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Research Article

Chaos Synchronization between Two Different Fractional Systems of Lorenz Family

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This work investigates chaos synchronization between two different fractional order chaotic systems of Lorenz family. The fractional order Lü system is controlled to be the fractional order Chen system, and the fractional order Chen system is controlled to be the fractional order Lorenz-like system. The analytical conditions for the synchronization of these pairs of different fractional order chaotic systems are derived by utilizing Laplace transform. Numerical simulations are used to verify the theoretical analysis using different values of the fractional order parameter.

1. Introduction

Fractional calculus has been known since the early 17th century [1, 2]. It has useful applications in many fields of science like physics [3], engineering [4], mathematical biology [5, 6], and finance [7, 8].

The fractional order derivatives have many definitions; one of them is the Riemann-Liouville definition [9] which is given by

$$D^{\alpha} f(t) = \frac{d^{l}}{dt^{l}} J^{l-\alpha} f(t), \quad \alpha > 0, \tag{1.1}$$

where J^{θ} is the θ-order Riemann-Liouville integral operator which is given as

$$J^{\theta} u(t) = \frac{1}{\Gamma(\theta)} \int_{0}^{t} (t-\tau)^{\theta-1} u(\tau) d\tau, \qquad \theta > 0. \tag{1.2}$$

However, the most common definition is the Caputo definition [10], since it is widely used in real applications:

$$D_*^\alpha f(t) = J^{l-\alpha} f^{(l)}(t), \tag{1.3}$$

where $f^{(l)}$ represents the l-order derivative of $f(t)$ and $l = [\alpha]$; this means that l is the first integer which is not less than α. The operator D_*^α is called "the Caputo differential operator of order α." Hence, I choose the Caputo type throughout this paper.

On the other hand, chaos has been studied and developed with much interest by scientists since the birth of Lorenz chaotic attractor in 1963 [11]. Chen attractor is similar to Lorenz attractor but not topologically equivalent [12]. Recently, Lü et al. found a new chaotic system which connects the Lorenz and Chen attractors, according to the conditions formulated by Vaněček and Čelikovský, and it is called Lü system [13]. Afterwards, chaos in fractional order dynamical systems has become an interesting topic. In [14] chaotic behaviors of the fractional order Lorenz system are studied. Moreover, chaotic behaviors have also been found in the fractional order Chen system [15] and the fractional order Lü system [16]. Furthermore, Chaos synchronization in fractional order chaotic systems starts to attract increasing attention [16–20]. However, it has been studied very well in the case of integer order chaotic systems, due to its potential applications in physical, chemical, and biological systems [21–24] and secure communications [25].

The generalized synchronization between two different fractional order systems is investigated in [26]. However, in this paper, I investigate the conditions of chaos synchronization between two different fractional order chaotic systems of Lorenz family by designing suitable linear controllers. I give examples to achieve chaos synchronization of two pairs of different fractional order chaotic systems (fractional Chen & fractional Lü, fractional Lorenz-like, and fractional Chen) in drive-response structure. Conditions for achieving chaos synchronization using linear control method are further discussed using Laplace transform theory.

2. Systems Description

The fractional order Chen system isgiven as follows:

$$\frac{d^\alpha x}{dt^\alpha} = a(y - x), \qquad \frac{d^\alpha y}{dt^\alpha} = (c - a)x - xz + cy, \qquad \frac{d^\alpha z}{dt^\alpha} = xy - bz. \tag{2.1}$$

Here and throughout, $(a, b, c) = (35, 3, 28)$ where α is the fractional order. In the following I choose $\alpha = 0.9$ at which system (2.1) exhibits chaotic attractor (see Figure 1).

The fractional order Lü system is given as follows

$$\frac{d^\alpha x}{dt^\alpha} = r(y - x), \qquad \frac{d^\alpha y}{dt^\alpha} = -xz + py, \qquad \frac{d^\alpha z}{dt^\alpha} = xy - qz. \tag{2.2}$$

Here and throughout, $(r, p, q) = (35, 28, 3)$. By choosing $\alpha = 0.9$, system (2.2) has chaotic attractor (see Figure 2).

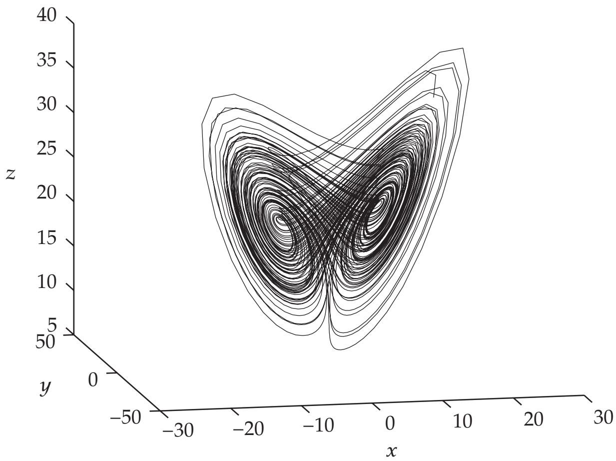

Figure 1: Chaotic attractor of the fractional order Chen system (2.1) with $\alpha = 0.9$ and $(a, b, c) = (35, 3, 28)$.

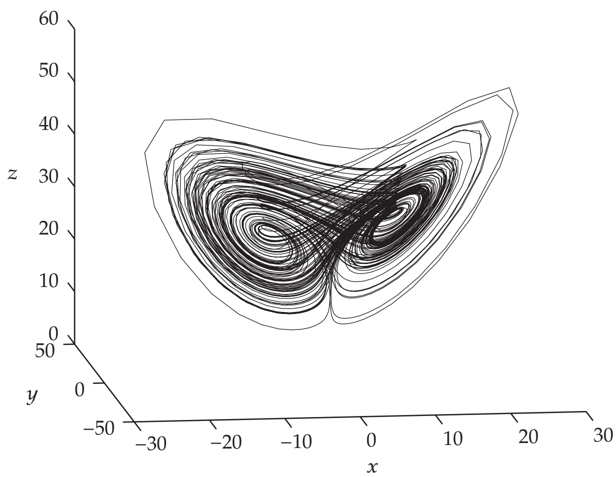

Figure 2: Chaotic attractor of the fractional order Lü system (2.2) with $\alpha = 0.9$ and $(r, p, q) = (35, 28, 3)$.

The fractional order Lorenz-like system [27] is described by

$$\frac{d^\alpha x}{dt^\alpha} = \sigma(y - x), \qquad \frac{d^\alpha y}{dt^\alpha} = \rho x - xz + \gamma y, \qquad \frac{d^\alpha z}{dt^\alpha} = xy - \beta z, \tag{2.3}$$

which has a chaotic attractor as shown in Figure 3 when $\beta = 2.8$, $\gamma = 10.6$, $\rho = 14$, $\sigma = 20$, and $\alpha = 0.9$.

It should be also noted that, the systems (2.1), (2.2), and (2.3) are still chaotic at the fractional order values $\alpha = 0.95$ and $\alpha = 0.99$.

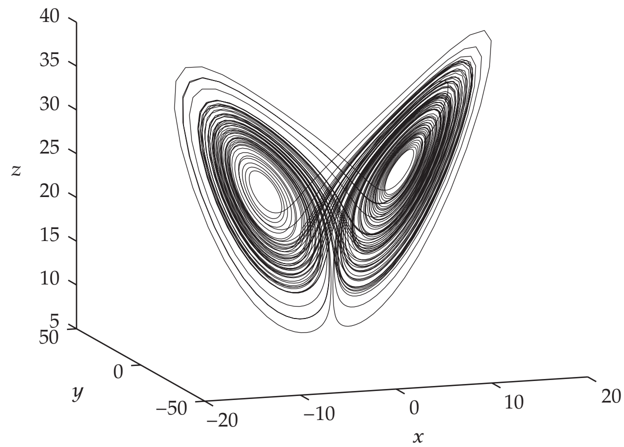

Figure 3: Chaotic attractor of the fractional order Lorenz-like system (2.3) with $\alpha = 0.9$ and $(\beta, \gamma, \rho, \sigma) = (2.8, 10.6, 14, 20)$.

3. Synchronization between Two Different Fractional Order Systems

Consider the master-slave (or drive-response) synchronization scheme of two autonomous different fractional order chaotic systems:

$$\frac{d^\alpha X}{dt^\alpha} = f(X), \qquad \frac{d^\alpha Y}{dt^\alpha} = g(Y) + U(t), \tag{3.1}$$

where α is the fractional order, $X \in R^n$, $Y \in R^n$ represent the states of the drive and response systems, respectively, $f : R^n \to R^n$, $g : R^n \to R^n$ are the vector fields of the drive and response systems, respectively. The aim is to choose a suitable linear control function $U(t) = (u_1, \ldots, u_n)^T$ such that the states of the drive and response systems are synchronized (i.e., $\lim_{t\to\infty}\|X - Y\| = 0$, where $\|\cdot\|$ is the Euclidean norm).

3.1. Synchronization between Chen and Lü Fractional Order Systems

In this subsection, the goal is to achieve chaos synchronization between the fractional order Chen system and the fractional order Lü system by using the fractional order Chen system to drive the fractional order Lü system. The drive and response systems are given as follows:

$$\frac{d^\alpha x_m}{dt^\alpha} = a(y_m - x_m), \qquad \frac{d^\alpha y_m}{dt^\alpha} = (c - a)x_m - x_m z_m + c y_m, \qquad \frac{d^\alpha z_m}{dt^\alpha} = x_m y_m - b z_m, \tag{3.2}$$

$$\frac{d^\alpha x_s}{dt^\alpha} = r(y_s - x_s) + u_1, \qquad \frac{d^\alpha y_s}{dt^\alpha} = -x_s z_s + p y_s + u_2, \qquad \frac{d^\alpha z_s}{dt^\alpha} = x_s y_s - q z_s + u_3, \tag{3.3}$$

where u_1, u_2, and u_3 are the linear control functions. Define the error variables as follows:

$$e_1 = x_s - x_m, \qquad e_2 = y_s - y_m, \qquad e_3 = z_s - z_m. \tag{3.4}$$

By subtracting (3.2) from (3.3) and using (3.4), we obtain

$$\begin{aligned}
\frac{d^{\alpha}e_1}{dt^{\alpha}} &= r(e_2 - e_1) + (r - a)(y_m - x_m) + u_1, \\
\frac{d^{\alpha}e_2}{dt^{\alpha}} &= pe_2 - z_m e_1 - x_m e_3 - e_1 e_3 - (c - a)x_m + (p - c)y_m + u_2, \\
\frac{d^{\alpha}e_3}{dt^{\alpha}} &= -qe_3 + y_m e_1 + x_m e_2 + e_1 e_2 - (q - b)z_m + u_3.
\end{aligned} \tag{3.5}$$

Now, by letting

$$\begin{aligned}
u_1 &= (a - r)(y_m - x_m), \\
u_2 &= (c - a)x_m + (c - p)y_m - k_1(y_s - y_m), \\
u_3 &= (q - b)z_m - k_2(z_s - z_m),
\end{aligned} \tag{3.6}$$

where $k_1, k_2 \geq 0$, then the error system (3.5) is reduced to

$$\begin{aligned}
\frac{d^{\alpha}e_1}{dt^{\alpha}} &= r(e_2 - e_1), \\
\frac{d^{\alpha}e_2}{dt^{\alpha}} &= (p - k_1)e_2 - z_m e_1 - x_m e_3 - e_1 e_3, \\
\frac{d^{\alpha}e_3}{dt^{\alpha}} &= -(q + k_2)e_3 + y_m e_1 + x_m e_2 + e_1 e_2.
\end{aligned} \tag{3.7}$$

By taking the Laplace transform in both sides of (3.7), letting $E_i(s) = L\{e_i(t)\}$ where $(i = 1, 2, 3)$, and applying $L\{d^{\alpha}e_i/dt^{\alpha}\} = s^{\alpha}E_i(s) - s^{\alpha-1}e_i(0)$, we obtain

$$\begin{aligned}
s^{\alpha}E_1(s) - s^{\alpha-1}e_1(0) &= r(E_2(s) - E_1(s)), \\
s^{\alpha}E_2(s) - s^{\alpha-1}e_2(0) &= (p - k_1)E_2(s) - L\{x_m e_3\} - L\{z_m e_1\} - E_1(s)E_3(s), \\
s^{\alpha}E_3(s) - s^{\alpha-1}e_3(0) &= -(q + k_2)E_3(s) + L\{y_m e_1\} + L\{x_m e_2\} + E_1(s)E_2(s).
\end{aligned} \tag{3.8}$$

Proposition 3.1. *If $E_1(s)$, $E_2(s)$ are bounded and $p - k_1 \neq 0$, then the drive and response systems* (3.2) *and* (3.3) *will be synchronized under a suitable choice of k_1 and k_2.*

Proof. Rewrite (3.8) as follows:

$$\begin{aligned}
E_1(s) &= \frac{rE_2(s)}{s^{\alpha} + r} + \frac{s^{\alpha-1}e_1(0)}{s^{\alpha} + r}, \\
E_2(s) &= -\frac{L\{z_m e_1\}}{s^{\alpha} - p + k_1} - \frac{L\{x_m e_3\}}{s^{\alpha} - p + k_1} - \frac{E_1(s)E_3(s)}{s^{\alpha} - p + k_1} + \frac{s^{\alpha-1}e_2(0)}{s^{\alpha} - p + k_1}, \\
E_3(s) &= \frac{L\{y_m e_1\}}{s^{\alpha} + q + k_2} + \frac{L\{x_m e_2\}}{s^{\alpha} + q + k_2} + \frac{E_1(s)E_2(s)}{s^{\alpha} + q + k_2} + \frac{s^{\alpha-1}e_3(0)}{s^{\alpha} + q + k_2}.
\end{aligned} \tag{3.9}$$

Using the final value theorem of the Laplace transform, it follows that

$$\begin{aligned}
\lim_{t\to\infty} e_1(t) &= \lim_{s\to 0^+} sE_1(s) = \lim_{s\to 0^+} sE_2(s) = \lim_{t\to\infty} e_2(t),\\
\lim_{t\to\infty} e_2(t) &= \lim_{s\to 0^+} sE_2(s) = \frac{1}{p-k_1}\lim_{s\to 0^+} sL\{x_m e_3\} + \frac{1}{p-k_1}\lim_{s\to 0^+} sL\{z_m e_1\} + \frac{1}{p-k_1}\lim_{t\to\infty} e_1(t)\cdot\lim_{t\to\infty} e_3(t),\\
\lim_{t\to\infty} e_3(t) &= \lim_{s\to 0^+} sE_3(s) = \frac{1}{q+k_2}\lim_{s\to 0^+} sL\{y_m e_1\} + \frac{1}{q+k_2}\lim_{s\to 0^+} sL\{x_m e_2\} + \frac{1}{q+k_2}\lim_{t\to\infty} e_1(t)\cdot\lim_{t\to\infty} e_2(t).
\end{aligned} \tag{3.10}$$

Since $E_1(s),\ E_2(s)$ are bounded and $p - k_1 \neq 0$ then $\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty} e_2(t) = 0$. Now, owing to the attractiveness of the attractors of systems (2.1) and (2.2), there exists $\eta > 0$ such that $|x_i(t)| \le \eta < \infty$, $|y_i(t)| \le \eta < \infty$, and $|z_i(t)| \le \eta < \infty$ where i refers to the index of the drive or response variables. Therefore, $\lim_{t\to\infty} e_3(t) = 0$. This implies that

$$\lim_{t\to\infty} e_i(t) = 0, \quad i = 1, 2, 3. \tag{3.11}$$

Consequently, the synchronization between the drive and response systems (3.2) and (3.3) is achieved. □

3.1.1. Numerical Results

An efficient method for solving fractional order differential equations is the predictor-correctors scheme or more precisely, PECE (Predict, Evaluate, Correct, Evaluate) technique which has been investigated in [28, 29], and represents a generalization of the Adams-Bashforth-Moulton algorithm. It is used throughout this paper.

Based on the above mentioned discretization scheme, the drive and response systems (3.2) and (3.3) are integrated numerically with the fractional orders $\alpha = 0.9,\ 0.95,\ 0.99$ and using the initial values $x_m(0) = 15$, $y_m(0) = 20$, $z_m(0) = 29$ and $x_s(0) = 10$, $y_s(0) = 15$, $z_s(0) = 25$. From Figure 4, it is clear that the synchronization is achieved for all these values of fractional order when $k_1 = 20$ and $k_2 = 10$.

3.2. Synchronization between Lorenz-Like and Chen Fractional Order Systems

In this case it is assumed that, the fractional order Lorenz-like system drives the fractional order Chen system. The drive and response systems are defined as follows:

$$\frac{d^\alpha x_m}{dt^\alpha} = \sigma(y_m - x_m), \qquad \frac{d^\alpha y_m}{dt^\alpha} = \rho x_m - x_m z_m + \gamma y_m, \qquad \frac{d^\alpha z_m}{dt^\alpha} = x_m y_m - \beta z_m, \tag{3.12}$$

$$\frac{d^\alpha x_s}{dt^\alpha} = a(y_s - x_s) + v_1, \qquad \frac{d^\alpha y_s}{dt^\alpha} = (c-a)x_s - x_s z_s + c y_s + v_2, \qquad \frac{d^\alpha z_s}{dt^\alpha} = x_s y_s - b z_s + v_3, \tag{3.13}$$

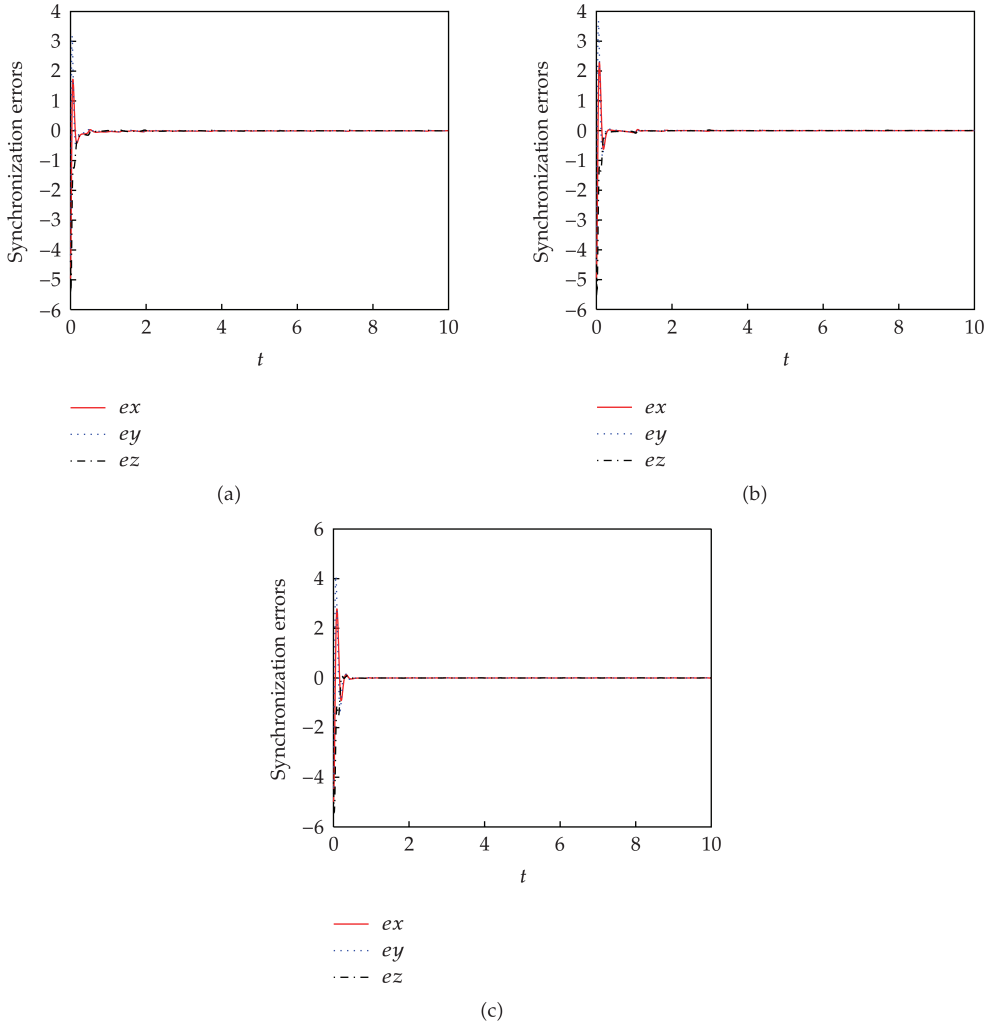

Figure 4: Synchronization errors of the drive system (3.2) and response system (3.3) using $k_1 = 20$, $k_2 = 10$ and fractional orders: (a) $\alpha = 0.9$, (b) $\alpha = 0.95$, and (c) $\alpha = 0.99$.

where v_1, v_2, and v_3 are the linear control functions. The error variables are given by

$$e_1 = x_s - x_m, \qquad e_2 = y_s - y_m, \qquad e_3 = z_s - z_m. \tag{3.14}$$

By subtracting (3.12) from (3.13) and using (3.14), we get

$$\begin{aligned} \frac{d^\alpha e_1}{dt^\alpha} &= a(e_2 - e_1) + (a - \sigma)(y_m - x_m) + v_1, \\ \frac{d^\alpha e_2}{dt^\alpha} &= ce_2 - z_m e_1 - x_m e_3 - e_1 e_3 + (c - a)x_s - \rho x_m + (c - \gamma)y_m + v_2, \\ \frac{d^\alpha e_3}{dt^\alpha} &= -be_3 + y_m e_1 + x_m e_2 + e_1 e_2 + (\beta - b)z_m + v_3. \end{aligned} \tag{3.15}$$

Now, by choosing

$$v_1 = (\sigma - a)(y_m - x_m), \qquad v_2 = \rho x_m + (a - c)x_s + (\gamma - c)y_m - k_1 e_2, \qquad v_3 = (b - \beta)z_m - k_2 e_3, \tag{3.16}$$

where $k_1, k_2 \geq 0$, then the error system (3.15) is rewritten as

$$\begin{aligned} \frac{d^{\alpha} e_1}{dt^{\alpha}} &= a(e_2 - e_1), \\ \frac{d^{\alpha} e_2}{dt^{\alpha}} &= (c - k_1)e_2 - z_m e_1 - x_m e_3 - e_1 e_3, \\ \frac{d^{\alpha} e_3}{dt^{\alpha}} &= -(b + k_2)e_3 + y_m e_1 + x_m e_2 + e_1 e_2. \end{aligned} \tag{3.17}$$

Take Laplace transform in both sides of (3.17), let $E_i(s) = L\{e_i(t)\}$, where $(i = 1, 2, 3)$, and apply $L\{d^{\alpha} e_i / dt^{\alpha}\} = s^{\alpha} E_i(s) - s^{\alpha-1} e_i(0)$. After that, by doing similar analysis like the previous subsection, we obtain

$$\begin{aligned} \lim_{t\to\infty} e_1(t) &= \lim_{s\to 0^+} sE_1(s) = \lim_{s\to 0^+} sE_2(s) = \lim_{t\to\infty} e_2(t), \\ \lim_{t\to\infty} e_2(t) &= \lim_{s\to 0^+} sE_2(s) = \frac{1}{c - k_1} \lim_{s\to 0^+} sL\{x_m e_3\} + \frac{1}{c - k_1} \lim_{s\to 0^+} sL\{z_m e_1\} + \frac{1}{c - k_1} \lim_{t\to\infty} e_1(t) \cdot \lim_{t\to\infty} e_3(t), \\ \lim_{t\to\infty} e_3(t) &= \lim_{s\to 0^+} sE_3(s) = \frac{1}{b + k_2} \lim_{s\to 0^+} sL\{y_m e_1\} + \frac{1}{b + k_2} \lim_{s\to 0^+} sL\{x_m e_2\} + \frac{1}{b + k_2} \lim_{t\to\infty} e_1(t) \cdot \lim_{t\to\infty} e_2(t). \end{aligned} \tag{3.18}$$

If we assume that $c - k_1 \neq 0$ and $E_1(s)$, $E_2(s)$ are bounded, then it follows that $\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty} e_2(t) = 0$. Now, owing to the attractiveness of the attractors of systems (2.1) and (2.3), there exists $\xi > 0$ such that $|x_i(t)| \leq \xi < \infty$, $|y_i(t)| \leq \xi < \infty$, and $|z_i(t)| \leq \xi < \infty$ where i refers to the index of the drive or response variables. Therefore, $\lim_{t\to\infty} e_3(t) = 0$. Consequently,

$$\lim_{t\to\infty} e_i(t) = 0, \quad i = 1, 2, 3. \tag{3.19}$$

Thus, the states of the drive system (3.12) are synchronized with the states of the response system (3.13), as the controllers (3.16) are activated.

3.2.1. Numerical Results

Numerical simulations are carried out to integrate the drive and response systems (3.12) and (3.13) using the predictor-correctors scheme, with the fractional orders $\alpha = 0.9,\ 0.95,\ 0.99$ and the initial values $x_m(0) = 10$, $y_m(0) = 16$, $z_m(0) = 25$ and $x_s(0) = 15$, $y_s(0) = 20$, $z_s(0) = 29$. Thus, the drive and response systems (3.12) and (3.13) are synchronized in such a successful way for all at the above-mentioned fractional orders values, using the linear controllers (3.16) with $k_1 = 20$ and $k_2 = 10$ (see Figure 5).

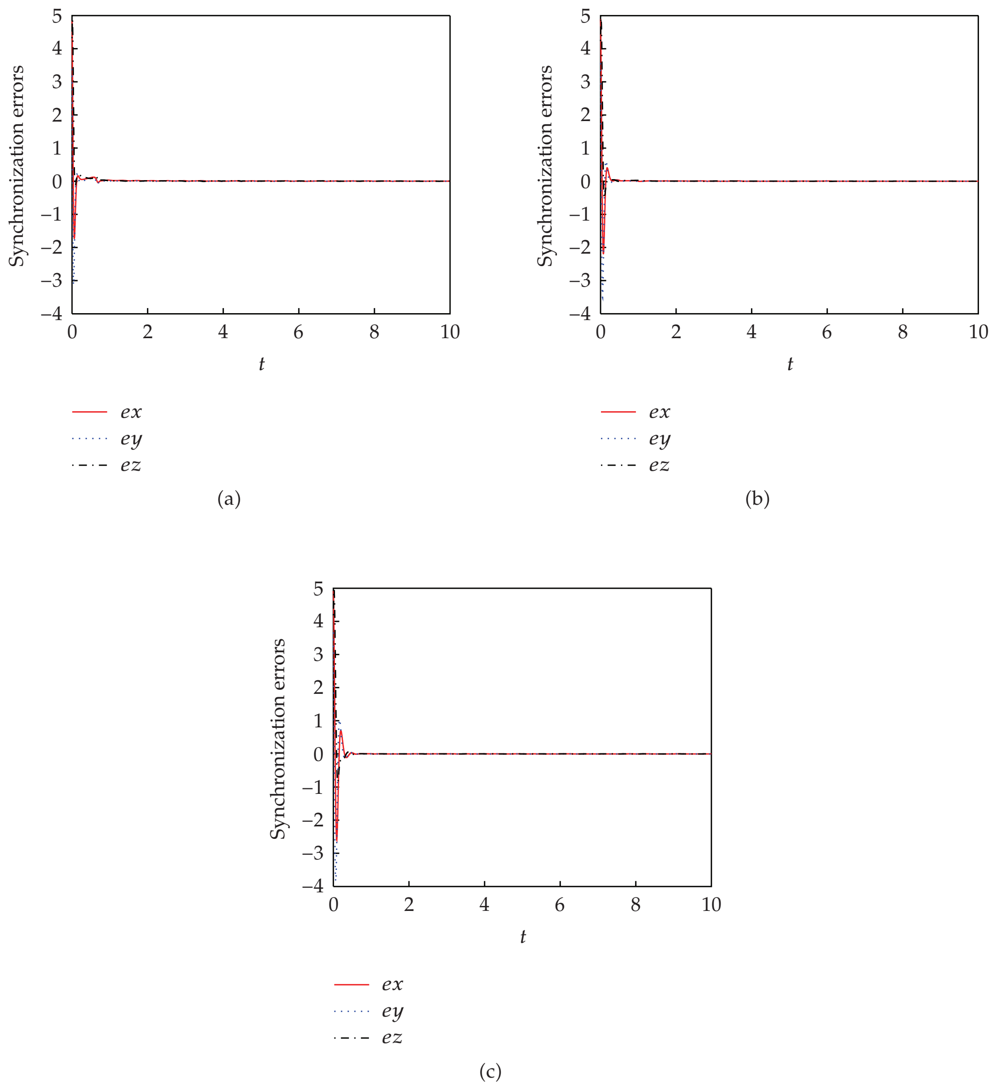

Figure 5: Synchronization errors of the drive system (3.12) and response system (3.13) using $k_1 = 20$, $k_2 = 10$ and fractional orders: (a) $\alpha = 0.9$, (b) $\alpha = 0.95$, and (c) $\alpha = 0.99$.

4. Conclusion

Chaos synchronization between two different fractional order chaotic systems has been studied using linear control technique. Fractional order Chen system has been used to drive fractional order Lü system, and fractional order Lorenz-like system has been used to drive fractional order Chen system. Conditions for chaos synchronization have been investigated theoretically by using Laplace transform. Numerical simulations have been carried out using different fractional order values to show the effectiveness of the proposed synchronization techniques.

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