates

The discretization of problem (1.1) is carried out in two steps. In the first step, let us define the grid space

$$\begin{gathered}\widetilde{\Omega}_h = \{x = x_r = (h_1 r_1, \ldots, h_m r_m),\ r = (r_1, \ldots, r_m),\ 0 \leq r_j \leq N_j,\ h_j N_j = 1,\ j = 1, \ldots, m\}, \\ \Omega_h = \widetilde{\Omega}_h \cap \Omega, \quad S_h = \widetilde{\Omega}_h \cap S.\end{gathered} \tag{2.1}$$

We introduce the Banach space $L_{2h} = L_2(\widetilde{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 r_1, \ldots, h_m r_m)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_2(\widetilde{\Omega}_h)} = \left(\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_m\right)^{1/2}. \tag{2.2}$$

To the differential operator A^x generated by problem (1.1), we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{m} \left(a_r(x) u_{\overline{x}_r}^h\right)_{x_r, j_r} \tag{2.3}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\widetilde{\Omega}_h)$. With the help of A_h^x we arrive at the initial boundary value problem

$$\begin{gathered}\frac{d^2 v^h(t,x)}{dt^2} + A_h^x v^h(t,x) + D_t^{1/2} v^h(t,x) = f^h(t,x), \quad 0 \leq t \leq 1,\ x \in \Omega_h, \\ v^h(0,x) = 0, \quad \frac{d v^h(0,x)}{dt} = 0, \quad x \in \widetilde{\Omega}\end{gathered} \tag{2.4}$$

for an finite system of ordinary fractional differential equations.

In the second step, applying the first order of approximation formula $(1/\sqrt{\pi})\sum_{m=1}^{k}(\Gamma(k-m+(1/2))/(k-m)!)((u(t_k)-u(t_{k-1}))/\tau^{1/2})$ for $D_t^{1/2}u(t)$ (see [10]) and using the first order of accuracy stable difference scheme for hyperbolic equations (see [25]), one can present the first order of acuraccy difference scheme

$$\begin{gathered}
\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_{k+1}^h+\frac{1}{\sqrt{\pi}}\sum_{m=1}^{k}\frac{\Gamma(k-m+1/2)}{(k-m)!}\frac{\left(u_m^h-u_{m-1}^h\right)}{\tau^{1/2}}=f_k^h(x), \quad x\in\widetilde{\Omega}_h, \\
f_k^h(x)=f(t_k,x_n),\ t_k=k\tau,\ 1\le k\le N-1,\ N\tau=1, \\
\frac{u_1^h(x)-u_0^h(x)}{\tau}=0,\ u_0^h(x)=0,\ x\in\widetilde{\Omega}_h
\end{gathered} \tag{2.5}$$

for the approximate solution of problem (2.4). Here $\Gamma(k-m+1/2)=\int_0^\infty t^{k-m-1/2}e^{-t}dt$.

Theorem 2.1. *Let τ and $|h|$ be sufficiently small numbers. Then, the solutions of difference scheme* (2.5) *satisfy the following stability estimates:*

$$\begin{gathered}
\max_{1\le k\le N}\left\|u_k^h\right\|_{L_{2h}}+\max_{1\le k\le N}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\le C_1\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}, \\
\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{1\le k\le N}\sum_{r=1}^{m}\left\|\left(u_k^h\right)_{\bar{x}_r x_r, j_r}\right\|_{L_{2h}} \\
\le C_2\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right].
\end{gathered} \tag{2.6}$$

Here C_1 and C_2 do not depend on τ, h, and $f_k^h, 1\le k<N-1$.

The proof of Theorem 2.1 is based on the self-adjointness and positive definitness of operator A_h^x in L_{2h} and on the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 2.2. *For the solutions of the elliptic difference problem*

$$\begin{gathered}
A_h^x u^h(x)=\omega^h(x), \quad x\in\Omega_h, \\
u^h(x)=0, \quad x\in S_h,
\end{gathered} \tag{2.7}$$

the following coercivity inequality holds [26]:

$$\sum_{r=1}^{m}\left\|u_{x_r\bar{x}_r,j_r}^h\right\|_{L_{2h}}\le C\left\|\omega^h\right\|_{L_{2h}}. \tag{2.8}$$

Remark 2.3. The stability estimates of Theorem 2.1 are satisfied in the case of operator

$$Au = -\sum_{k=1}^{n} a_k(x)\frac{\partial^2 u}{\partial x_k^2} + \sum_{k=1}^{n} b_k(x)\frac{\partial u}{\partial x_k} + c(x)u \tag{2.9}$$

with Dirichlet condition $u = 0$ in S and $D_t^{\alpha} = D_{0+}^{\alpha}$ is the standard Riemann-Lioville's derivative of order α, $0 \le \alpha < 1$. In this case, A is not self-adjoint operator in H. Nevertheless, $Au = A_0 u + Bu$ and A_0 is a self-adjoint positive definite operator in H and BA_0^{-1} is bounded in H. The proof of this statement is based on the abstract results of [25] and difference analogy of integral inequality.

Remark 2.4. The stability estimates of Theorem 2.1 permit us to obtain the estimate of convergence of difference scheme of the first order of accuracy for approximate solutions of the initial-boundary value problem

$$\begin{gathered}
\frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{r=1}^{n} a_r(x) u_{x_r x_r} + \sum_{r=1}^{n} b_r(x) u_{x_r} + D_t^{\alpha} u(t,x) = f(t,x; u(t,x), u_t(t,x), u_{x_1}(t,x), \dots, u_{x_n}(t,x)),\\
x = (x_1, \dots, x_n) \in \Omega, \quad 0 < t < 1,\\
u(0,x) = 0, \quad \frac{\partial u(0,x)}{\partial t} = 0, \quad x \in \overline{\Omega},\\
u(t,x) = 0, \quad x \in S
\end{gathered} \tag{2.10}$$

for semilinear fractional hyperbolic partial differential equations.

Note that, one has not been able to obtain a sharp estimate for the constants figuring in the stability estimates of Theorem 2.1. Therefore, our interest in the present paper is studying the difference scheme (2.5) by numerical experiments. Applying this difference scheme, the numerical methods are proposed in the following section for solving the one-dimensional fractional hyperbolic partial differential equation. The method is illustrated by numerical experiments.

3. Numerical Results

For the numerical result, the mixed problem

$$\begin{gathered}
D_t^2 u(t,x) - u_{xx}(t,x) + D_t^{1/2} u(t,x) = f(t,x),\\
f(t,x) = \left(2 - \frac{8t^{3/2}}{3\sqrt{\pi}} + (\pi t)^2\right)\sin \pi x, \quad 0 < t, \ x < 1,\\
u(0,x) = 0, \quad u_t(0,x) = 0, \quad 0 \le x \le 1,\\
u(t,0) = u(t,1) = 0, \quad 0 \le t \le 1
\end{gathered} \tag{3.1}$$

for the one-dimensional fractional hyperbolic partial differential equation is considered. Applying difference scheme (2.5), we obtain

$$\begin{gathered}
\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+\frac{1}{\sqrt{\Pi}}\sum_{m=1}^{k}\frac{\Gamma(k-m+1/2)}{(k-m)!}\left(\frac{u_n^m-u_n^{m-1}}{\tau^{1/2}}\right)=\varphi_n^k,\\
\varphi_n^k=f(t_k,x_n),\quad 1\le k\le N-1,\ 1\le n\le M-1,\\
u_n^0=0,\quad \tau^{-1}\left(u_n^1-u_n^0\right)=0,\ 0\le n\le M,\\
u_0^k=u_M^k=0,\quad 0\le k\le N.
\end{gathered}\tag{3.2}$$

We get the system of equations in the matrix form

$$\begin{gathered}
AU_{n+1}+BU_n+CU_{n-1}=D\varphi_n,\quad 1\le n\le M-1,\\
U_0=\widetilde{0},\quad U_M=\widetilde{0},
\end{gathered}\tag{3.3}$$

where

$$\widetilde{0}=\begin{bmatrix}0\\0\\0\\0\\\cdots\\0\\0\end{bmatrix}_{(N+1)\times(1)},$$

$$A=\begin{bmatrix}0&0&0&0&\cdots&0&0\\0&0&0&0&\cdots&0&0\\0&0&a&0&\cdots&0&0\\0&0&0&a&\cdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&\cdots&a&0\\0&0&0&0&\cdots&0&0\end{bmatrix}_{(N+1)\times(N+1)},$$

$$B=\begin{bmatrix} b_{1,1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_{2,1} & b_{2,2} & 0 & 0 & \cdots & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 & \cdots & 0 & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{N,1} & b_{N,2} & b_{N,3} & b_{N,4} & \cdots & b_{N,N} & 0 \\ b_{N+1,1} & b_{N+1,2} & b_{N+1,3} & b_{N+1,4} & \cdots & b_{N+1,N} & b_{N+1,N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$C=A,$$

$$D=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$U_s=\begin{bmatrix} U_s^0 \\ U_s^1 \\ U_s^2 \\ U_s^3 \\ \cdots \\ U_s^{N-1} \\ U_s^N \end{bmatrix}_{(N+1)\times(1)}, \quad s=n-1,n,n+1,$$

$$a=-\frac{1}{h^2}, \quad b_{1,1}=1, \quad b_{2,1}=-\frac{1}{\tau}, \quad b_{2,2}=\frac{1}{\tau}, \quad b_{3,1}=\frac{1}{\tau^2}+\frac{1}{\tau^{1/2}},$$

$$b_{3,2}=-\frac{2}{\tau^2}-\frac{1}{\tau^{1/2}}, \quad b_{3,2}=\frac{1}{\tau^2}+\frac{2}{h^{1/2}},$$

$$b_{k+2,1}=\frac{1}{\sqrt{\pi}}\frac{\Gamma(k-1+1/2)}{\Gamma(k)\tau^{1/2}}, \quad 2\le k\le N-1,$$

$$b_{k+2,k+1}=-\frac{2}{\tau^2}-\frac{1}{\tau^{1/2}}, \quad 1\le k\le N-1,$$

$$b_{k+2,k}=\frac{1}{\tau^2}+\frac{1}{\sqrt{\pi}}\left(-\frac{\Gamma(3/2)}{\Gamma(2)}+\frac{\Gamma(1/2)}{\Gamma(1)}\right)\frac{1}{\tau^{1/2}}, \quad 2\le k\le N-1,$$

$$b_{k+2,k+2}=\frac{1}{\tau^2}+\frac{2}{h^2}, \quad 1\le k\le N-1,$$

$$b_{k+2,i+1} = \frac{1}{\sqrt{\pi}}\left(-\frac{\Gamma(k-i+1/2)}{\Gamma(k-(i-1))} + \frac{\Gamma(k-(i+1)+1/2)}{\Gamma(k-(i-1)-1)}\right)\frac{1}{\tau^{1/2}}, \quad 3 \le k \le N-1,\ 1 \le i \le k-2,$$

$$\varphi_n^k = \left(2 - \frac{8(k\tau)^{3/2}}{3\sqrt{\pi}} + (\pi k\tau)^2\right)\sin\pi(nh)),$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \cdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}. \tag{3.4}$$

So, we have the second-order difference equation with respect to n matrix coefficients. This type system was developed by Samarskii and Nikolaev [27]. To solve this difference equation we have applied a procedure of modified Gauss elimination method for difference equation with respect to k matrix coefficients. Hence, we seek a solution of the matrix equation in the following form:

$$U_j = \alpha_{j+1}U_{j+1} + \beta_{j+1}, \tag{3.5}$$

$n = M-1, \ldots, 2, 1$, where α_j $(j = 1, \ldots, M)$ are $(N+1)\times(N+1)$ square matrices and β_j $(j = 1, \ldots, M)$ are $(N+1)\times 1$ column matrices defined by

$$\begin{gathered} \alpha_{n+1} = -(B + C\alpha_n)^{-1}A, \\ \beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n), \quad n = 1, 2, \ldots, M-1, \end{gathered} \tag{3.6}$$

where

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\beta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}_{(N+1)\times 1}. \tag{3.7}$$

Table 1: The difference scheme.

	$M = 80$ $N = 20$	$M = 80$ $N = 40$	$M = 80$ $N = 80$
The values of C_{t1}	1.0379	1.0667	1.0800
The values of C_{t2}	0.6265	0.6186	0.6140

Now, we will give the results of the numerical analysis. First, as we noted above one can not obtain a sharp estimate for the constants C_1 and C_2 figuring in the stability estimates of Theorem 2.1. We have

$$C_1 = \max_{f,u}(C_{t1}),$$
$$C_2 = \max_{f,u}(C_{t2}), \tag{3.8}$$

where

$$C_{t1} = \max_{1\le k\le N}\left\|u_k^h\right\|_{L_{2h}} + \max_{1\le k\le N}\left\|\tau^{-1}\left(u_k^h - u_{k-1}^h\right)\right\|_{L_{2h}}\left(\max_{1\le k\le N-1}\left\|f_k^h\right\|_{L_{2h}}\right)^{-1},$$
$$C_{t2} = \left[\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right)\right\|_{L_{2h}} + \max_{1\le k\le N-1}\sum_{r=1}^{n}\left\|\left(u_{k+1}^h\right)_{\overline{x}_r,x_r,j_r}\right\|_{L_{2h}}\right] \tag{3.9}$$
$$\times\left(\max_{2\le k\le N-1}\left\|\tau^{-1}\left(f_k^h - f_{k-1}^h\right)\right\|_{L_{2h}} + \left\|f_1^h\right\|_{L_{2h}}\right)^{-1}.$$

The constants C_{t1} and C_{t2} in the case of numerical solution of initial-boundary value problem (3.1) are computed.

The numerical solutions are recorded for different values of N and M, u_n^k represents the numerical solutions of this difference scheme at (t_k, x_n). The constants C_{t1} and C_{t2} are given in Table 1 for $N = 20, 40, 80$, and $M = 80$, respectively.

Recall that we have not been able to obtain a sharp estimate for the constants C_1 and C_2 figuring in the stability estimates. The numerical results in the Tables 1 and 2 give $C_{t1} \cong 1.00$ and $C_{t2} \cong 0.62$, respectively. That means the constants C_1 and C_2 figuring in the stability estimates in the case of numerical solution of initial-boundary value problem (3.1) of this difference scheme is stable with no large constants.

Second, for the accurate comparison of the difference scheme considered, the errors computed by

$$E_0 = \max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\left|u(t_k, x_n) - u_n^k\right|^2 h\right)^{1/2},$$
$$E_1 = \max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\left|u_{tt}(t_k, x_n) - \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2}\right|^2 h\right)^{1/2} \tag{3.10}$$

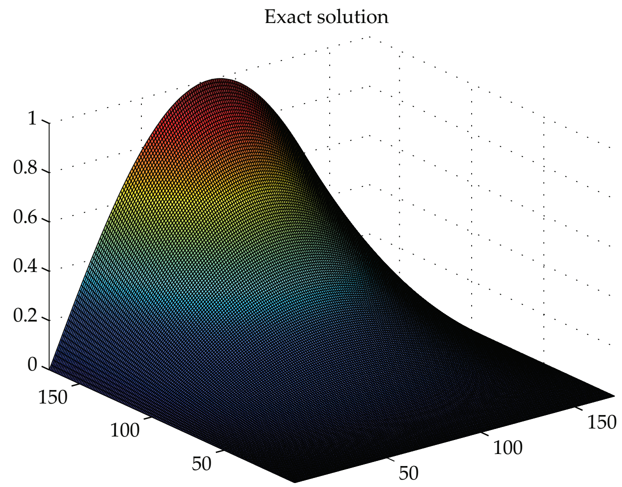

Figure 1: The exact solution.

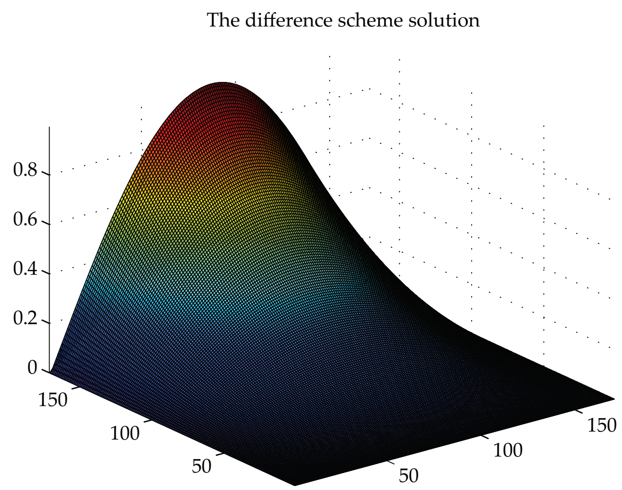

Figure 2: Difference Scheme.

Table 2: The difference scheme.

Method	$M = 80$ $N = 20$	$M = 80$ $N = 40$	$M = 80$ $N = 80$
Comparison of errors (E_0) for approximate solutions	0.0452	0.0235	0.0125
Comparison of errors (E_1) for approximate solutions	0.2702	0.1452	0.0779

of the numerical solution are recorded for higher values of N and M, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n). The errors E_0 and E_1 results are shown in Table 2 for $N = 20, 40, 80$ and $M = 80$, respectively.

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