

Research Article

An Inverse Quadratic Eigenvalue Problem for Damped Structural Systems

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We first give the representation of the general solution of the following inverse quadratic eigenvalue problem (IQEP): given $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_p\} \in \mathbf{C}^{p \times p}$, $X = [x_1, \dots, x_p] \in \mathbf{C}^{n \times p}$, and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \dots, l$, and $\lambda_k \in \mathbf{R}$, $x_k \in \mathbf{R}^n$ for $k = 2l + 1, \dots, p$, find real-valued symmetric $(2r + 1)$ -diagonal matrices M, D and K such that $MX\Lambda^2 + DX\Lambda + KX = 0$. We then consider an optimal approximation problem: given real-valued symmetric $(2r + 1)$ -diagonal matrices $M_a, D_a, K_a \in \mathbf{R}^{n \times n}$, find $(\widehat{M}, \widehat{D}, \widehat{K}) \in \mathbf{S}_E$ such that $\|\widehat{M} - M_a\|^2 + \|\widehat{D} - D_a\|^2 + \|\widehat{K} - K_a\|^2 = \inf_{(M,D,K) \in \mathbf{S}_E} (\|M - M_a\|^2 + \|D - D_a\|^2 + \|K - K_a\|^2)$, where \mathbf{S}_E is the solution set of IQEP. We show that the optimal approximation solution $(\widehat{M}, \widehat{D}, \widehat{K})$ is unique and derive an explicit formula for it.

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1. Introduction

Throughout this paper, we will adopt the following notation. $\mathbf{C}^{m \times n}$ and $\mathbf{R}^{m \times n}$ denote the set of all $m \times n$ complex and real matrices, respectively. $\mathbf{SR}^{n \times n}$ denotes the set of all symmetric matrices in $\mathbf{R}^{n \times n}$. A^T and A^+ stand for the transpose and the Moore-Penrose generalized inverse of a real matrix A . I_n represents the identity matrix of size n ; $\bar{\alpha}$ denotes the conjugate of the complex number α . For $A, B \in \mathbf{R}^{m \times n}$, an inner product in $\mathbf{R}^{m \times n}$ is defined by $(A, B) = \text{trace}(B^T A)$, then $\mathbf{R}^{m \times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. Given two matrices $A = [a_{ij}] \in \mathbf{R}^{m \times n}$ and $B = [b_{ij}] \in \mathbf{R}^{p \times q}$, the Kronecker product of A and B is defined by $A \otimes B = [a_{ij}B] \in \mathbf{R}^{mp \times nq}$, and the stretching function $\text{Vec}(A)$ is defined by $\text{Vec}(A) = [a_1^T, a_2^T, \dots, a_n^T]^T \in \mathbf{R}^{mn}$, where a_i , $i = 1, \dots, n$, is the i th column vector of A . Furthermore, for a matrix $A \in \mathbf{R}^{m \times n}$, let E_A and F_A stand for the two orthogonal projectors $E_A = I_m - AA^+$ and $F_A = I_n - A^+A$.

Using finite element techniques, vibrating structures such as beams, buildings, bridges, highways, and large space structures can be discretized to matrix second-order models (referred to as analytical models). A matrix second-order model of the free motion of a vibrating system is a system of differential equations of the form

$$M_a \ddot{x}(t) + D_a \dot{x}(t) + K_a x(t) = 0, \quad (1.1)$$

where M_a , D_a , and K_a are the $n \times n$ analytical mass, damping, and stiffness matrices. The system represented by (1.1) is called damped structural system. It is well known that all solutions of (1.1) can be obtained via the algebraic equation

$$(\lambda^2 M_a + \lambda D_a + K_a)x = 0. \quad (1.2)$$

Complex numbers λ and nonzero complex vectors x for which this relation holds are, respectively, the eigenvalues and eigenvectors of the system. It is known that (1.2) has $2n$ finite eigenvalues over the complex field, provided that the leading matrix coefficient M_a is nonsingular.

Due to the complexity of the structure, the finite element model is only an approximation to the practical structure. On the other hand, a part of the natural frequencies (eigenvalues) and corresponding mode shapes (eigenvectors) of the structure can be obtained experimentally by performing vibration tests [1]. Generally speaking, very often natural frequencies and mode shapes of an analytical model described by (1.2) do not match very well with experimentally measured frequencies and mode shapes. Thus, engineers would like to improve the analytical model of the structure such that the updated model predicts the observed dynamic behavior. Then, the updated model may be considered to be a better dynamic representation of the structure. This model can be used with greater confidence for the analysis of the structure under different boundary conditions or with physical structural changes.

For undamped systems (i.e., $D_a = 0$), various techniques for updating mass and stiffness matrices using measured response data have been discussed by Baruch [2], Baruch and Bar-Izhack [3], Berman [4], Berman and Nagy [5], and Wei [6, 7]. For damped structural systems, the theory and computation were first proposed by Friswell et al. [8, 9]; they applied the ideas in [2, 3] to minimize changes between the analytical and updated model subject to the spectral constraints. Kuo et al. [10] have recently proposed a direct method to close the weaknesses in [8] which seems more efficient and reliable. All these existing methods can reproduce the given set of measured data while updated matrices symmetry, but the connectivity of the original finite element model is not necessarily preserved, causing the addition of unwanted load paths.

The purpose of the work presented in this paper is to develop a new method for finite element model updating problems which preserves the connectivity of the original model. Assume that M_a , D_a , and K_a are real-valued symmetric $(2r + 1)$ -diagonal matrices. Thus, the problem of updating mass, damping, and stiffness matrices simultaneously can be mathematically formulated as follows.

Problem IQEP. Given matrices $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_p\} \in \mathbf{C}^{p \times p}$, $X = [x_1, \dots, x_p] \in \mathbf{C}^{n \times p}$, and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \dots, l$, and $\lambda_k \in \mathbf{R}$, $x_k \in \mathbf{R}^n$ for $k = 2l + 1, \dots, p$, find real-valued symmetric $(2r + 1)$ -diagonal matrices M , D , and K such that

$$MX\Lambda^2 + DX\Lambda + KX = 0. \quad (1.3)$$

Problem II. Let \mathbf{S}_E be the solution set of IQEP. Find $(\widehat{M}, \widehat{D}, \widehat{K}) \in \mathbf{S}_E$ such that

$$\|\widehat{M} - M_a\|^2 + \|\widehat{D} - D_a\|^2 + \|\widehat{K} - K_a\|^2 = \inf_{(M,D,K) \in \mathbf{S}_E} (\|M - M_a\|^2 + \|D - D_a\|^2 + \|K - K_a\|^2). \quad (1.4)$$

The paper is organized as follows. In Section 2, using the Kronecker product and stretching function of matrices, we give an explicit representation of the solution set \mathbf{S}_E of Problem IQEP. In Section 3, we show that there exists a unique solution in Problem II and present the expression of the unique solution $(\widehat{M}, \widehat{D}, \widehat{K})$. Finally, in Section 4, a numerical algorithm to acquire the optimal approximation solution under the Frobenius norm sense is described, and a numerical example is provided.

2. The solution of Problem IQEP

To begin with, we introduce a lemma [11].

Lemma 2.1. *If $L \in \mathbf{R}^{m \times q}$, $b \in \mathbf{R}^m$, then $Ly = b$ has a solution $y \in \mathbf{R}^q$ if and only if $LL^+b = b$. In this case, the general solution of the equation can be described as $y = L^+b + (I_q - L^+L)z$, where $z \in \mathbf{R}^q$ is an arbitrary vector.*

Let S_0 be the set of all $n \times n$ real-valued symmetric $(2r + 1)$ -diagonal matrices, then S_0 is a linear subspace of $\mathbf{SR}^{n \times n}$, and the dimension of S_0 is $N = (1/2)(2n - r)(r + 1)$.

Define Y_{ij} as

$$Y_{ij} = \begin{cases} \frac{\sqrt{2}}{2}(e_i e_j^T + e_j e_i^T), & i = 1, \dots, n-1; j = i+1, \dots, t_i, \\ e_i e_i^T, & i = j = 1, \dots, n, \end{cases} \quad (2.1)$$

where $t_i = \min\{i + r, n\}$ and e_i , $i = 1, \dots, n$, is the i th column vector of the identity matrix I_n . It is easy to verify that $\{Y_{ij}\}$ forms an orthonormal basis of the subspace S_0 , that is,

$$(Y_{ij}, Y_{kl}) = \begin{cases} 0, & i \neq k \text{ or } j \neq l, \\ 1, & i = k \text{ and } j = l. \end{cases} \quad (2.2)$$

Now, if $M, D, K \in \mathbf{SR}^{n \times n}$ are $(2r + 1)$ -diagonal matrices, then M, D, K can be expressed as

$$M = \sum_{i,j} \alpha_{ij} Y_{ij}, \quad D = \sum_{i,j} \beta_{ij} Y_{ij}, \quad K = \sum_{i,j} \gamma_{ij} Y_{ij}, \quad (2.3)$$

where the real numbers $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$, $i = 1, \dots, n; j = i, \dots, t_i$, $t_i = \min\{i + r, n\}$, are yet to be determined.

Define a matrix T_p as

$$T_p = \text{diag} \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \dots, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, I_{p-2l} \right\} \in \mathbf{C}^{p \times p}, \quad (2.4)$$

where $i = \sqrt{-1}$. It is easy to verify that T_p is a unitary matrix, that is, $\overline{T_p}^T T_p = I_p$. Using this transformation matrix, we have

$$\tilde{\Lambda} = \overline{T_p}^T \Lambda T_p = \text{diag} \left\{ \begin{bmatrix} \zeta_1 & \eta_1 \\ -\eta_1 & \zeta_1 \end{bmatrix}, \dots, \begin{bmatrix} \zeta_{2l-1} & \eta_{2l-1} \\ -\eta_{2l-1} & \zeta_{2l-1} \end{bmatrix}, \lambda_{2l+1}, \dots, \lambda_p \right\} \in \mathbf{R}^{p \times p}, \quad (2.5)$$

$$\tilde{X} = X T_p = [\sqrt{2}y_1, \sqrt{2}z_1, \dots, \sqrt{2}y_{2l-1}, \sqrt{2}z_{2l-1}, x_{2l+1}, \dots, x_p] \in \mathbf{R}^{n \times p}, \quad (2.6)$$

where ζ_j and η_j are, respectively, the real part and the imaginary part of the complex number λ_j ; y_j and z_j , are, respectively, the real part and the imaginary part of the complex vector x_j for $j = 1, 3, \dots, 2l-1$.

It follows from (2.5) and (2.6) that (1.3) can be equivalently written as

$$M \tilde{X} \tilde{\Lambda}^2 + D \tilde{X} \tilde{\Lambda} + K \tilde{X} = 0. \quad (2.7)$$

Substituting (2.3) into (2.7), we have

$$\sum_{i,j} \alpha_{ij} Y_{ij} \tilde{X} \tilde{\Lambda}^2 + \sum_{i,j} \beta_{ij} Y_{ij} \tilde{X} \tilde{\Lambda} + \sum_{i,j} \gamma_{ij} Y_{ij} \tilde{X} = 0. \quad (2.8)$$

When setting

$$\begin{aligned} \alpha &= [\alpha_{11}, \dots, \alpha_{1,r+1}, \dots, \alpha_{n-r,n-r}, \dots, \alpha_{n-r,n}, \dots, \alpha_{n-1,n-1}, \alpha_{n-1,n}, \alpha_{n,n}]^T, \\ \beta &= [\beta_{11}, \dots, \beta_{1,r+1}, \dots, \beta_{n-r,n-r}, \dots, \beta_{n-r,n}, \dots, \beta_{n-1,n-1}, \beta_{n-1,n}, \beta_{n,n}]^T, \\ \gamma &= [\gamma_{11}, \dots, \gamma_{1,r+1}, \dots, \gamma_{n-r,n-r}, \dots, \gamma_{n-r,n}, \dots, \gamma_{n-1,n-1}, \gamma_{n-1,n}, \gamma_{n,n}]^T, \\ G &= [\text{Vec}(Y_{11}), \dots, \text{Vec}(Y_{1,r+1}), \dots, \text{Vec}(Y_{n-r,n-r}), \dots, \\ &\quad \text{Vec}(Y_{n-r,n}), \dots, \text{Vec}(Y_{n-1,n-1}), \text{Vec}(Y_{n-1,n}), \text{Vec}(Y_{n,n})] \in \mathbf{R}^{n^2 \times N}, \\ A &= \left((\tilde{\Lambda}^2)^T \tilde{X}^T \otimes I_n \right) G, \quad B = \left(\tilde{\Lambda}^T \tilde{X}^T \otimes I_n \right) G, \quad C = \left(\tilde{X}^T \otimes I_n \right) G. \end{aligned} \quad (2.9)$$

We see that (2.8) is equivalent to

$$A\alpha + B\beta + C\gamma = 0. \quad (2.10)$$

It follows from Lemma 2.1 that (2.10) with unknown vector α has a solution if and only if

$$E_A B \beta = -E_A C \gamma. \quad (2.11)$$

Using Lemma 2.1 again, we know that (2.11) with respect to β has a solution if and only if

$$E_H E_A C \gamma = 0, \quad (2.12)$$

where $H = E_A B$. It follows from Lemma 2.1 that (2.12) with respect to γ is always solvable, and the general solution to the equation is

$$\gamma = F_W u, \quad (2.13)$$

where $W = E_H E_A C$, and $u \in \mathbf{R}^N$ is an arbitrary vector. Substituting (2.13) into (2.11) and applying Lemma 2.1, we obtain

$$\beta = -H^+ E_A C F_W u + F_H v, \quad (2.14)$$

where $v \in \mathbf{R}^N$ is an arbitrary vector. Inserting (2.14) and (2.13) into (2.10) yields

$$\alpha = A^+ B H^+ E_A C F_W u - A^+ B F_H v - A^+ C F_W u + F_A q, \quad (2.15)$$

where $q \in \mathbf{R}^N$ is an arbitrary vector.

As a summary of the above discussion, we have proved the following result.

Theorem 2.2. *Suppose that $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_p\} \in \mathbf{C}^{p \times p}$, $X = [x_1, \dots, x_p] \in \mathbf{C}^{n \times p}$, and both Λ and X are closed under complex conjugation. The real matrices $\tilde{\Lambda}$ and \tilde{X} are given by (2.5) and (2.6). Let $\{Y_{ij}\}$, G, A, B, C be given as in (2.1), (2.9). Write $N = (1/2)(2n-r)(r+1)$, $H = E_A B$, $W = E_H E_A C$. Then the solution set \mathbf{S}_E of problem IQEP can be expressed as*

$$\mathbf{S}_E = \{(M, D, K) \in \mathbf{SR}^{n \times n} \times \mathbf{SR}^{n \times n} \times \mathbf{SR}^{n \times n} \mid M = S(\alpha \otimes I_n), D = S(\beta \otimes I_n), K = S(\gamma \otimes I_n)\}, \quad (2.16)$$

where

$$S = [Y_{11}, \dots, Y_{1,r+1}, \dots, Y_{n-r,n-r}, \dots, Y_{n-r,n}, \dots, Y_{n-1,n-1}, Y_{n-1,n}, Y_{n,n}] \in \mathbf{R}^{n \times nN}, \quad (2.17)$$

α, β, γ are, respectively, given by (2.15), (2.14), and (2.13) with $u, v, q \in \mathbf{R}^N$ being arbitrary vectors.

3. The solution of Problem II

It follows from Theorem 2.2 that the set \mathbf{S}_E is always nonempty. It is easy to verify that \mathbf{S}_E is a closed convex subset of $\mathbf{SR}^{n \times n} \times \mathbf{SR}^{n \times n} \times \mathbf{SR}^{n \times n}$. From the best approximation theorem (see [12]), we know there exists a unique solution $(\widehat{M}, \widehat{D}, \widehat{K})$ in \mathbf{S}_E such that (1.4) holds.

We now focus our attention on seeking the unique solution $(\widehat{M}, \widehat{D}, \widehat{K})$ in \mathbf{S}_E . For the real-valued symmetric $(2r+1)$ -diagonal matrices M_a, D_a , and K_a , it is easily seen that M_a, D_a , and K_a can be expressed as the linear combinations of the orthonormal basis $\{Y_{ij}\}$, that is,

$$M_a = \sum_{i,j} \delta_{ij} Y_{ij}, \quad D_a = \sum_{i,j} \xi_{ij} Y_{ij}, \quad K_a = \sum_{i,j} \varphi_{ij} Y_{ij}, \quad (3.1)$$

where $\delta_{ij}, \xi_{ij}, \varphi_{ij}$, $i = 1, \dots, n; j = i, \dots, t_i$, $t_i = \min\{i+r, n\}$, are uniquely determined by the elements of M_a, D_a , and K_a . Let

$$\begin{aligned} \delta &= [\delta_{11}, \dots, \delta_{1,r+1}, \dots, \delta_{n-r,n-r}, \dots, \delta_{n-r,n}, \dots, \delta_{n-1,n-1}, \delta_{n-1,n}, \delta_{n,n}]^T, \\ \xi &= [\xi_{11}, \dots, \xi_{1,r+1}, \dots, \xi_{n-r,n-r}, \dots, \xi_{n-r,n}, \dots, \xi_{n-1,n-1}, \xi_{n-1,n}, \xi_{n,n}]^T, \\ \varphi &= [\varphi_{11}, \dots, \varphi_{1,r+1}, \dots, \varphi_{n-r,n-r}, \dots, \varphi_{n-r,n}, \dots, \varphi_{n-1,n-1}, \varphi_{n-1,n}, \varphi_{n,n}]^T. \end{aligned} \quad (3.2)$$

Then, for any triple of matrices $(M, D, K) \in \mathbf{S}_E$ in (2.16), by the relations of (2.2) and (3.1), we see that

$$\begin{aligned}
f &= \|M - M_a\|^2 + \|D - D_a\|^2 + \|K - K_a\|^2 \\
&= \left\| \sum_{i,j} (\alpha_{ij} - \delta_{ij}) Y_{ij} \right\|^2 + \left\| \sum_{i,j} (\beta_{ij} - \xi_{ij}) Y_{ij} \right\|^2 + \left\| \sum_{i,j} (\gamma_{ij} - \varphi_{ij}) Y_{ij} \right\|^2 \\
&= \left(\sum_{i,j} (\alpha_{ij} - \delta_{ij}) Y_{ij}, \sum_{i,j} (\alpha_{ij} - \delta_{ij}) Y_{ij} \right) + \left(\sum_{i,j} (\beta_{ij} - \xi_{ij}) Y_{ij}, \sum_{i,j} (\beta_{ij} - \xi_{ij}) Y_{ij} \right) \\
&\quad + \left(\sum_{i,j} (\gamma_{ij} - \varphi_{ij}) Y_{ij}, \sum_{i,j} (\gamma_{ij} - \varphi_{ij}) Y_{ij} \right) \\
&= \sum_{i,j} (\alpha_{ij} - \delta_{ij}) \left(Y_{ij}, \sum_{i,j} (\alpha_{ij} - \delta_{ij}) Y_{ij} \right) + \sum_{i,j} (\beta_{ij} - \xi_{ij}) \left(Y_{ij}, \sum_{i,j} (\beta_{ij} - \xi_{ij}) Y_{ij} \right) \\
&\quad + \sum_{i,j} (\gamma_{ij} - \varphi_{ij}) \left(Y_{ij}, \sum_{i,j} (\gamma_{ij} - \varphi_{ij}) Y_{ij} \right) \\
&= \sum_{i,j} (\alpha_{ij} - \delta_{ij})^2 + \sum_{i,j} (\beta_{ij} - \xi_{ij})^2 + \sum_{i,j} (\gamma_{ij} - \varphi_{ij})^2 \\
&= \|\alpha - \delta\|^2 + \|\beta - \xi\|^2 + \|\gamma - \varphi\|^2.
\end{aligned} \tag{3.3}$$

Substituting (2.13), (2.14), and (2.15) into the relation of f , we have

$$\begin{aligned}
f &= \|Ju - A^+BF_Hv + F_Aq - \delta\|^2 + \|-Lu + F_Hv - \xi\|^2 + \|F_Wu - \varphi\|^2 \\
&= u^T J^T Ju - 2u^T J^T A^+BF_Hv - 2u^T J^T \delta + v^T F_H B^T (A^+)^T A^+BF_Hv + 2v^T F_H B^T (A^+)^T \delta \\
&\quad + q^T F_A q - 2q^T F_A \delta + \delta^T \delta + u^T L^T Lu + 2u^T L^T \xi + v^T F_H v - 2v^T F_H \xi + \xi^T \xi \\
&\quad + u^T F_W u - 2u^T F_W \varphi + \varphi^T \varphi,
\end{aligned} \tag{3.4}$$

where $L = H^+ E_A C F_W$, $J = A^+ B L - A^+ C F_W$. Therefore,

$$\begin{aligned}
\frac{\partial f}{\partial u} &= 2J^T Ju - 2J^T A^+BF_Hv - 2J^T \delta + 2L^T Lu + 2L^T \xi + 2F_W u - 2F_W \varphi, \\
\frac{\partial f}{\partial v} &= -2F_H B^T (A^+)^T Ju + 2F_H B^T (A^+)^T A^+BF_Hv + 2F_H B^T (A^+)^T \delta + 2F_H v - 2F_H \xi, \\
\frac{\partial f}{\partial q} &= 2F_A q - 2F_A \delta.
\end{aligned} \tag{3.5}$$

Clearly, $f = \|M - M_a\|^2 + \|D - D_a\|^2 + \|K - K_a\|^2 = \min$ if and only if

$$\frac{\partial f}{\partial u} = 0, \quad \frac{\partial f}{\partial v} = 0, \quad \frac{\partial f}{\partial q} = 0. \quad (3.6)$$

Note that $LF_W = L$, $JF_W = J$. Therefore, $\partial f / \partial u = 0$ implies that

$$F_W u = P^{-1} J^T A^+ B F_H v + P^{-1} (J^T \delta - L^T \xi + F_W \varphi), \quad (3.7)$$

where $P = J^T J + L^T L + I_N$. Substituting (3.7) into $\partial f / \partial v = 0$ yields

$$F_H v = Q^+ F_H B^T (A^+)^T J P^{-1} (J^T \delta - L^T \xi + F_W \varphi) - Q^+ (F_H B^T (A^+)^T \delta - F_H \xi), \quad (3.8)$$

where $Q = I_N + F_H B^T (A^+)^T A^+ B F_H - F_H B^T (A^+)^T J P^{-1} J^T A^+ B F_H$.

Clearly, $\partial f / \partial q = 0$ is equivalent to

$$F_A q = F_A \delta. \quad (3.9)$$

Upon substituting (3.7) and (3.9) into (2.13), (2.14), and (2.15), we obtain

$$\begin{aligned} \hat{\alpha} &= (J P^{-1} J^T A^+ B - A^+ B) F_H v + J P^{-1} (J^T \delta - L^T \xi + F_W \varphi) + F_A \delta, \\ \hat{\beta} &= (I_N - L P^{-1} J^T A^+ B) F_H v - L P^{-1} (J^T \delta - L^T \xi + F_W \varphi), \\ \hat{\gamma} &= P^{-1} J^T A^+ B F_H v + P^{-1} (J^T \delta - L^T \xi + F_W \varphi), \end{aligned} \quad (3.10)$$

where $F_H v$ is given by (3.8).

By now, we have proved the following result.

Theorem 3.1. *Let the real-valued symmetric $(2r + 1)$ -diagonal matrices M_a, D_a , and K_a be given. Then, Problem II has a unique solution, and the unique solution of Problem II can be expressed as*

$$\widehat{M} = S(\hat{\alpha} \otimes I_n), \quad \widehat{D} = S(\hat{\beta} \otimes I_n), \quad \widehat{K} = S(\hat{\gamma} \otimes I_n), \quad (3.11)$$

where $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$ are given by (3.10).

4. A numerical example

Based on Theorems 2.2 and 3.1, we can describe an algorithm for solving problem IQEP and Problem II as follows.

Example 4.1. Consider a five-DOF system modelled analytically with mass, damping, and stiffness matrices given by

$$M_a = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}, \quad D_a = \begin{bmatrix} 11.0 & -8.0 & 0 & 0 & 0 \\ -8.0 & 14.0 & -3.5 & 0 & 0 \\ 0 & -3.5 & 13.0 & -7.8 & 0 \\ 0 & 0 & -7.8 & 13.5 & -9.0 \\ 0 & 0 & 0 & -9.0 & 15.4 \end{bmatrix}, \quad K_a = \begin{bmatrix} 100 & -20 & 0 & 0 & 0 \\ -20 & 120 & -35 & 0 & 0 \\ 0 & -35 & 80 & -12 & 0 \\ 0 & 0 & -12 & 95 & -40 \\ 0 & 0 & 0 & -40 & 124 \end{bmatrix}. \quad (4.1)$$

- (1) Input $M_a, D_a, K_a, \Lambda, X$.
- (2) Compute the unitary transformation matrix T_p by (2.4).
- (3) Compute real matrices $\tilde{\Lambda}, \tilde{X}$ by (2.5) and (2.6), respectively.
- (4) Form the orthonormal basis $\{Y_{ij}\}$ by (2.1).
- (5) Compute G, A, B, C according to (2.9).
- (6) Compute the matrix $H = E_A B, W = E_H E_A C$.
- (7) Compute $L = H^+ E_A C F_W, J = A^+ B L - A^+ C F_W, P = J^T J + L^T L + I_N$.
- (8) Compute the matrix $Q = I_N + F_H B^T (A^+)^T A^+ B F_H - F_H B^T (A^+)^T J P^{-1} J^T A^+ B F_H$.
- (9) Form vectors δ, ξ, φ by (3.1) and (3.2).
- (10) Compute $F_H v$ by (3.8).
- (11) Compute $S, \hat{\alpha}, \hat{\beta}, \hat{\gamma}$ by (2.17) and (3.10), respectively.
- (12) Compute the unique solution $(\hat{M}, \hat{D}, \hat{K})$ of Problem II by (3.11).

Algorithm 1

That is, M_a, D_a, K_a are symmetric 3-diagonal matrices. The measured eigenvalue and eigenvector matrices $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $X = [x_1, x_2, x_3, x_4]$ are given by

$$\Lambda = \text{diag}\{-3.9525 + 6.8578i, -3.9525 - 6.8578i, -0.1622 + 3.3465i, -0.1622 - 3.3465i\},$$

$$X = \begin{bmatrix} 0.1099 + 0.4083i & 0.1099 - 0.4083i & -0.0283 + 0.1570i & -0.0283 - 0.1570i \\ 0.0684 - 0.1754i & 0.0684 + 0.1754i & 0.0525 + 0.3742i & 0.0525 - 0.3742i \\ -0.1272 - 0.2703i & -0.1272 + 0.2703i & 0.0594 + 0.6146i & 0.0594 - 0.6146i \\ 0.3403 + 0.5185i & 0.3403 - 0.5185i & -0.1132 + 0.5663i & -0.1132 - 0.5663i \\ -0.3783 - 0.4108i & -0.3783 + 0.4108i & -0.0862 + 0.3309i & -0.0862 - 0.3309i \end{bmatrix}. \quad (4.2)$$

Using Algorithm 1, we obtain the unique solution of Problem II as follows:

$$\hat{M} = \begin{bmatrix} 1.6010 & 0.3152 & 0 & 0 & 0 \\ 0.3152 & 2.7909 & 1.5318 & 0 & 0 \\ 0 & 1.5318 & 2.9332 & 0.9771 & 0 \\ 0 & 0 & 0.9771 & 3.7779 & 1.2094 \\ 0 & 0 & 0 & 1.2094 & 3.6053 \end{bmatrix},$$

$$\hat{D} = \begin{bmatrix} 9.0659 & -6.4506 & 0 & 0 & 0 \\ -6.4506 & -5.5001 & 7.3677 & 0 & 0 \\ 0 & 7.3677 & 4.1831 & -6.4524 & 0 \\ 0 & 0 & -6.4524 & 10.2736 & -5.8430 \\ 0 & 0 & 0 & -5.8430 & 10.3568 \end{bmatrix}, \quad (4.3)$$

$$\hat{K} = \begin{bmatrix} 98.7229 & -25.5781 & 0 & 0 & 0 \\ -25.5781 & 124.0044 & -32.9787 & 0 & 0 \\ 0 & -32.9787 & 77.7282 & -12.0909 & 0 \\ 0 & 0 & -12.0909 & 99.5825 & -32.9904 \\ 0 & 0 & 0 & -32.9904 & 120.7917 \end{bmatrix}.$$

We define the residual as

$$\text{res}(\lambda_i, x_i) = \|(\lambda_i^2 \widehat{M} + \lambda_i \widehat{D} + \widehat{K})x_i\|, \quad (4.4)$$

and the numerical results shown in the following table.

(λ_i, x_i)	(λ_1, x_1)	(λ_2, x_2)	(λ_3, x_3)	(λ_4, x_4)
$\text{res}(\lambda_i, x_i)$	1.9584e-013	1.9584e-013	9.0344e-014	9.0344e-014

Therefore, the prescribed eigenvalues (the diagonal elements of the matrix Λ) and eigenvectors (the column vectors of the matrix X) are embedded in the new model $(\lambda^2 \widehat{M} + \lambda \widehat{D} + \widehat{K})x = 0$ and the updated matrices $\widehat{M}, \widehat{D}, \widehat{K}$ are also symmetric 3-diagonal matrices, which implies that the structural connectivity information of the analytical model is preserved.

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