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Research Article

Explicit Solution of the Inverse Eigenvalue Problem of Real Symmetric Matrices and Its Application to Electrical Network Synthesis

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A novel procedure for explicit construction of the entries of real symmetric matrices with assigned spectrum and the entries of the corresponding orthogonal modal matrices is presented. The inverse eigenvalue problem of symmetric matrices with some specific sign patterns (including hyperdominant one) is explicitly solved too. It has been shown to arise thereof a possibility of straightforward solving the inverse eigenvalue problem of symmetric hyperdominant matrices with assigned nonnegative spectrum. The results obtained are applied thereafter in synthesis of driving-point immittance functions of transformerless, common-ground, two-element-kind *RLC* networks and in generation of their equivalent realizations.

1. Introduction

During the past few decades,many papers [1–16] have studied the inverse eigenvalue problems (IEPs) of various types. The solution existence of the specific IEPs was generally considered in [1, 3–8, 10, 11, 13, 14] without explicit formulation of the corresponding procedure for solution construction, whereas in [2, 9, 12, 15, 16] this has been accomplished. The main result of [16] is the proof that IEP of symmetric hyperdominant (hd) matrices with assigned nonnegative spectrum has at least one solution which has also been constructed. This settled an old IEP opened in [17].Hyperdominant matrices have nonnegative diagonal and nonpositive off-diagonal entries and nonnegative hd margins of rows (hd margin of a row is the sum of entries in that row). The tool used in [16] to construct the nth-order hd matrix with assigned spectrum was the nth-order orthogonal Hessenberg matrix constructed as a special product of $n-1$ plane rotations [15]. Hessenberg matrices naturally arise in study of symmetric tridiagonal matrices, skew symmetric, and orthogonal matrices [13, 14, 18]. A matrix is upper (lower) Hessenberg if its entry (k, m) vanishes whenever $k > m+1$ $(m > k+1)$.

In practical work, it is commonly assumed to be better not to form Hessenberg matrices explicitly, but to keep them as products of plane rotations. On the other hand, explicit construction of real symmetric matrices with nonnegative spectrum, which either have hd sign pattern or are truly hd, is proved to be an inevitable task in considering the synthesis of driving-point immittance functions of passive, transformerless, common-ground, two element-kind *RLC* networks and in generation of their equivalent realizations [17–19]. *RLC* networks are comprised solely of resistors (R), inductors (L), and capacitors (C). Driving-point immittance function of a lumped, time invariant, linear electrical network is either a driving-point impedance $Z(s)$, or a driving-point admittance $Y(s) = Z^{-1}(s)$ $(s = \sigma + j\cdot\omega$ is the complex frequency; σ, ω are real numbers; $j := \sqrt{-1})$. It is well known that a *real rational function* in s can be driving-point immittance function of *RLC* network *if and only if* it is *positive real* function in s; or similarly, a *necessary* condition for a stable square matrix $\mathbf{W}(s)$ of real rational functions in s to be driving-point immittance matrix of a passive *RLC* network is that $\mathbf{W}(s)$ be *positive real matrix* [20, 21]. A few tests for ascertaining *positive real* properties of functions and/or matrices can be found in [20, 21]. In [22] it has been pointed out the role of hd matrices in synthesis of both passive and active, transformerless, common-ground multiports. Unlike [16], this paper presents explicit construction of entries of real symmetric matrices with arbitrarily assigned spectrum and the entries of the corresponding orthogonal modal matrices. It also presents explicit construction of real symmetric matrices with assigned spectrum and with specific sign patterns (including hd one). Thereof, a solution to the IEP of symmetric, truly hd matrices with assigned nonnegative spectrum is produced. Some of the obtained results are then applied in synthesis of driving-point immitances of transformerless, common-ground, two-element-kind *RLC* networks and in generation of their equivalent realizations. The two proposed realization procedures are illustrated by an example.Throughout the paper $\oplus$ denotes direct sum, $\mathbf{x}^T$ denotes transpose of $\mathbf{x}$, bold capital letters denote matrices and $\mathbf{I}_k$ stands for the kth-order unit or identity matrix.

2. Explicit solution to the IEP of real symmetric matrices by using canonic orthogonal transformations

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be assigned spectrum of the sought real symmetric matrices and let $\mathbf{G}_1 := \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be $n \times n$ spectral matrix. Consider a set of 2×2 orthogonal matrices $\mathbf{P}_k \in \{\mathbf{A}_k, \mathbf{B}_k, \mathbf{C}_k, \mathbf{D}_k\}$ $(k = 1, \ldots, n-1)$:

$$\mathbf{P}_k := \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}, \quad \mathbf{A}_k := \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix}, \quad \mathbf{B}_k := \begin{bmatrix} \cos\theta_k & \sin\theta_k \\ \sin\theta_k & -\cos\theta_k \end{bmatrix},$$
$$\mathbf{C}_k := \begin{bmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{bmatrix}, \quad \mathbf{D}_k := \begin{bmatrix} -\cos\theta_k & \sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix}, \quad \theta_k \in \left[0, \frac{\pi}{2}\right], \tag{2.1}$$

which are either rotators ($\mathbf{A}_k$ and $\mathbf{C}_k$) or reflectors ($\mathbf{B}_k$ and $\mathbf{D}_k$). A useful set of $n\times n$ orthogonal matrices is

$$\mathbf{U}_1 := \mathbf{P}_1 \oplus \mathbf{I}_{n-2}, \qquad \mathbf{U}_k := \mathbf{I}_{k-1} \oplus \mathbf{P}_k \oplus \mathbf{I}_{n-k-1} \quad (k = 2, \ldots, n-2), \qquad \mathbf{U}_{n-1} := \mathbf{I}_{n-2} \oplus \mathbf{P}_{n-1}. \tag{2.2}$$

From the following two matrix recurrent relations

$$\mathbf{G}_{k+1} := \mathbf{U}_k \mathbf{G}_k \mathbf{U}_k^{\mathrm{T}}, \quad \mathbf{S}_{k+1} := \mathbf{U}_{n-k} \mathbf{S}_k \mathbf{U}_{n-k}^{\mathrm{T}} \quad (k = 1, \dots, n-1), \tag{2.3}$$

we readily obtain $n \times n$ real symmetric matrices $\mathbf{G}_n$ and $\mathbf{S}_n$, which are both congruent and similar to $\mathbf{G}_1$

$$\mathbf{G}_n = \mathbf{U}\mathbf{G}_1\mathbf{U}^{\mathrm{T}}, \quad \mathbf{U} := \mathbf{U}_{n-1}\mathbf{U}_{n-2}\cdots\mathbf{U}_1; \qquad \mathbf{S}_n = \mathbf{V}\mathbf{G}_1\mathbf{V}^{\mathrm{T}}, \quad \mathbf{V} := \mathbf{V}_1\mathbf{V}_2\cdots\mathbf{V}_{n-1}. \tag{2.4}$$

Columns of the orthogonal modal matrix $\mathbf{U}$ ($\mathbf{V}$) correspond to eigenvectors of $\mathbf{G}_n$ ($\mathbf{S}_n$). Out of $(n-1)!$ different possibilities of using (2.3) in generation of $\mathbf{G}_n$ and $\mathbf{S}_n$, only the two selected by (2.4) produce explicit expressions of entries of $\mathbf{G}_n$ and $\mathbf{S}_n$ in terms of $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and the entries of $\mathbf{P}_k$ ($k = 1, \dots, n-1$). $\mathbf{U}$ ($\mathbf{V}$) from (2.4) will be shown later to take on lower (upper) Hessenberg form with the entries explicitly expressed too. For the sake of brevity, we will restrict our consideration only to the first of relations (2.3), bearing on mind the possibility of treating the second one similarly. For $k = 1$ and $k = 2$ we readily obtain $\mathbf{G}_2$ and $\mathbf{G}_3$, by using (2.1) and (2.3):

$$\mathbf{G}_2 = \left(\mathbf{P}_1 \oplus \mathbf{I}_{n-2}\right)\mathbf{G}_1\left(\mathbf{P}_1^{\mathrm{T}} \oplus \mathbf{I}_{n-2}\right)$$

$$= \left[\begin{array}{cc|c} a_1 & b_1 & \mathbf{0}_{2,n-2} \\ c_1 & d_1 & \\ \hline \mathbf{0}_{2,n-2}^{\mathrm{T}} & & \mathbf{I}_{n-2} \end{array}\right] \left[\begin{array}{cc|cccc} \lambda_1 & 0 & & \mathbf{0}_{2,n-2} & & \\ 0 & \lambda_2 & & & & \\ \hline & & \lambda_3 & & & \\ & & & & \mathbf{0} & \\ \mathbf{0}_{2,n-2}^{\mathrm{T}} & & & \ddots & & \\ & & \mathbf{0} & & & \\ & & & & & \lambda_n \end{array}\right] \left[\begin{array}{cc|c} a_1 & c_1 & \mathbf{0}_{2,n-2} \\ b_1 & d_1 & \\ \hline \mathbf{0}_{2,n-2}^{\mathrm{T}} & & \mathbf{I}_{n-2} \end{array}\right]$$

$$= \left[\begin{array}{cc|cccc} \lambda_1 a_1^2 + \lambda_2 b_1^2 & (\lambda_1 - \lambda_2) a_1 c_1 & & \mathbf{0}_{2,n-2} & & \\ (\lambda_1 - \lambda_2) a_1 c_1 & \lambda_1 c_1^2 + \lambda_2 d_1^2 & & & & \\ \hline & & \lambda_3 & & & \\ & & & & \mathbf{0} & \\ \mathbf{0}_{2,n-2}^{\mathrm{T}} & & & \ddots & & \\ & & \mathbf{0} & & & \\ & & & & & \lambda_n \end{array}\right],$$

$$\left\{\begin{array}{c} \text{More generally, } \forall k = 1, \dots, n-1 \text{ it holds} \\ a_k b_k + c_k d_k = a_k c_k + b_k d_k = 0 \\ a_k^2 + b_k^2 = c_k^2 + d_k^2 = a_k^2 + c_k^2 = b_k^2 + d_k^2 = 1 \end{array}\right\},$$

$$
\begin{aligned}
\mathbf{G}_3 &= \left(1 \oplus \mathbf{P}_2 \oplus \mathbf{I}_{n-3}\right)\mathbf{G}_2\left(1 \oplus \mathbf{P}_2^{\mathrm{T}} \oplus \mathbf{I}_{n-3}\right) \\
&= \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & a_2 & b_2 & \mathbf{0}_{3,n-3} \\ 0 & c_2 & d_2 & \\ \hline & \mathbf{0}_{3,n-3}^{\mathrm{T}} & & \mathbf{I}_{n-3} \end{array}\right]
\left[\begin{array}{c|c} \begin{array}{c|c} \lambda_1 a_1^2 + \lambda_2 b_1^2 & (\lambda_1 - \lambda_2) a_1 c_1 \\ \hline (\lambda_1 - \lambda_2) a_1 c_1 & \lambda_1 c_1^2 + \lambda_2 d_1^2 \end{array} & \mathbf{0}_{2,n-2} \\ \hline \mathbf{0}_{2,n-2}^{\mathrm{T}} & \begin{matrix} \lambda_3 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{matrix} \end{array}\right]
\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & a_2 & c_2 & \mathbf{0}_{3,n-3} \\ 0 & b_2 & d_2 & \\ \hline & \mathbf{0}_{3,n-3}^{\mathrm{T}} & & \mathbf{I}_{n-3} \end{array}\right] \\
&= \left[\begin{array}{c|c} \begin{array}{c|c|c} \lambda_1 a_1^2 + \lambda_2 b_1^2 & (\lambda_1 - \lambda_2) a_1 a_2 c_1 & (\lambda_1 - \lambda_2) a_1 c_1 c_2 \\ \hline (\lambda_1 - \lambda_2) a_1 a_2 c_1 & (\lambda_1 c_1^2 + \lambda_2 d_1^2) a_2^2 + \lambda_3 b_2^2 & (\lambda_1 c_1^2) + \lambda_2 d_1^2 - \lambda_3 a_2 c_2 \\ \hline (\lambda_1 - \lambda_2) a_1 c_1 c_2 & (\lambda_1 c_1^2 + \lambda_2 d_1^2 - \lambda_3) a_2 c_2 & (\lambda_1 c_1^2 + \lambda_2 d_1^2) c_2^2 + \lambda_3 d_2^2 \end{array} & \mathbf{0}_{3,n-3} \\ \hline \mathbf{0}_{3,n-3}^{\mathrm{T}} & \begin{matrix} \lambda_4 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{matrix} \end{array}\right].
\end{aligned}
\tag{2.5}
$$

Let $\lambda_1^* := \lambda_1$, $\varepsilon_1 := (\lambda_1^* - \lambda_2) a_1$, and $\mathbf{x}_2 := c_1 \varepsilon_1$, and let us firstly introduce in (2.5) the following notation:

$$
\begin{gathered}
\mathbf{M}_2 := \lambda_1 a_1^2 + \lambda_2 b_1^2, \qquad \lambda_2^* := \lambda_1^* c_1^2 + \lambda_2 d_1^2, \qquad \mathbf{D}_2^* := \operatorname{diag}\left(\lambda_2^*, \lambda_3\right), \qquad \mathbf{D}_2 := \operatorname{diag}\left(\lambda_4, \ldots, \lambda_n\right), \\
\mathbf{A}_2 := \left[\ \mathbf{x}_2 \mid 0\ \right]^{\mathrm{T}}, \qquad \mathbf{M}_3 := \left[\begin{array}{c|c} \lambda_1 a_1^2 + \lambda_2 b_1^2 & (\lambda_1 - \lambda_2) a_1 a_2 c_1 \\ \hline (\lambda_1 - \lambda_2) a_1 a_2 c_1 & (\lambda_1 c_1^2 + \lambda_2 d_1^2) a_2^2 + \lambda_3 b_2^2 \end{array}\right], \\
\lambda_3^* := \left(\lambda_1 c_1^2 + \lambda_2 d_1^2\right) c_2^2 + \lambda_3 d_2^2, \qquad \mathbf{D}_3^* := \operatorname{diag}\left(\lambda_3^*, \lambda_4\right), \\
\mathbf{D}_3 := \operatorname{diag}\left(\lambda_5, \ldots, \lambda_n\right), \qquad \mathbf{A}_3 := \left[\begin{array}{c|c} (\lambda_1 - \lambda_2) a_1 c_1 c_2 & (\lambda_1 c_1^2 + \lambda_2 d_1^2 - \lambda_3) a_2 c_2 \\ \hline 0 & 0 \end{array}\right].
\end{gathered}
\tag{2.6}
$$

Thereafter, observing the partition of $\mathbf{G}_2$ and $\mathbf{G}_3$ obtained in (2.5)

$$
\mathbf{G}_2 = \left[\begin{array}{c|c|c} \mathbf{M}_2 & \mathbf{A}_2^{\mathrm{T}} & \mathbf{0}_{1,n-3} \\ \hline \mathbf{A}_2 & \mathbf{D}_2^* & \mathbf{0}_{2,n-3} \\ \hline \mathbf{0}_{1,n-3}^{\mathrm{T}} & \mathbf{0}_{2,n-3}^{\mathrm{T}} & \mathbf{D}_2 \end{array}\right], \qquad \mathbf{G}_3 = \left[\begin{array}{c|c|c} \mathbf{M}_3 & \mathbf{A}_3^{\mathrm{T}} & \mathbf{0}_{2,n-4} \\ \hline \mathbf{A}_3 & \mathbf{D}_3^* & \mathbf{0}_{2,n-4} \\ \hline \mathbf{0}_{2,n-4}^{\mathrm{T}} & \mathbf{0}_{2,n-4}^{\mathrm{T}} & \mathbf{D}_3 \end{array}\right], \tag{2.7}
$$

it can readily be anticipated the partition of subsequent matrices $\mathbf{G}_k$ $(k = 4, \ldots, n-2)$ as follows:

$$
\mathbf{G}_k = \left[\begin{array}{c|c|c} \mathbf{M}_k & \mathbf{A}_k^{\mathrm{T}} & \mathbf{0}_{k-1,n-k-1} \\ \hline \mathbf{A}_k & \mathbf{D}_k^* & \mathbf{0}_{2,n-k-1} \\ \hline \mathbf{0}_{k-1,n-k-1}^{\mathrm{T}} & \mathbf{0}_{2,n-k-1}^{\mathrm{T}} & \mathbf{D}_k \end{array}\right], \qquad \mathbf{x}_k := \left[x_{k,1} \ \ x_{k,2} \cdots x_{k,k-1}\right], \quad \mathbf{A}_k := \begin{bmatrix} \mathbf{x}_k \\ \mathbf{0}_{1,k-1} \end{bmatrix}, \tag{2.8}
$$

where $\mathbf{M}_k$ is the symmetric $(k-1)\times(k-1)$ matrix, $\mathbf{x}_k$ is $1\times(k-1)$ row vector, $\mathbf{A}_k$ is $2\times(k-1)$ matrix, λ_k^* is modified eigenvalue λ_k, $\mathbf{D}_k^* := \operatorname{diag}(\lambda_k^*,\lambda_{k+1})$ and $\mathbf{D}_k := \operatorname{diag}(\lambda_{k+2},\ldots,\lambda_n)$. For $k=2,\ldots,n-3$ from (2.1)–(2.3), (2.8) it follows that

$$\mathbf{G}_{k+1} = \left(\mathbf{I}_{k-1}\oplus\mathbf{P}_k\oplus\mathbf{I}_{n-k-1}\right)\mathbf{G}_k\left(\mathbf{I}_{k-1}\oplus\mathbf{P}_k^{\mathrm{T}}\oplus\mathbf{I}_{n-k-1}\right) = \left[\begin{array}{c|c|c} \mathbf{M}_k & \mathbf{A}_k^{\mathrm{T}}\mathbf{P}_k^{\mathrm{T}} & \mathbf{0}_{k-1,n-k-1} \\ \hline \mathbf{P}_k\mathbf{A}_k & \mathbf{P}_k\mathbf{D}_k^*\mathbf{P}_k^{\mathrm{T}} & \mathbf{0}_{2,n-k-1} \\ \hline \mathbf{0}_{k-1,n-k-1}^{\mathrm{T}} & \mathbf{0}_{2,n-k-1}^{\mathrm{T}} & \mathbf{D}_k \end{array}\right],$$

$$\mathbf{P}_k\mathbf{A}_k = \left[\begin{array}{c|c|c} a_k x_{k,1} & \cdots & a_k x_{k,k-1} \\ \hline c_k x_{k,1} & \cdots & c_k x_{k,k-1} \end{array}\right], \qquad \mathbf{P}_k\mathbf{D}_k^*\mathbf{P}_k^{\mathrm{T}} = \left[\begin{array}{c|c} \lambda_k^* a_k^2 + \lambda_{k+1} b_k^2 & \left(\lambda_k^* - \lambda_{k+1}\right) a_k c_k \\ \hline \left(\lambda_k^* - \lambda_{k+1}\right) a_k c_k & \lambda_k^* c_k^2 + \lambda_{k+1} d_k^2 \end{array}\right]. \tag{2.9}$$

For $k=2,\ldots,n-3$, let us define: $\lambda_{k+1}^* := \lambda_k^* c_k^2 + \lambda_{k+1} d_k^2$, $\varepsilon_k := (\lambda_k^* - \lambda_{k+1}) a_k$, $\psi_{kk} := \lambda_k^* a_k^2 + \lambda_{k+1} b_k^2$ and thereafter $\mathbf{D}_{k+1} := \operatorname{diag}(\lambda_{k+3},\ldots,\lambda_n)$ and $\mathbf{D}_{k+1}^* := \operatorname{diag}(\lambda_k^* c_k^2 + \lambda_{k+1} d_k^2, \lambda_{k+2})$. Then, from (2.8)-(2.9) it follows the identification

$$\mathbf{M}_{k+1} := \left[\begin{array}{c|c} \mathbf{M}_k & a_k\mathbf{x}_k^{\mathrm{T}} \\ \hline a_k\mathbf{x}_k & \psi_{kk} \end{array}\right], \qquad \mathbf{A}_{k+1} = \begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{0}_{1,k} \end{bmatrix} := \left[\begin{array}{c|c} c_k\mathbf{x}_k & c_k\varepsilon_k \\ \hline \mathbf{0}_{1,k} & 0 \end{array}\right] = \left[\begin{array}{c|c|c|c} c_k x_{k,1} & \cdots & c_k x_{k,k-1} & c_k\varepsilon_k \\ \hline 0 & \cdots & 0 & 0 \end{array}\right], \tag{2.10}$$

which enables the partition of $\mathbf{G}_{k+1}$ in (2.9) to be like that of $\mathbf{G}_k$ in (2.8), and that partition of $\mathbf{x}_{k+1}$ be rather simple

$$\mathbf{G}_{k+1} = \left[\begin{array}{c|c|c} \mathbf{M}_{k+1} & \mathbf{A}_{k+1}^{\mathrm{T}} & \mathbf{0}_{k,n-k-2} \\ \hline \mathbf{A}_{k+1} & \mathbf{D}_{k+1}^* & \mathbf{0}_{2,n-k-2} \\ \hline \mathbf{0}_{k,n-k-2}^{\mathrm{T}} & \mathbf{0}_{2,n-k-2}^{\mathrm{T}} & \mathbf{D}_{k+1} \end{array}\right], \quad \mathbf{x}_{k+1} := c_k\left[\ \mathbf{x}_k \mid \varepsilon_k\ \right], \quad k=2,\ldots,n-3. \tag{2.11}$$

Let $\psi_{11} := \mathbf{M}_2$. Having uncovered the partition pattern of $\mathbf{M}_{k+1}$ $(k=2,\ldots,n-3)$, we can pursue partitioning of $\mathbf{M}_{n-2}$ backwardly from $\mathbf{M}_{n-2}$ to $\mathbf{M}_2$, by using (2.10). Afterwards, we can produce $\mathbf{G}_{n-2}$, by using (2.10)-(2.11). The results are

$$\mathbf{M}_{n-2} = \left[\begin{array}{ccc|c} \begin{array}{cc|c} \begin{array}{c|c} \psi_{11} & a_2\mathbf{x}_2^{\mathrm{T}} \\ \hline a_2\mathbf{x}_2 & \psi_{22} \end{array} & & a_3\mathbf{x}_3^{\mathrm{T}} \\ \hline a_3\mathbf{x}_3 & & \psi_{33} \end{array} & \cdots & & a_{n-3}\mathbf{x}_{n-3}^{\mathrm{T}} \\ \vdots & \ddots & & \\ \hline & a_{n-3}\mathbf{x}_{n-3} & & \psi_{n-3,n-3} \end{array}\right], \qquad \mathbf{G}_{n-2} = \left[\begin{array}{cc|c} \begin{array}{c|c} \begin{array}{c|c} \mathbf{M}_{n-2} & \mathbf{x}_{n-2}^{\mathrm{T}} \\ \hline \mathbf{x}_{n-2} & \lambda_{n-2}^* \end{array} & \mathbf{0}_{n-2,1} \\ \hline \mathbf{0}_{1,n-2} & \lambda_{n-1} \end{array} & & \mathbf{0}_{n-1,1} \\ \hline \mathbf{0}_{1,n-1} & & \lambda_n \end{array}\right]. \tag{2.12}$$

Since $\mathbf{G}_{n-1} := (\mathbf{I}_{n-3}\oplus\mathbf{P}_{n-2}\oplus 1)\mathbf{G}_{n-2}(\mathbf{I}_{n-3}\oplus\mathbf{P}_{n-2}^{\mathrm{T}}\oplus 1)$ and

$$\mathbf{P}_{n-2}\begin{bmatrix} \lambda_{n-2}^* & 0 \\ 0 & \lambda_{n-1} \end{bmatrix}\mathbf{P}_{n-2}^{\mathrm{T}} = \left[\begin{array}{c|c} \lambda_{n-2}^* a_{n-2}^2 + \lambda_{n-1} b_{n-2}^2 & \left(\lambda_{n-2}^* - \lambda_{n-1}\right) a_{n-2} c_{n-2} \\ \hline \left(\lambda_{n-2}^* - \lambda_{n-1}\right) a_{n-2} c_{n-2} & \lambda_{n-2}^* c_{n-2}^2 + \lambda_{n-1} d_{n-2}^2 \end{array}\right], \tag{2.13}$$

then after defining $\psi_{n-2,n-2} := \lambda_{n-2}^{*}a_{n-2}^{2} + \lambda_{n-1}b_{n-2}^{2}$, $\varepsilon_{n-2} := (\lambda_{n-2}^{*} - \lambda_{n-1})a_{n-2}$, $\lambda_{n-1}^{*} := \lambda_{n-2}^{*}c_{n-2}^{2} + \lambda_{n-1}d_{n-2}^{2}$ and $\mathbf{x}_{n-1} := c_{n-2}[\ \mathbf{x}_{n-2} \mid \varepsilon_{n-2}\ ]$, it follows from (2.12)-(2.13)

$$\mathbf{G}_{n-1} = \left[\begin{array}{c|c} \begin{array}{c|c} \begin{array}{c|c} \begin{array}{c|c} \begin{array}{cc} \begin{array}{c|c} \begin{array}{c|c} \psi_{11} & a_2\mathbf{x}_2^{\mathrm{T}} \\ \hline a_2\mathbf{x}_2 & \psi_{22} \end{array} & a_3\mathbf{x}_3^{\mathrm{T}} \\ \hline a_3\mathbf{x}_3 & \psi_{33} \end{array} & \cdots \\ \vdots & \ddots \end{array} & a_{n-3}\mathbf{x}_{n-3}^{\mathrm{T}} \\ \hline a_{n-3}\mathbf{x}_{n-3} & \psi_{n-3,n-3} \end{array} & a_{n-2}\mathbf{x}_{n-2}^{\mathrm{T}} \\ \hline a_{n-2}\mathbf{x}_{n-2} & \psi_{n-2,n-2} \end{array} & \mathbf{x}_{n-1}^{\mathrm{T}} \\ \hline \mathbf{x}_{n-1} & \lambda_{n-1}^{*} \end{array} & \mathbf{0}_{n-1,1} \\ \hline \mathbf{0}_{1,n-1} & \lambda_n \end{array}\right]. \tag{2.14}$$

Since $\mathbf{G}_n := (\mathbf{I}_{n-2} \oplus \mathbf{P}_{n-1})\mathbf{G}_{n-1}(\mathbf{I}_{n-2} \oplus \mathbf{P}_{n-1}^{\mathrm{T}})$ and

$$\mathbf{P}_{n-1}\begin{bmatrix} \lambda_{n-1}^{*} & 0 \\ 0 & \lambda_n \end{bmatrix}\mathbf{P}_{n-1}^{\mathrm{T}} = \left[\begin{array}{c|c} \lambda_{n-1}^{*}a_{n-1}^{2} + \lambda_n b_{n-1}^{2} & \left(\lambda_{n-1}^{*} - \lambda_n\right)a_{n-1}c_{n-1} \\ \hline \left(\lambda_{n-1}^{*} - \lambda_n\right)a_{n-1}c_{n-1} & \lambda_{n-1}^{*}c_{n-1}^{2} + \lambda_n d_{n-1}^{2} \end{array}\right], \tag{2.15}$$

then on introducing $\psi_{n-1,n-1} := \lambda_{n-1}^{*}a_{n-1}^{2} + \lambda_n b_{n-1}^{2}$, $\varepsilon_{n-1} := (\lambda_{n-1}^{*} - \lambda_n)a_{n-1}$, and $\lambda_n^{*} := \lambda_{n-1}^{*}c_{n-1}^{2} + \lambda_n d_{n-1}^{2}$, we obtain from (2.14)-(2.15) the partition of $\mathbf{G}_n$ which is amenable to the production of its entries in explicit form and is suitable for further discussion about solving some specific IEPs

$$\mathbf{G}_{n} = \left[\begin{array}{c|c|c} \begin{array}{c|c} \begin{array}{cc} \begin{array}{c|c} \begin{array}{c|c} \psi_{11} & a_2\mathbf{x}_2^{\mathrm{T}} \\ \hline a_2\mathbf{x}_2 & \psi_{22} \end{array} & a_3\mathbf{x}_3^{\mathrm{T}} \\ \hline a_3\mathbf{x}_3 & \psi_{33} \end{array} & \cdots \\ \vdots & \ddots \end{array} & a_{n-2}\mathbf{x}_{n-2}^{\mathrm{T}} \\ \hline a_{n-2}\mathbf{x}_{n-2} & \psi_{n-2,n-2} \end{array} & a_{n-1}\mathbf{x}_{n-1}^{\mathrm{T}} & c_{n-1}\mathbf{x}_{n-1}^{\mathrm{T}} \\ \hline a_{n-1}\mathbf{x}_{n-1} & \psi_{n-1,n-1} & c_{n-1}\varepsilon_{n-1} \\ \hline c_{n-1}\mathbf{x}_{n-1} & c_{n-1}\varepsilon_{n-1} & \lambda_n^{*} \end{array}\right]. \tag{2.16}$$

For $k = 2,\ldots,n$, we consecutively obtain from $\lambda_1^{*} := \lambda_1$ and $\lambda_k^{*} := \lambda_{k-1}^{*}c_{k-1}^{2} + \lambda_k d_{k-1}^{2}$ that generally it holds

$$\lambda_k^{*} = \left(c_1c_2\cdots c_{k-1}\right)^2\lambda_1 + \left(d_1c_2\cdots c_{k-1}\right)^2\lambda_2 + \cdots + \left(d_{k-2}c_{k-1}\right)^2\lambda_{k-1} + d_{k-1}^{2}\lambda_k, \quad k = 2,\ldots,n. \tag{2.17}$$

Since $\psi_{11} = \lambda_1 a_1^2 + \lambda_2 b_1^2$ and $\psi_{kk} := \lambda_k^{*}a_k^2 + \lambda_{k+1}b_k^2$ $(k = 2,\ldots,n-1)$, then from (2.17) it follows that

$$\begin{aligned} \psi_{kk} = {} & \left(c_1c_2\cdots c_{k-1}a_k\right)^2\lambda_1 + \left(d_1c_2\cdots c_{k-1}a_k\right)^2\lambda_2 + \cdots + \left(d_{k-2}c_{k-1}a_k\right)^2\lambda_{k-1} \\ & + \left(d_{k-1}a_k\right)^2\lambda_k + b_k^2\lambda_{k+1}, \quad k = 2,\ldots,n-1. \end{aligned} \tag{2.18}$$

Observe that it is not necessary to calculate "ψ"s from (2.18), but only the modified eigenvalues from (2.17) since it holds $\varepsilon_k = (\lambda_k^{*} - \lambda_{k+1})a_k$ and $\psi_{kk} = \lambda_k^{*}a_k^2 + \lambda_{k+1}b_k^2 =$

$(\lambda_k^* - \lambda_{k+1})a_k^2 + (a_k^2 + b_k^2)\lambda_{k+1} = a_k\varepsilon_k + \lambda_{k+1}$ $(k = 1, \ldots, n-1)$. As it is $\mathbf{x}_2 = c_1\varepsilon_1$, then for $k = 2, \ldots, n-2$ from (2.10) it follows that

$$
\begin{aligned}
\mathbf{x}_{k+1} &= c_k\left[\, \mathbf{x}_k \mid \varepsilon_k \,\right] \\
&= c_k\left[\, c_{k-1}\left[\, \mathbf{x}_{k-1} \mid \varepsilon_{k-1} \,\right] \mid \varepsilon_k \,\right] \\
&= \left[\, c_k c_{k-1}\mathbf{x}_{k-1} \mid c_k c_{k-1}\varepsilon_{k-1} \mid c_k\varepsilon_k \,\right] \\
&= \cdots = \left[\, c_k c_{k-1}\cdots c_2\mathbf{x}_2 \mid c_k c_{k-1}\cdots c_2\varepsilon_2 \mid \cdots \mid c_k c_{k-1}\varepsilon_{k-1} \mid c_k\varepsilon_k \,\right] \\
&= \left[\, c_k c_{k-1}\cdots c_2 c_1\varepsilon_1 \mid c_k c_{k-1}\cdots c_2\varepsilon_2 \mid \cdots \mid c_k c_{k-1}\varepsilon_{k-1} \mid c_k\varepsilon_k \,\right]
\end{aligned}
\tag{2.19}
$$

$$
a_k\mathbf{x}_k = \left[\, a_k c_{k-1}c_{k-2}\cdots c_2 c_1\varepsilon_1 \mid a_k c_{k-1}c_{k-2}\cdots c_2\varepsilon_2 \mid \cdots \mid a_k c_{k-1}c_{k-2}\varepsilon_{k-2} \mid a_k c_{k-1}\varepsilon_{k-1} \,\right], \quad k = 2, \ldots, n-1.
\tag{2.20}
$$

The real symmetric matrix $\mathbf{G}_n$ with assigned spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and the explicitly expressed entries can be derived from (2.16) and (2.20), bearing on mind that "ψ"s and "ε"s are calculated by using $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\mathbf{P}_k$, modified eigenvalues (2.17) and $\varepsilon_k = (\lambda_k^* - \lambda_{k+1})a_k$ $(k = 1, \ldots, n-1)$:

$$
\mathbf{G}_n = \left[\begin{array}{c|c|c|c|c|c|c}
\psi_{11} & a_2c_1\varepsilon_1 & a_3c_2c_1\varepsilon_1 & a_4c_3c_2c_1\varepsilon_1 & \cdots & \mathcal{P} & \mathcal{S} \\ \hline
a_2c_1\varepsilon_1 & \psi_{22} & a_3c_2\varepsilon_2 & a_4c_3c_2\varepsilon_2 & \cdots & \mathcal{U} & \mathcal{F} \\ \hline
a_3c_2c_1\varepsilon_1 & a_3c_2\varepsilon_2 & \vdots & \vdots & \cdots & \vdots & \vdots \\ \hline
a_4c_3c_2c_1\varepsilon_1 & a_4c_3c_2\varepsilon_2 & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline
\mathcal{P} & \mathcal{U} & \cdots & \cdots & \cdots & \psi_{n-1,n-1} & c_{n-1}\varepsilon_{n-1} \\ \hline
\mathcal{S} & \mathcal{F} & \cdots & \cdots & \cdots & c_{n-1}\varepsilon_{n-1} & \lambda_n^*
\end{array}\right],
\tag{2.21}
$$

where $\mathcal{P}$ denotes $a_{n-1}c_{n-2}\cdots c_2c_1\varepsilon_1$, $\mathcal{S}$ denotes $c_{n-1}c_{n-2}\cdots c_2c_1\varepsilon_1$, $\mathcal{U}$ denotes $a_{n-1}c_{n-2}\cdots c_2\varepsilon_2$, and $\mathcal{F}$ denotes $c_{n-1}c_{n-2}\cdots c_2\varepsilon_2$. Entries of $\mathbf{G}_n = \mathbf{G}_n^{\mathrm{T}} = [g_{km}]_{n\times n}$ are $g_{kk} = \psi_{kk}$ $(k = 1, \ldots, n-1)$, $g_{nn} = \lambda_n^*$, $g_{km} = a_k c_{k-1}c_{k-2}\cdots c_m\varepsilon_m$ $(k > m;\ k = 2, \ldots, n-1)$ and $g_{nm} = c_{n-1}c_{n-2}\cdots c_m\varepsilon_m$ $(m = 1, \ldots, n-1)$. They are calculated according to the following steps:

(a) Select arbitrarily the entries $\{a_k, b_k, c_k, d_k\}$ of 2×2 orthogonal matrices $\mathbf{P}_k$ $(k = 1, \ldots, n-1)$, given by (2.1);

(b) with $\lambda_1^* := \lambda_1$, calculate the modified eigenvalues λ_k^* $(k = 2, \ldots, n)$, by using (2.17);

(c) calculate $\varepsilon_k = (\lambda_k^* - \lambda_{k+1})a_k$ and $\psi_{kk} = a_k\varepsilon_k + \lambda_{k+1}$ $(k = 1, \ldots, n-1)$;

(d) calculate the entries of $\mathbf{G}_n$, by using (2.21).

Matrix $\mathbf{U}$ (2.4) is $n \times n$ orthogonal modal matrix established from eigenvectors of $\mathbf{G}_n$. We will now prove that $\mathbf{U}$ is not only orthogonal, but also lower Hessenberg with explicitly expressed entries. Let us firstly produce $\mathbf{U}_1^{\mathrm{T}}\mathbf{U}_2^{\mathrm{T}}$ and $\mathbf{U}_1^{\mathrm{T}}\mathbf{U}_2^{\mathrm{T}}\mathbf{U}_3^{\mathrm{T}}$, whose partition will enable us to anticipate the partition of $\mathbf{U}_1^{\mathrm{T}}\mathbf{U}_2^{\mathrm{T}}\mathbf{U}_3^{\mathrm{T}}\cdots\mathbf{U}_k^{\mathrm{T}}$ $(k = 4,\dots,n-1)$

$$\mathbf{U}_1^{\mathrm{T}}\mathbf{U}_2^{\mathrm{T}} = \left[\begin{array}{ccc|c} a_1 & a_2c_1 & c_2c_1 & \\ b_1 & a_2d_1 & c_2d_1 & \mathbf{0}_{3,n-3} \\ 0 & b_2 & d_2 & \\ \hline & \mathbf{0}_{3,n-3}^{\mathrm{T}} & & \mathbf{I}_{n-3} \end{array}\right], \qquad \mathbf{U}_1^{\mathrm{T}}\mathbf{U}_2^{\mathrm{T}}\mathbf{U}_3^{\mathrm{T}} = \left[\begin{array}{cccc|c} a_1 & a_2c_1 & a_3c_2c_1 & c_3c_2c_1 & \\ b_1 & a_2d_1 & a_3c_2d_1 & c_3c_2d_1 & \mathbf{0}_{4,n-4} \\ 0 & b_2 & a_3d_2 & c_3d_2 & \\ 0 & 0 & b_3 & d_3 & \\ \hline & & \mathbf{0}_{4,n-4}^{\mathrm{T}} & & \mathbf{I}_{n-4} \end{array}\right]. \tag{2.22}$$

If we now suppose that $\mathbf{U}_1^{\mathrm{T}}\mathbf{U}_2^{\mathrm{T}}\cdots\mathbf{U}_k^{\mathrm{T}} := \mathbf{H}_{k+1}\oplus\mathbf{I}_{n-k-1}$ $(k = 2,\dots,n-1)$, where $\mathbf{H}_{k+1}$ is orthogonal $(k+1)\times(k+1)$ upper Hessenberg matrix

$$\mathbf{H}_{k+1} = \left[\begin{array}{c|c|c|c|c|c} a_1 & a_2c_1 & a_3c_2c_1 & \cdots & a_kc_{k-1}\cdots c_2c_1 & c_kc_{k-1}\cdots c_2c_1 \\ \hline b_1 & a_2d_1 & a_3c_2d_1 & \cdots & a_kc_{k-1}\cdots c_2d_1 & c_kc_{k-1}\cdots c_2d_1 \\ \hline 0 & b_2 & a_3d_2 & \cdots & a_kc_{k-1}\cdots c_3d_2 & c_kc_{k-1}\cdots c_3d_2 \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline 0 & \cdots & 0 & b_{k-1} & a_kd_{k-1} & c_kd_{k-1} \\ \hline 0 & \cdots & \cdots & 0 & b_k & d_k \end{array}\right], \tag{2.23}$$

then since according to (2.2), it holds $\mathbf{U}_{k+1} := \mathbf{I}_k \oplus \mathbf{P}_{k+1} \oplus \mathbf{I}_{n-k-2}$, we may write further for $k = 2,\dots,n-2$

$$\begin{aligned} \mathbf{U}_1^{\mathrm{T}}\mathbf{U}_2^{\mathrm{T}}\cdots\mathbf{U}_k^{\mathrm{T}}\mathbf{U}_{k+1}^{\mathrm{T}} &= \left(\mathbf{H}_{k+1}\oplus\mathbf{I}_{n-k-1}\right)\mathbf{U}_{k+1}^{\mathrm{T}} \\ &= \left(\mathbf{H}_{k+1}\oplus 1\oplus\mathbf{I}_{n-k-2}\right)\left(\mathbf{I}_k\oplus\mathbf{P}_{k+1}^{\mathrm{T}}\oplus\mathbf{I}_{n-k-2}\right) \\ &= \left[\left(\mathbf{H}_{k+1}\oplus 1\right)\left(\mathbf{I}_k\oplus\mathbf{P}_{k+1}^{\mathrm{T}}\right)\right]\oplus\mathbf{I}_{n-k-2} \\ &= \mathbf{H}_{k+2}\oplus\mathbf{I}_{n-k-2}, \quad \text{where } \mathbf{H}_{k+2} := \left(\mathbf{H}_{k+1}\oplus 1\right)\left(\mathbf{I}_k\oplus\mathbf{P}_{k+1}^{\mathrm{T}}\right). \end{aligned} \tag{2.24}$$

By using (2.2), (2.23)-(2.24), it follows that

$$\mathbf{H}_{k+2} = \left[\begin{array}{c|c|c|c|c|c|c} a_1 & a_2c_1 & a_3c_2c_1 & \cdots & a_kc_{k-1}\cdots c_2c_1 & c_kc_{k-1}\cdots c_2c_1 & 0 \\ \hline b_1 & a_2d_1 & a_3c_2d_1 & \cdots & a_kc_{k-1}\cdots c_2d_1 & c_kc_{k-1}\cdots c_2d_1 & 0 \\ \hline 0 & b_2 & a_3d_2 & \cdots & a_kc_{k-1}\cdots c_3d_2 & c_kc_{k-1}\cdots c_3d_2 & 0 \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline 0 & \cdots & 0 & b_{k-1} & a_kd_{k-1} & c_kd_{k-1} & 0 \\ \hline 0 & \cdots & \cdots & 0 & b_k & d_k & 0 \\ \hline 0 & \cdots & \cdots & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c|cc} \mathbf{I}_k & \mathbf{0}_{k,2} & \\ \hline \mathbf{0}_{k,2}^{\mathrm{T}} & a_{k+1} & c_{k+1} \\ & b_{k+1} & d_{k+1} \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c|c|c} a_1 & a_2c_1 & a_3c_2c_1 & \cdots & a_kc_{k-1}\cdots c_2c_1 & a_{k+1}c_kc_{k-1}\cdots c_2c_1 & c_{k+1}c_kc_{k-1}\cdots c_2c_1 \\ \hline b_1 & a_2d_1 & a_3c_2d_1 & \cdots & a_kc_{k-1}\cdots c_2d_1 & a_{k+1}c_kc_{k-1}\cdots c_2d_1 & c_{k+1}c_kc_{k-1}\cdots c_2d_1 \\ \hline 0 & b_2 & a_3d_2 & \cdots & a_kc_{k-1}\cdots c_3d_2 & a_{k+1}c_kc_{k-1}\cdots c_3d_2 & c_{k+1}c_kc_{k-1}\cdots c_3d_2 \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline 0 & \cdots & 0 & b_{k-1} & a_kd_{k-1} & a_{k+1}c_kd_{k-1} & c_{k+1}c_kd_{k-1} \\ \hline 0 & \cdots & \cdots & 0 & b_k & a_{k+1}d_k & c_{k+1}d_k \\ \hline 0 & \cdots & \cdots & 0 & 0 & b_{k+1} & d_{k+1} \end{array}\right], \tag{2.25}$$

and thereby it is proved our previous assumption that $\mathbf{U}_1^{\mathrm{T}}\mathbf{U}_2^{\mathrm{T}}\cdots\mathbf{U}_k^{\mathrm{T}} := \mathbf{H}_{k+1}\oplus\mathbf{I}_{n-k-1}$ ($k = 2,\dots,n-1$), where $\mathbf{H}_{k+1}$ (2.23) is the orthogonal upper Hessenberg $(k+1)\times(k+1)$ matrix with entries expressed explicitly. And finally, for $k = n-1$ from $\mathbf{U}_1^{\mathrm{T}}\mathbf{U}_2^{\mathrm{T}}\cdots\mathbf{U}_k^{\mathrm{T}} := \mathbf{H}_{k+1}\oplus\mathbf{I}_{n-k-1}$ and (2.4), (2.23), we obtain $\mathbf{H}_n = \mathbf{U}^{\mathrm{T}} = \mathbf{U}_1^{\mathrm{T}}\mathbf{U}_2^{\mathrm{T}}\cdots\mathbf{U}_{n-1}^{\mathrm{T}}$ and

$$\mathbf{U}=\mathbf{H}_n^{\mathrm{T}}=\mathbf{U}_{n-1}\mathbf{U}_{n-2}\cdots\mathbf{U}_1=\left[\begin{array}{c|c|c|c|c|c|c}
a_1 & b_1 & 0 & 0 & \cdots & 0 & 0\\ \hline
a_2c_1 & a_2d_1 & b_2 & 0 & \cdots & 0 & 0\\ \hline
a_3c_2c_1 & a_3c_2d_1 & a_3d_2 & \vdots & \cdots & \vdots & \vdots\\ \hline
a_4c_3c_2c_1 & a_4c_3c_2d_1 & a_4c_3d_2 & \vdots & \vdots & \vdots & \vdots\\ \hline
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \hline
a_{n-1}c_{n-2}\cdots c_2c_1 & a_{n-1}c_{n-2}\cdots c_2d_1 & \cdots & \cdots & \cdots & a_{n-1}d_{n-2} & b_{n-1}\\ \hline
c_{n-1}c_{n-2}\cdots c_2c_1 & c_{n-1}c_{n-2}\cdots c_2d_1 & \cdots & \cdots & \cdots & c_{n-1}d_{n-2} & d_{n-1}
\end{array}\right]. \tag{2.26}$$

The entries of the orthogonal lower Hessenberg matrix $\mathbf{U} = [u_{km}]$ $(k,m = 1,\dots,n)$ are defined as follows:

$$\begin{gathered}
u_{km} = 0 \quad (m > k+1;\ k,m = 1,\dots,n), \qquad u_{k,k+1} = b_k \quad (k = 1,\dots,n-1), \qquad u_{11} = a_1,\\
u_{kk} = a_k d_{k-1} \quad (k = 2,\dots,n-1), \qquad u_{k,1} = a_k c_{k-1}c_{k-2}\cdots c_2c_1 \quad (k = 2,\dots,n-1),\\
u_{n,1} = c_{n-1}c_{n-2}\cdots c_1, \qquad u_{n,k} = c_{n-1}c_{n-2}\cdots c_k d_{k-1} \quad (k = 2,\dots,n-1),\\
u_{km} = a_k c_{k-1}\cdots c_m d_{m-1} \quad (m+1 \le k \le n-1;\ m = 2,\dots,n-1).
\end{gathered} \tag{2.27}$$

By using the similar arguments as in derivation of entries of matrix $\mathbf{U}$, the orthogonal matrix $\mathbf{V}$ which is to be produced by using (2.4) can be shown to take on upper Hessenberg form. Proving of this fact goeswith similar paces that were used for obtaining $\mathbf{U}$ and it is left to the reader.

3. The explicit solution of the IEP of real symmetric matrices with some specific sign patterns

Let the real eigenvalues from the spectrum $\{\lambda_1,\lambda_2,\dots,\lambda_n\}$ be arbitrarily enumerated, thereby establishing the sequence $\{\lambda_k\}$ $(k = 1,\dots,n)$. The nonnegative sequence will be denoted by $\{\lambda_k\} \ge 0$, and the nonpositive one by $\{\lambda_k\} \le 0$ $(k = 1,\dots,n)$. Firstly, we will prove two lemmas.

Lemma 3.1. *If the sequence* $\{\lambda_k\} \ge 0$ $(k = 1,\dots,n)$ *is increasing [decreasing], then in* (2.21) $\lambda_n^* \ge 0$, $\psi_{mm} \ge 0$, *and the sequence* $\{a_m\varepsilon_m\} \le 0[\{a_m\varepsilon_m\} \ge 0]$ $(m = 1,\dots,n-1)$.

Proof. Since $\{\lambda_k\} \ge 0$ $(k = 1,\dots,n)$, then it is trivial to see from (2.17) and (2.18) that all diagonal entries of $\mathbf{G}_n$ are nonnegative, that is, $\lambda_n^* \ge 0$ and $\psi_{mm} \ge 0$ $(m = 1,\dots,n-1)$ no matter whether the sequence $\{\lambda_k\} \ge 0$ is increasing or decreasing. By virtue of orthogonality of $\mathbf{P}_k$, we have $c_k^2 + d_k^2 = 1$ $(k = 1, \ldots, n)$. If $\{\lambda_k\}$ $(k = 1, \ldots, n)$ is *increasing* sequence, then for $m = 1$ we have $a_1\varepsilon_1 = (\lambda_1^* - \lambda_2)a_1^2 = (\lambda_1 - \lambda_2)a_1^2 \le 0$ and for $m = 2, \ldots, n-1$ we obtain

$$\begin{aligned}
&d_{m-1}^2\lambda_m - \lambda_{m+1} \\
&\qquad \le d_{m-1}^2\lambda_m - \lambda_m = \lambda_m\left(1 - d_{m-1}^2\right) = -c_{m-1}^2\lambda_m \le -c_{m-1}^2\lambda_{m-1}, \\
&\left(d_{m-2}c_{m-1}\right)^2\lambda_{m-1} + d_{m-1}^2\lambda_m - \lambda_{m+1} \\
&\qquad \le \left(d_{m-2}c_{m-1}\right)^2\lambda_{m-1} - c_{m-1}^2\lambda_{m-1} = -\left(c_{m-2}c_{m-1}\right)^2\lambda_{m-1} \le -\left(c_{m-2}c_{m-1}\right)^2\lambda_{m-2}, \\
&\left(d_{m-3}c_{m-2}c_{m-1}\right)^2\lambda_{m-2} + \left(d_{m-2}c_{m-1}\right)^2\lambda_{m-1} + d_{m-1}^2\lambda_m - \lambda_{m+1} \\
&\qquad \le -\left(c_{m-3}c_{m-2}c_{m-1}\right)^2\lambda_{m-2} \le -\left(c_{m-3}c_{m-2}c_{m-1}\right)^2\lambda_{m-3}, \\
&\left(d_1c_2\cdots c_{m-1}\right)^2\lambda_2 + \cdots + \left(d_{m-2}c_{m-1}\right)^2\lambda_{m-1} + d_{m-1}^2\lambda_m - \lambda_{m+1} \\
&\qquad \le -\left(c_1\cdots c_{m-1}\right)^2\lambda_2 \le -\left(c_1\cdots c_{m-1}\right)^2\lambda_1.
\end{aligned} \tag{3.1}$$

From (2.17) and the last of inequalities (3.1) it follows $\lambda_m^* \le \lambda_{m+1}$ and $a_m\varepsilon_m = (\lambda_m^* - \lambda_{m+1})a_m^2 \le 0$ $(m = 2, \ldots, n-1)$. If $\{\lambda_k\}$ $(k = 1, \ldots, n)$ is *decreasing* sequence, then for $m = 1$ we have $a_1\varepsilon_1 = (\lambda_1^* - \lambda_2)a_1^2 = (\lambda_1 - \lambda_2)a_1^2 \ge 0$ and for $m = 2, \ldots, n-1$ we obtain

$$\begin{aligned}
&d_{m-1}^2\lambda_m - \lambda_{m+1} \\
&\qquad \ge d_{m-1}^2\lambda_m - \lambda_m = -\lambda_m\left(1 - d_{m-1}^2\right) = -c_{m-1}^2 \ge -c_{m-1}^2\lambda_{m-1}, \\
&\left(d_{m-2}c_{m-1}\right)^2\lambda_{m-1} + d_{m-1}^2\lambda_m - \lambda_{m+1} \\
&\qquad \ge \left(d_{m-2}c_{m-1}\right)^2\lambda_{m-1} - c_{m-1}^2\lambda_{m-1} = -\left(c_{m-2}c_{m-1}\right)^2\lambda_{m-1} \ge -\left(c_{m-2}c_{m-1}\right)^2\lambda_{m-2}, \\
&\left(d_{m-3}c_{m-2}c_{m-1}\right)^2\lambda_{m-2} + \left(d_{m-2}c_{m-1}\right)^2\lambda_{m-1} + d_{m-1}^2\lambda_m - \lambda_{m+1} \\
&\qquad \ge -\left(c_{m-3}c_{m-2}c_{m-1}\right)^2\lambda_{m-2} \ge -\left(c_{m-3}c_{m-2}c_{m-1}\right)^2\lambda_{m-3}, \\
&\left(d_1c_2\cdots c_{m-1}\right)^2\lambda_2 + \cdots + \left(d_{m-2}c_{m-1}\right)^2\lambda_{m-1} + d_{m-1}^2\lambda_m - \lambda_{m+1} \\
&\qquad \ge -\left(c_1\cdots c_{m-1}\right)^2\lambda_2 \ge -\left(c_1\cdots c_{m-1}\right)^2\lambda_1.
\end{aligned} \tag{3.2}$$

From (2.17) and the last of inequalities (3.2) it follows $\lambda_m^* \ge \lambda_{m+1}$ and $a_m\varepsilon_m = (\lambda_m^* - \lambda_{m+1})a_m^2 \ge 0$ $(m = 2, \ldots, n-1)$. This completes the proof of lemma. For a *nonpositive* sequence, an analogous lemma can be formulated. □

Lemma 3.2. *If the sequence* $\{\lambda_k\} \leq 0$ $(k = 1, \ldots, n)$ *is increasing [decreasing], then in* (2.21) $\lambda_n^* \leq 0$, $\psi_{mm} \leq 0$ *and the sequence* $\{a_m\varepsilon_m\} \leq 0[\{a_m\varepsilon_m\} \geq 0]$ $(m = 1, \ldots, n-1)$.

Proof. It is similar to that of Lemma 3.1, but in this case the diagonal entries of $\mathbf{G}_n$ are nonpositive, that is, $\lambda_n^* \leq 0$ and $\psi_{mm} \leq 0$ $(m = 1, \ldots, n-1)$, no matter whether the sequence $\{\lambda_k\} \leq 0$ is increasing or decreasing (see (2.18)). □

Now, we shall formulate a new theorem related to explicit solving of IEP of real symmetric matrices with some specific sign patterns.

Theorem 3.3. *If* θ_k $(k = 1, \ldots, n-1)$ *are arbitrarily selected angles from the range* $[0, \pi/2]$, *then the entries of real symmetric matrices* $\mathbf{G}_n$ *with assigned spectrum* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, *produced by* (2.21), *can attain the following twelve sign patterns (zero entries are permitted), depending on selection of matrices* $\mathbf{P}_k$ $(k = 1, \ldots, n-1)$ *(see* (2.1)*).*

Case 1.

$$\mathbf{P}_k = \mathbf{A}_k = \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix}, \quad \text{or} \quad \mathbf{P}_k = \mathbf{B}_k = \begin{bmatrix} \cos\theta_k & \sin\theta_k \\ \sin\theta_k & -\cos\theta_k \end{bmatrix} \Longrightarrow$$

$$\mathbf{G}_n = \begin{bmatrix} + & & & \\ & + & + & \\ & & \ddots & \\ & + & & \\ & & & + \end{bmatrix}, \qquad \mathbf{G}_n = \begin{bmatrix} - & & & \\ & - & + & \\ & & \ddots & \\ & + & & \\ & & & - \end{bmatrix},$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \qquad \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 \leq 0 \tag{3.3}$$

$$\mathbf{G}_n = \begin{bmatrix} + & & & \\ & + & - & \\ & & \ddots & \\ & - & & \\ & & & + \end{bmatrix}, \qquad \mathbf{G}_n = \begin{bmatrix} - & & & \\ & - & - & \\ & & \ddots & \\ & - & & \\ & & & - \end{bmatrix}.$$

$$\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 \geq 0 \qquad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 0$$

Case 2.

$$\mathbf{P}_k = \mathbf{C}_k = \begin{bmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{bmatrix} \Longrightarrow$$

$$\mathbf{G}_n = \begin{bmatrix} + & - & + & \cdots & (-1)^{n-1} \\ - & + & - & \cdots & (-1)^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{n-2} & \cdots & \cdots & + & - \\ (-1)^{n-1} & \cdots & \cdots & - & + \end{bmatrix}, \qquad \mathbf{G}_n = \begin{bmatrix} - & - & + & \cdots & (-1)^{n-1} \\ - & - & - & \cdots & (-1)^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{n-2} & \cdots & \cdots & - & - \\ (-1)^{n-1} & \cdots & \cdots & - & - \end{bmatrix},$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \qquad \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 \leq 0$$

$$\mathbf{G}_n = \begin{bmatrix} + & + & - & \cdots & (-1)^n \\ + & + & + & \cdots & (-1)^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{n-1} & \cdots & \cdots & + & + \\ (-1)^n & \cdots & \cdots & + & + \end{bmatrix}, \qquad \mathbf{G}_n = \begin{bmatrix} - & + & - & \cdots & (-1)^n \\ + & - & + & \cdots & (-1)^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (-1)^{n-1} & \cdots & \cdots & - & + \\ (-1)^n & \cdots & \cdots & + & - \end{bmatrix}.$$

$$\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 \geq 0 \qquad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 0 \tag{3.4}$$

Case 3.

$$\mathbf{P}_k = \mathbf{D}_k = \begin{bmatrix} -\cos\theta_k & \sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix} \Longrightarrow$$

$$\begin{gathered} \mathbf{G}_n = \begin{bmatrix} + & + & + & \cdots & + & - \\ + & + & + & \cdots & + & - \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ + & + & + & \cdots & + & - \\ - & - & - & \cdots & - & + \end{bmatrix}, \qquad \mathbf{G}_n = \begin{bmatrix} - & + & + & \cdots & + & - \\ + & - & + & \cdots & + & - \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ + & + & \cdots & \cdots & - & - \\ - & - & \cdots & \cdots & - & - \end{bmatrix}, \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \qquad \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 \leq 0 \\ \mathbf{G}_n = \begin{bmatrix} + & - & - & \cdots & - & + \\ - & + & - & \cdots & - & + \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ - & - & \cdots & \cdots & + & + \\ + & + & \cdots & \cdots & + & + \end{bmatrix}, \qquad \mathbf{G}_n = \begin{bmatrix} - & - & - & \cdots & - & + \\ - & - & - & \cdots & - & + \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ - & - & \cdots & \cdots & - & + \\ + & + & \cdots & \cdots & + & + \end{bmatrix}. \\ \lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 \geq 0 \qquad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 0 \end{gathered} \tag{3.5}$$

Proof. If $\theta_k \in [0, \pi/2]$, then the signs of a_k and c_k depend solely on selection of canonic orthogonal matrices $\mathbf{P}_k$ $(k = 1, \ldots, n-1)$. For any sign of sequence $\{\lambda_m\}$ $(m = 1, \ldots, n)$ and its monotonicy realized through enumeration of its members, one can readily check the sign patterns stated above: by using (2.18) to determine signs of the diagonal entries in $\mathbf{G}_n$ and by using Lemma 3.1 or Lemma 3.2 to determine signs of ε_k $(k = 1, \ldots, n-1)$. Observe that only in Case 1 when $\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1 \geq 0$, that is, when the sequence $\{\lambda_m\}$ $(m = 1, \ldots, n)$ is nonnegative and increasing (but not strictly), matrix $\mathbf{G}_n$ is produced with hd sign pattern, including the possible presence of zero entries. $\mathbf{G}_n$ may attain a sparse structure if, for example, some eigenvalues are equal. To see that, let us firstly suppose $\lambda_1 = \cdots = \lambda_k = \lambda$. Then from (2.17)-(2.18) it follows that $\lambda_1^* = \cdots = \lambda_k^* = \lambda$, $\psi_{11} = \cdots = \psi_{k-1,k-1} = \lambda$ and $\varepsilon_1 = \cdots = \varepsilon_{k-1} = 0$, thus obviously making the matrix $\mathbf{G}_n$ (2.21) with sparse structure. By using (2.17)-(2.18), (2.21) and both Lemmas, we can readily infer that if $\theta_k \in (0, \pi/2)$ $(k = 1, \ldots, n-1)$ and the sequence $\{\lambda_m\}$ $(m = 1, \ldots, n)$ is *strictly monotone*, then matrix $\mathbf{G}_n$ (2.21) is produced with no zero entries in all three considered cases. □

Remark 3.4. Let $\lambda_1\lambda_2 \cdots \lambda_n \neq 0$. Then, since $\mathbf{G}_n = \mathbf{U}\mathbf{G}_1\mathbf{U}^{\mathrm{T}}$ and $\mathbf{U}^{-1} = \mathbf{U}^{\mathrm{T}}$ (recall that $\mathbf{U}$ is orthogonal), it follows that $\mathbf{G}_n^{-1} = (\mathbf{U}^{\mathrm{T}})^{-1}\mathbf{G}_1^{-1}\mathbf{U}^{-1} = \mathbf{U}\mathbf{G}_1^{-1}\mathbf{U}^{\mathrm{T}}$. Also, when the sequence $\{\lambda_m\}$ $(m = 1, \ldots, n)$ is increasing (decreasing), then the sequence $\{\lambda_m^{-1}\}$ $(m = 1, \ldots, n)$ is decreasing (increasing). These facts and Theorem 3.3 offer a possibility of determining the sign pattern of $\mathbf{G}_n^{-1}$ without really inverting $\mathbf{G}_n$. Furthermore, by using (2.17)-(2.18), (2.21), $\mathbf{G}_n^{-1}$ can be calculated explicitly, also without really inverting $\mathbf{G}_n$.

Theorem 3.5. *Let the positive increasing sequence $\{\lambda_m\}$ $(m = 1,\ldots,n)$ be the spectrum of $\mathbf{G}_n$ produced by using Case 1 of Theorem 3.3. Then there always exists such a diagonal matrix $\mathbf{D} :=$ diag$(d_1, d_2,\ldots,d_n)$ with positive diagonal entries which makes $\mathbf{DG}_n\mathbf{D}$ truly hyperdominant.*

Proof. If $\mathbf{G}_1 := \operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$, then by Case 1 of Theorem 3.3, the nonsingular matrix $\mathbf{G}_n = \mathbf{UG}_1\mathbf{U}^{\mathrm{T}}$ will have hd sign pattern and by Remark 3.4$\mathbf{G}_n^{-1} = \mathbf{UG}_1^{-1}\mathbf{U}^{\mathrm{T}}$ will be nonnegative matrix. Since $d_m > 0$ $(m = 1,\ldots,n)$, then the nonsingular symmetric matrix $\mathbf{DG}_n\mathbf{D}$ is produced with hd sign pattern, but it may not be truly hd, unless hd margin of each of its rows (or columns) is nonnegative (recall that hd margin of a row or a column is sum of all entries in that row or column). If $\mathbf{G}_n = [g_{km}]$ $(k,m = 1,\ldots,n)$, then hd margin p_k of the kth row (or the kth column) in $\mathbf{DG}_n\mathbf{D}$ is given by

$$p_k = \sum_{m=1}^{n} g_{km}d_md_k = d_k\alpha_k, \quad \text{where } \alpha_k := \sum_{m=1}^{n} g_{km}d_m,\ k = 1,\ldots,n. \tag{3.6}$$

Let we arbitrarily select $\alpha_k > 0$ $(k = 1,\ldots,n)$ and let $\mathbf{a} := [\alpha_1\ \ \alpha_2\cdots\alpha_n]^{\mathrm{T}}$, col$(\mathbf{D}) := [d_1\ \ d_2\cdots d_n]^{\mathrm{T}}$ and $\mathbf{p} := [p_1\ \ p_2\cdots p_n]^{\mathrm{T}}$. Then, from (3.6) it follows that $\mathbf{G}_n$ col$(\mathbf{D}) = \mathbf{a}$, that is, col$(\mathbf{D}) = \mathbf{G}_n^{-1}\mathbf{a} > \mathbf{0}_{n,1}$ and $\mathbf{p} = \mathbf{Da} > \mathbf{0}_{n,1}$. This not only means that $\mathbf{DG}_n\mathbf{D}$ has hd sign pattern, but that it is truly hd furthermore. Obviously, as much as "α"s are assumed greater, the greater will be row (column) hd margins of $\mathbf{DG}_n\mathbf{D}$. This completes the proof of theorem. □

4. Explicit solution of IEP of hd matrices with uncommitted and with assigned nonnegative spectrum

Theorem 4.1. *Let θ_k $(k = 1,\ldots,n-1)$ be a set of angles selected from the range $[0,\pi/2]$ and let $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ be* uncommitted *nonnegative spectrum of the real symmetric matrix $\mathbf{G}_n = \mathbf{UG}_1\mathbf{U}^{\mathrm{T}}$ $[\mathbf{G}_1 = \operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)]$ which is to be produced as truly hd. Suppose that through enumeration of eigenvalues the sequence $\{\lambda_m\} \geq 0$ $(m = 1,\ldots,n)$ is made increasing. Then, matrix $\mathbf{G}_n$ given by* (2.21) *will be truly hyperdominant if λ_1 is sufficiently great.*

Proof. Since by assumption the conditions of Theorem 3.3 (Case 1) are satisfied, then $\mathbf{G}_n$ produced by using (2.21) has hd sign pattern. As it is $\varepsilon_k = (\lambda_k^* - \lambda_{k+1})a_k$ $(k = 1,\ldots,n-1)$, then from (2.17)-(2.18), (2.21) it follows that hd margin p_m of the mth row (or column) from $\mathbf{G}_n$ $(m = 1,\ldots,n)$ can be in general represented as

$$\begin{gathered} p_m = \alpha_1^{(m)}\lambda_1 + \alpha_2^{(m)}\lambda_2 + \cdots + \alpha_m^{(m)}\lambda_m + \alpha_{m+1}^{(m)}\lambda_{m+1} \quad (m = 1,\ldots,n-1), \\ p_n = \alpha_1^{(n)}\lambda_1 + \alpha_2^{(n)}\lambda_2 + \cdots + \alpha_{n-1}^{(n)}\lambda_{n-1} + \alpha_n^{(n)}\lambda_n, \end{gathered} \tag{4.1}$$

where "α" coefficients are defined as follows:

$$\begin{aligned} \alpha_1^{(1)} &:= a_1\big(a_1 + a_2c_1 + a_3c_2c_1 + \cdots + a_{n-1}c_{n-2}\cdots c_2c_1 + c_{n-1}c_{n-2}\cdots c_2c_1\big), \quad m = 1, \\ \alpha_2^{(1)} &:= b_1\big(b_1 + a_2d_1 + a_3c_2d_1 + \cdots + a_{n-1}c_{n-2}\cdots c_2d_1 + c_{n-1}c_{n-2}\cdots c_2d_1\big), \end{aligned}$$

$$\begin{aligned} \alpha_1^{(m)} := \big(a_mc_{m-1}\cdots c_2c_1\big)\big(&a_1 + a_2c_1 + a_3c_2c_1 + \cdots + a_mc_{m-1}\cdots c_2c_1 \\ &+ \cdots + a_{n-1}c_{n-2}\cdots c_2c_1 + c_{n-1}c_{n-2}\cdots c_2c_1\big), \end{aligned}$$

$$
\begin{aligned}
\alpha_2^{(m)} &:= (a_m c_{m-1}\cdots c_2 d_1)(b_1 + a_2 d_1 + a_3 c_2 c_1 + \cdots + a_m c_{m-1}\cdots c_2 d_1 + \cdots + a_{n-1}c_{n-2}\cdots c_2 d_1 \\
&\qquad + c_{n-1}c_{n-2}\cdots c_2 d_1), \quad p = 3,\ldots,m, \\
\alpha_p^{(m)} &:= (a_m c_{m-1}\cdots c_p d_{p-1})(b_{p-1} + a_p d_{p-1} + a_{p+1}c_p d_{p-1} + \cdots + a_{n-1}c_{n-2}\cdots c_p d_{p-1} \\
&\qquad + c_{n-1}c_{n-2}\cdots c_p d_{p-1}), \quad m = 2,\ldots,n-1,\ p = 3,\ldots,m \\
\alpha_m^{(m)} &:= (a_m d_{m-1})(b_{m-1} + a_m d_{m-1} + a_{m+1}c_m d_{m-1} + \cdots + a_{n-1}c_{n-2}\cdots c_m d_{m-1} + c_{n-1}c_{n-2}\cdots c_m d_{m-1}), \\
\alpha_{m+1}^{(m)} &:= b_m(b_m + a_{m+1}d_m + a_{m+2}c_{m+1}d_m + \cdots + a_{n-1}c_{n-2}\cdots c_{m+1}d_m + c_{n-1}c_{n-2}\cdots c_{m+1}d_m), \\
&\qquad m = 2,\ldots,n-2, \\
\alpha_n^{(n-1)} &:= b_{n-1}(b_{n-1} + d_{n-1}), \\
\alpha_1^{(n)} &:= (c_{n-1}c_{n-2}\cdots c_2 c_1)(a_1 + a_2 c_1 + a_3 c_2 c_1 + \cdots + a_{n-1}c_{n-2}\cdots c_2 c_1 + c_{n-1}c_{n-2}\cdots c_2 c_1), \\
\alpha_2^{(n)} &:= (c_{n-1}c_{n-2}\cdots c_2 d_1)(b_1 + a_2 d_1 + a_3 c_2 d_1 + \cdots + a_{n-1}c_{n-2}\cdots c_2 d_1 + c_{n-1}c_{n-2}\cdots c_2 d_1) \\
\alpha_p^{(n)} &:= (c_{n-1}c_{n-2}\cdots c_p d_{p-1})(b_{p-1} + a_p d_{p-1} + a_{p+1}c_p d_{p-1} + \cdots + a_{n-1}c_{n-2}\cdots c_p d_{p-1} \\
&\qquad + c_{n-1}c_{n-2}\cdots c_p d_{p-1}), \quad p = 3,\ldots,n-1. \\
\alpha_n^{(n)} &:= d_{n-1}(b_{n-1} + d_{n-1}).
\end{aligned} \tag{4.2}
$$

According to Case 1 of Theorem 3.3, both a_k and c_k are nonnegative when $\theta_k \in [0,\pi/2]$ ($k = 1,\ldots,n-1$). Then, from (4.2) we see that $\alpha_1^{(m)} \geq 0$ ($m = 1,\ldots,n$), whereas other "α"s may be nonpositive. Since "α"s depend only on selection of "θ"s, then by presuming $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda \neq 0$, we obtain from (2.21) $\mathbf{G}_n = \lambda \mathbf{I}_n$ and $p_m = \lambda$ ($m = 1,\ldots,n$) and from (4.1) we conclude that in general it holds:

$$
\sum_{p=1}^{m+1} \alpha_p^{(m)} \equiv 1 \quad (m = 1,\ldots,n-1), \qquad \sum_{q=1}^{n} \alpha_q^{(n)} \equiv 1. \tag{4.3}
$$

Although $\mathbf{G}_n$ is produced with hd sign pattern, it will not be truly hd unless each of its row (column) hd margins is nonnegative, that is, $\mathbf{p} = [p_1 \;\; p_2 \cdots p_n]^{\mathrm{T}} \geq \mathbf{0}_{n,1}$. The column vector $\mathbf{p}$ with entries (4.1) can be written as

$$
\mathbf{p} = \begin{bmatrix}
\alpha_1^{(1)} & \alpha_2^{(1)} & 0 & 0 & \cdots & 0 \\
\alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_1^{(n-1)} & \alpha_2^{(n-1)} & \alpha_3^{(n-1)} & \cdots & \cdots & \alpha_n^{(n-1)} \\
\alpha_1^{(n)} & \alpha_2^{(n)} & \alpha_3^{(n)} & \cdots & \cdots & \alpha_n^{(n)}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
-1 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
0 & -1 & 1 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
\lambda_1 \\ \lambda_2 - \lambda_1 \\ \lambda_3 - \lambda_2 \\ \cdots \\ \lambda_{n-1} - \lambda_{n-2} \\ \lambda_n - \lambda_{n-1}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & \alpha_2^{(1)} & 0 & 0 & \cdots & 0 \\
1 & \sum_{r=2}^{3} \alpha_r^{(2)} & \alpha_3^{(2)} & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \sum_{s=2}^{n} \alpha_s^{(n-1)} & \sum_{u=3}^{n} \alpha_u^{(n-1)} & \cdots & \cdots & \alpha_n^{(n-1)} \\
1 & \sum_{t=2}^{n} \alpha_t^{(n)} & \sum_{w=3}^{n} \alpha_w^{(n)} & \cdots & \cdots & \alpha_n^{(n)}
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\ \lambda_2 - \lambda_1 \\ \lambda_3 - \lambda_2 \\ \cdots \\ \lambda_{n-1} - \lambda_{n-2} \\ \lambda_n - \lambda_{n-1}
\end{bmatrix}. \tag{4.4}
$$

From (4.4) we finally obtain

$$p_1 = \lambda_1 + (\lambda_2 - \lambda_1)\alpha_2^{(1)}, \qquad p_2 = \lambda_1 + (\lambda_2 - \lambda_1)\sum_{r=2}^{3}\alpha_r^{(2)} + (\lambda_3 - \lambda_2)\alpha_3^{(2)}, \ldots,$$
$$p_n = \lambda_1 + (\lambda_2 - \lambda_1)\sum_{t=2}^{n}\alpha_t^{(n)} + (\lambda_3 - \lambda_2)\sum_{w=3}^{n}\alpha_w^{(n)} + \cdots + (\lambda_n - \lambda_{n-1})\alpha_n^{(n)}. \tag{4.5}$$

Since "α"s and "$\sum$"s in (4.5) are not certainly nonnegative and since the sequence $\{\lambda_m\}$ $(m = 1, \ldots, n)$ is increasing, then firstly by arbitrary selection of differences $\lambda_2 - \lambda_1 \geq 0, \lambda_3 - \lambda_2 \geq 0, \ldots, \lambda_{n-1} - \lambda_{n-2} \geq 0$ and $\lambda_n - \lambda_{n-1} \geq 0$ and thereafter a sufficiently great $\lambda_1 \geq 0$, all hd margins p_m $(m = 1, \ldots, n)$ can be made nonnegative, that is, the matrix $\mathbf{G}_n$ can be always produced as truly hyperdominant. This completes the proof of this theorem. □

Presentation of explicit solution to the IEP of truly hd matrices with assigned nonnegative spectrum is now in order. It has been proved in [16] that this IEP always has at least one solution and that infinitely many others can be produced thereof by using Givens rotations. Solution of that IEP is important in electrical network synthesis of driving-point immittance functions and matrices of both passive and active, common-ground, transformerless, two-element-kind *RLC* networks and in generation of various classes of canonic and noncanonic equivalent realizations [19, 22]. In [16] we have proved the existence of solution to the IEP of hd matrices with assigned nonnegative spectrum, but here we shall present the explicit construction of solution matrix entries by using other arguments. This represents the explicit solution of the problem opened in [17].

Theorem 4.2. *For any set of real nonnegative numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ there always exists at least one (and infinitely many) $n \times n$ real symmetric hyperdominant matrices having these numbers as eigenvalues. In other words, IEP of symmetric hd matrices with assigned nonnegative spectrum always has at least one solution.*

Proof. We will take the same assumptions as in Theorem 4.1, except for $\theta_k \in (0, \pi/2)$ $(k = 1, \ldots, n-1)$. Through enumeration of eigenvalues, the nonnegative sequence $\{\lambda_m\}$ $(m = 1, \ldots, n)$ is made increasing. Then, according to Theorem 3.3 (Case 1), the symmetric matrix $\mathbf{G}_n = \mathbf{U}\mathbf{G}_1\mathbf{U}^{\mathrm{T}}$ $[\mathbf{G}_1 = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)]$ with spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and the entries determined by (2.21), is produced with hd sign pattern, no matter what selection of θ_ks $(k = 1, \ldots, n-1)$ has been made. Observe that in Case 1 $a_k = \cos\theta_k$ and $c_k = \sin\theta_k$. To make $\mathbf{G}_n = \mathbf{U}\mathbf{G}_1\mathbf{U}^{\mathrm{T}}$ truly hd, we will prove the existence of such "θ"s that make all "α"s (and hence all "p"s) in (4.1) nonnegative. Let we introduce the following positive sequence $\{M_m\}$ $(m = 1, \ldots, n)$

$$\begin{aligned}
M_1 &:= a_1 + a_2c_1 + a_3c_2c_1 + \cdots + a_{n-1}c_{n-2}\cdots c_2c_1 + c_{n-1}c_{n-2}\cdots c_2c_1\\
M_2 &:= a_2 + a_3c_2 + a_4c_3c_3 + \cdots + a_{n-1}c_{n-2}\cdots c_3c_2 + c_{n-1}c_{n-2}\cdots c_3c_2\\
M_3 &:= a_3 + a_4c_3 + a_5c_4c_3 + \cdots + a_{n-1}c_{n-2}\cdots c_4c_3 + c_{n-1}c_{n-2}\cdots c_4c_3\\
M_{n-2} &:= a_{n-2} + a_{n-1}c_{n-2} + c_{n-1}c_{n-2},\\
M_{n-1} &:= a_{n-1} + c_{n-1},\\
M_n &:= 1.
\end{aligned} \tag{4.6}$$

Then, by using (4.2) we obtain a consistent set of inequalities that ensure nonnegativity of all "α"s in (4.1)

$$\alpha_1^{(1)} = a_1 M_1 \geq 0, \qquad \alpha_2^{(1)} = 1 - a_1 M_1 = c_1(c_1 - a_1 M_2) \geq 0, \tag{4.7}$$

$$\alpha_1^{(k)} \geq 0, \qquad \alpha_p^{(k)} \geq 0 \Longleftrightarrow -c_{p-1} + a_{p-1} M_p \geq 0 \quad (p = 2, \ldots, k), \tag{4.8}$$

$$\alpha_{k+1}^{(k)} \geq 0 \Longleftrightarrow c_k - a_k M_{k+1} \geq 0 \quad (k = 2, \ldots, n-1), \tag{4.9}$$

$$\alpha_1^{(n)} \geq 0, \qquad \alpha_q^{(n)} \geq 0 \Longleftrightarrow -c_{q-1} + a_{q-1} M_q \geq 0 \quad (q = 2, \ldots, n-1), \tag{4.10}$$

$$\alpha_n^{(n)} \geq 0 \Longleftrightarrow -c_{n-1} + a_{n-1} \geq 0, \tag{4.11}$$

$$\alpha_n^{(n-1)} = b_{n-1}^2 + b_{n-1} d_{n-1} = c_{n-1}^2 - a_{n-1} c_{n-1} = c_{n-1}\left(c_{n-1} - a_{n-1}\right) \geq 0 \Longleftrightarrow c_{n-1} - a_{n-1} \geq 0. \tag{4.12}$$

For $p = 2$ from (4.8) we obtain $M_2 \geq c_1/a_1$ and from (4.7) $M_2 \leq c_1/a_1$. Then, $M_2 = c_1/a_1$, $\alpha_2^{(1)} = 0$, $\alpha_1^{(1)} = 1 - \alpha_2^{(1)} = 1$, $M_1 = 1/a_1$ and $p_1 = \lambda_1$. From (4.11)-(4.12) it follows that $a_{n-1} = c_{n-1}$ ($\leftrightarrow \theta_{n-1} = \pi/4$) and $\alpha_n^{(n-1)} = \alpha_n^{(n)} = 0$ [inequalty (4.12) is the same as (4.9) if $k = n-1$ ($M_n = 1$)]. For $k = 2, \ldots, (n-2)$, we obtain from (4.9) $M_{k+1} \leq c_k/a_k$ and for $q = 3, \ldots, (n-1)$, we obtain from (4.10) $M_q \geq c_{q-1}/a_{q-1}$. To summarize, we have proved that: (a) $M_r = c_{r-1}/a_{r-1}$, for $r = 2, \ldots, (n-1)$ and (b) $\{\alpha_1^{(k)} = 1, \alpha_s^{(k)} = 0\ [s = 2, \ldots, (k+1)]$ and $p_k = \lambda_1\}$, for $k = 1, \ldots, (n-1)$. And finally, from (4.10) we obtain $\alpha_q^{(n)} = 0$ for $q = 2, \ldots, n$, $\alpha_1^{(n)} = 1$ and $p_n = \lambda_1$. Since the matrix $\mathbf{G}_n$ has hd sign pattern and each of its row (column) hd margins is equal to $\lambda_1 \geq 0$, then $\mathbf{G}_n$ is truly hd matrix. This completes the proof of the theorem. □

Remark 4.3. It relates to calculation of entries of $\mathbf{G}_n$. In Theorem 4.2 it is proved that $M_1 = 1/a_1$ and $M_k = c_{k-1}/a_{k-1}$ ($k = 2, \ldots, n-1$). It is assumed $M_n = 1$. Since $\theta_{n-1} = \pi/4$, then $M_{n-1} = a_{n-1} + c_{n-1} = \cos\theta_{n-1} + \sin\theta_{n-1} = \sqrt{2}$. For $w = 1, \ldots, n-1$ it follows from (4.6)

$$\begin{aligned} M_w &= a_w + c_w M_{w+1} = a_w + c_w \frac{c_w}{a_w} = \frac{1}{a_w} = \sqrt{\frac{a_w^2 + c_w^2}{a_w^2}} = \sqrt{1 + \left(\frac{c_w}{a_w}\right)^2} \\ &= \sqrt{1 + M_{w+1}^2} = \sqrt{2 + M_{w+2}^2} = \cdots = \sqrt{n - w - 1 + M_{n-1}^2} = \sqrt{n - w + 1}, \\ a_w &= \cos\theta_w = \frac{1}{\sqrt{n - w + 1}}, \qquad c_w = \sin\theta_w = \sqrt{\frac{n - w}{n - w + 1}}. \end{aligned} \tag{4.13}$$

By using (2.17)-(2.18), (2.21), (4.13) we can easily calculate all entries of the (initial) hd matrix $\mathbf{G}_n$. Other hd matrices having the same spectrum can be produced thereof by application of Givens rotations, one at a time.

5. Application of the obtained results in electrical network synthesis

It is well known that synthesis methods of passive, common-ground, transformerless, two-element-kind *RLC* networks yield topological configurations which are severely restricted by the method chosen [19]. By using of the results above, a new class of non-canonic, driving-point immittance realizations of passive, common-ground, transformerless, two-element-kind *RLC* networks with minimum number of both nodes and elements of one kind can be generated with possibility of reduction in number of elements of other kind. The network synthesis is always performed by using normalization of both the complex frequency s and the impedance $Z(s)$. If Ω is a *selected* normalization frequency, then the normalized frequency is $s_n = s/\Omega$. Similarly, if R_0 is a *selected* normalization resistance, then the normalized impedance is $Z_n(s) = Z(s)/R_0$. Thereby we achieve [20]: (a) lesser dispersion of coefficients in normalized functions and (b) dimensionless manipulation of quantities. The normalized resistance of resistor R is $R_n := R/R_0$. The normalized impedance of an inductor L is $Z_{Ln}(s) = Ls/R_0 = (L\Omega/R_0)s_n = L_n s_n$ ($L_n := L\Omega/R_0$-normalized inductance). The normalized impedance of a capacitor C is $Z_{Cn}(s) = 1/(CR_0 s) = 1/[(R_0 C\Omega)s_n] = 1/(C_n s_n)$ ($C_n := R_0 C\Omega$-normalized capacitance). To physically realize a network after synthesis, a *denormalization* process must be performed. The actual parameter values of RLC elements are calculated as follows: $R = R_n R_0$, $L = L_n R_0/\Omega$, $C = C_n/(R_0\Omega)$. From now on it will be assumed that normalized synthesis is being carried out, but the lower index "n" we be dropped from component labels for brevity.

It is well known that if a real rational function in s can be realized as RL driving-point impedance $Z_{RL}(s)$, then it can be also realized as RC driving-point admittance $Y_{RC}(s)$ [20]. And similarly, if it can be realized as RL driving-point admittance $Y_{RL}(s)$, then it can also be realized as RC driving-point impedance $Z_{RC}(s)$. The $LC : RC$ transformation turns the synthesis of LC driving-point impedance $Z^{\square}_{LC}(s)$ to synthesis of RC driving-point impedance $Z_{RC}(s) = Z^{\square}_{LC}(\sqrt{s})/\sqrt{s}$ [20]. It also turns the synthesis of LC driving-point admittance $Y^{\square}_{LC}(s)$ to synthesis of RC driving-point admittance $Y_{RC}(s) = \sqrt{s}\cdot Y^{\square}_{LC}(\sqrt{s})$. These RC driving-point imittances are at first realized by prototype RC networks and thereof are produced the desired LC networks in the following way: capacitors in RC and LC networks remain the same, but the resistor from RC network turns to inductor in LC network with the same parameter value. Also, $LR : RC$ transformation turns the synthesis of RL driving-point impedance $Z^{\square}_{RL}(s)$ to synthesis of RC driving-point impedance $Z_{RC}(s) = Z^{\square}_{RL}(s)/s$. It also turns synthesis of RL driving-point admittance $Y^{\square}_{RL}(s)$ to synthesis of RC driving-point admittance $Y_{RC}(s) = sY^{\square}_{RL}(s)$. These RC imittances are realized by prototype RC networks and the desired RL networks are produced thereof in the following way: the resistor from RC network turns to inductor in LR network with the same parameter value, and the capacitor from RC network turns to resistor in LR network with reciprocal parameter value. Bearing all the aforementioned on mind, we can obviously restrict our consideration only to synthesis of driving-point impedance functions $Z_{RC}(s)$ of RC networks, which satisfy the following well known analytic necessary and sufficient conditions [20]: (a) $Z_{RC}(s)$ is real rational function in s, (b) It has only simple poles on negative real axis, or at the origin. At infinity it cannot have pole and (c) Residues of these poles are real and positive and $A_\infty := \lim_{s\to\infty} Z_{RC}(s) \geq 0$.

In general, the first canonic Foster's expansion (form) of $Z_{RC}(s)$ [20] reads

$$Z_{RC}(s) = A_\infty + \sum_{m=0}^{n} \frac{A_m}{s+s_m} \quad [A_\infty > 0;\ s_0 = 0,\ A_0 > 0;\ s_p, A_p > 0\ (p = 1, \ldots, n)], \tag{5.1}$$

where A_m is residue of the pole s_m ($m = 0, 1, \ldots, n$). The network which realizes driving-point impedance $Z_{RC}(s)$ (5.1) with minimum number of nodes ($= n+1$), minimum number of resistors ($= n+1$) and minimum number of capacitors ($= n+1$) is depicted in Figure 1. Observe that neither the resistors, nor the capacitors share common-node and hence the overall network realization is said to be non common-grounded.

Now, we will present our synthesis procedure. If for a given driving-point impedance $Z_{RC}(s)$ we found that $A_0 > 0$ and/or $A_\infty > 0$, then in the preamble of the realization

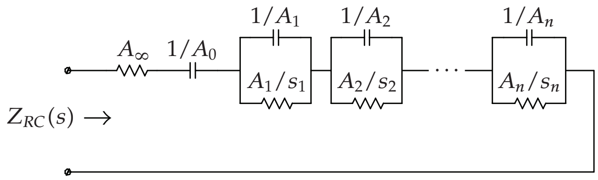

Figure 1: The first canonic Foster's realization of $Z_{RC}(s)$. Denoted are the normalized "values" of RC parameters.

procedure A_∞ and/or A_0/s should be at first extracted from (5.1) and realized by a series connection of resistor $R_\infty = A_\infty$ and capacitor $C_0 = 1/A_0$, thereby leaving for realization the driving-point impedance $\overline{Z}_{RC}(s) = Z_{RC}(s) - A_\infty - A_0/s$ with solely n poles lying on the negative real axis. In the sequel we will assume that $Z_{RC}(s)$ has only n such poles.

Let $\mathbf{C} = \text{diag}(C_1, C_2, \ldots, C_n)$ and $\mathbf{G} = \text{diag}(G_1, G_2, \ldots, G_n)$ be diagonal $n \times n$ matrices with strictly positive diagonal entries corresponding to the normalized capacitances and conductances, respectively. If we arbitrarily choose a nonsingular $n \times n$ matrix $\mathbf{T}$, then the reciprocal passive networks which realize $\mathbf{C}s + \mathbf{G}$ and $\mathbf{Y}(s) = \mathbf{T}(\mathbf{C}s + \mathbf{G})\mathbf{T}^{\text{T}}$ will have the same natural frequencies. By arbitrary selection of $n \times n$ nonsingular diagonal matrices $\boldsymbol{\delta} = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$, a broad class of nonsingular $n \times n$ matrices $\mathbf{T}$ can be generated with assumption $\mathbf{T} = \mathbf{V}\boldsymbol{\delta}\mathbf{U}$, where $\mathbf{U}$ and $\mathbf{V}$ are $n \times n$ orthogonal matrices. Since $\mathbf{Y}(s) = \mathbf{T}(\mathbf{C}s + \mathbf{G})\mathbf{T}^{\text{T}} = \mathbf{V}[\boldsymbol{\delta}(\mathbf{U}\mathbf{C}\mathbf{U}^{\text{T}})\boldsymbol{\delta}s + \boldsymbol{\delta}(\mathbf{U}\mathbf{G}\mathbf{U}^{\text{T}})\boldsymbol{\delta}]\mathbf{V}^{\text{T}}$, then

$$\mathbf{Z}(s) = [z_{mr}(s)]_{n\times n} = \mathbf{Y}^{-1}(s) = \left[\mathbf{T}(\mathbf{C}s + \mathbf{G})\mathbf{T}^{\text{T}}\right]^{-1} = \left(\mathbf{V}\boldsymbol{\delta}^{-1}\mathbf{U}\mathbf{C}^{1/2}\right)\left(s\mathbf{I}_n + \mathbf{G}\mathbf{C}^{-1}\right)^{-1}\left(\mathbf{V}\boldsymbol{\delta}^{-1}\mathbf{U}\mathbf{C}^{1/2}\right)^{\text{T}}. \tag{5.2}$$

Various network topologies can be produced by different choices of $\mathbf{U}$ and $\mathbf{V}$. But, only by selecting $\mathbf{C} = C\mathbf{I}_n$ $(C > 0)$ and $\mathbf{V} = \mathbf{I}_n$, the networks with minimum number of common-ground capacitors are produced; and only by selecting $\mathbf{G} = G\mathbf{I}_n$ $(G > 0)$ and $\mathbf{V} = \mathbf{I}_n$, the networks with minimum number of common-ground resitors are produced. Let us select $\mathbf{C} = C\mathbf{I}_n$ $(C > 0)$ and $\mathbf{V} = \mathbf{I}_n$, and let us assume in (5.1): $A_0 = A_\infty = 0$ and $s_n > s_{n-1} \cdots > s_1 > 0$. Since $\mathbf{U} = [u_{mr}]$ $(m, r = 1, \ldots, n)$, then from (5.2) it follows that

$$\mathbf{Y}(s) = C\boldsymbol{\delta}^2 s + \boldsymbol{\delta}(\mathbf{U}\mathbf{G}\mathbf{U}^{\text{T}})\boldsymbol{\delta}, \qquad z_{mr}(s) = \frac{1}{C\delta_m\delta_r}\sum_{p=1}^{n}\frac{u_{mp}u_{rp}}{s + G_p/C} \quad (m, r = 1, \ldots, n). \tag{5.3}$$

The matrices which are effectively realized by common-ground network with $n+1$ nodes $[(n+1)$th node is the common-ground$]$ are $C\boldsymbol{\delta}^2 s$ and $\boldsymbol{\delta}(\mathbf{U}\mathbf{G}\mathbf{U}^{\text{T}})\boldsymbol{\delta}$, provided that both are truly hd. According to (5.1) and (5.3) it holds $G_p = Cs_p$ $(p = 1, \ldots, n)$ and $G_n > G_{n-1} \cdots > G_1 > 0$. By using Theorem 3.3 [Case 1, $\mathbf{P}_k = \mathbf{A}_k$ and $\theta_k \in (0, \pi/2)$ $(k = 1, \ldots, n-1)$] we infer that $\mathbf{U}\mathbf{G}\mathbf{U}^{\text{T}}$ (2.21) is produced with hd sign pattern and no zero entries and with strictly positive inverse. Matrix $\mathbf{U}$ (2.26) is lower Hessenberg with nonnegative entries, except for negative "b"s. The same conclusions relating to $\mathbf{U}\mathbf{G}\mathbf{U}^{\text{T}}$ and $\mathbf{U}$ also hold if we apply Case 1 of Theorem 3.3 with $\mathbf{P}_k = \mathbf{B}_k$ and $\theta_k \in (0, \pi/2)$ $(k = 1, \ldots, n-1)$, except for "d"s in (2.26) are then negative and "b"s are positive. To realize $Z_{RC}(s)$ we must select in (5.3) either $m = r = n-1$ or $m = r = n$, thus obtaining either $Z_{RC}(s) = z_{n-1,n-1}(s)$, or $Z_{RC}(s) = z_{nn}(s)$. By assuming

$m = r = n[\leftrightarrow Z_{RC}(s) = z_{nn}(s)]$, $\mathbf{P}_k = \mathbf{A}_k$ and $\theta_k \in (0, \pi/2)$ $(k = 1, \dots, n-1)$, it follows from (2.26), (5.1), (5.3)

$$C\delta_n^2 = \frac{1}{\sum_{p=1}^{n} A_p}, \quad a_{i-1} = d_{i-1} = \left(\frac{A_i}{\sum_{q=1}^{i} A_q}\right)^{1/2}, \quad c_{i-1} = \left(\frac{\sum_{r=1}^{i-1} A_r}{\sum_{q=1}^{i} A_q}\right)^{1/2}, \quad (i = 2, \dots, n). \tag{5.4}$$

To prove the existence of a physical realization of both $C\boldsymbol{\delta}^2 s$ and $\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}$ we still have to determine the positive column vector $\mathrm{col}(\boldsymbol{\delta}) = [\delta_1 \ \ \delta_2 \cdots \delta_n]^{\mathrm{T}}$ which, according to Theorem 3.5, makes $\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}$ truly hd with possibly zero hd margins of at most $n-1$ rows. Let hd margin of the ith row in $\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}$ be p_i $(i = 1, \dots, n)$ and let $\mathbf{p} := [p_1 \ \ p_2 \cdots p_n]^{\mathrm{T}}$. If $\mathbf{u} := [1 \ \ 1 \cdots 1]^{\mathrm{T}}$ (n unities), then $\mathbf{p} = \boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}\mathbf{u}$. Let we introduce a column vector $\mathbf{p}^* := [p_1^* \ \ p_2^* \cdots p_n^*]^{\mathrm{T}}$ ($\|\mathbf{p}^*\|_{\mathrm{E}} > 0$) of *arbitrarily assumed real nonegative numbers*. Since $(\mathbf{UGU}^{\mathrm{T}})^{-1}$ is strictly positive, then it always can be find a diagonal matrix $\boldsymbol{\delta} = \mathrm{diag}(\delta_1, \delta_2, \dots, \delta_n)$ with positive diagonal entries, such that $\mathbf{p}^* = (\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}\mathbf{u} = (\mathbf{UGU}^{\mathrm{T}})\mathrm{col}(\boldsymbol{\delta})$. Herefrom, we obtain $\mathrm{col}(\boldsymbol{\delta}) = (\mathbf{UGU}^{\mathrm{T}})^{-1}\mathbf{p}^*$, that is, that $\delta_i > 0$ $(i = 1, \dots, n)$. Since $\mathbf{p} = \boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}\mathbf{u} = \boldsymbol{\delta}\mathbf{p}^*$, then it follows $p_i = \delta_i p_i^* \geq 0$ $(i = 1, \dots, n)$, bearing on mind that at most $n-1$ "p"s can be equal to zero. These "p"s indices correspond to indices of those rows (or columns) in $\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}$ which have zero hd margins. Then, from the overall network vanish resistors connecting common-ground to nodes with the same indices as that of rows (columns) with zero hd margins [22]. For different selections of $\mathbf{p}^*$, different algorithms and different topologically and parametrically equivalent realizations emerge. For example, if we select $\mathrm{col}(\boldsymbol{\delta}) = \mu[u_{11} \ \ u_{21} \cdots u_{n1}]^{\mathrm{T}}$ $(\mu > 0)$, where it is according to (2.26), (5.4)

$$u_{11} = \left[\frac{A_2}{A_1 + A_2}\right]^{1/2}, \qquad u_{p1} = \left[\frac{A_1 A_{p+1}}{(\sum_{q=1}^{n} A_q)(\sum_{r=1}^{p+1} A_r)}\right]^{1/2} \quad (p = 2, \dots, n-1), \qquad u_{n1} = \left[\frac{A_1}{\sum_{q=1}^{n} A_q}\right]^{1/2}, \tag{5.5}$$

then the column vector $\mathbf{p}$ of row (column) hd margins of matrix $\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}$ with hd sign pattern reads

$$\mathbf{p} = \boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}\mathbf{u} = \boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\mathrm{col}(\boldsymbol{\delta}) = G_1\mathrm{col}(\boldsymbol{\delta}^2) > \mathbf{0}_{n,1}. \tag{5.6}$$

This means that $\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}$ is truly hd. We will now present two algorithms for realization of driving-point impedances $Z_{RC}(s)$ which rely on the results developed above.

Algorithm 1. Realization of $Z_{RC}(s)$ with minimum number of common-ground capacitors and non-reduced number of resistors

(1^0) Commencing with A_i $(i = 1, \dots, n)$ calculate the entries of $\mathbf{U}$, by using (2.26) and (5.4).

(2^0) Arbitrarily select some $\mu > 0$ and then calculate $C = 1/\mu^2 A_1$ and $G_q = Cs_q$ $(q = 1, \dots, n)$.

(3^0) Calculate $\mathrm{col}(\boldsymbol{\delta}) = \mu[u_{11} \ \ u_{21} \cdots u_{n1}]^{\mathrm{T}}$, by using (5.5) and the entries of hd matrix $\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}$. Calculate $\mathbf{p}$, by using (5.6).

(4^0) Realize $\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}$ by common-ground, transformerless, conductance network. This can be done easily, almost by visual inspection of $\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}$ [22]. Attach to the ports

$$X_{LC}^{\square}(\omega) = \frac{(\omega^2-1)(\omega^2-3)(\omega^2-5)(\omega^2-8)}{\omega(\omega^2-2)(\omega^2-4)(\omega^2-6)}$$

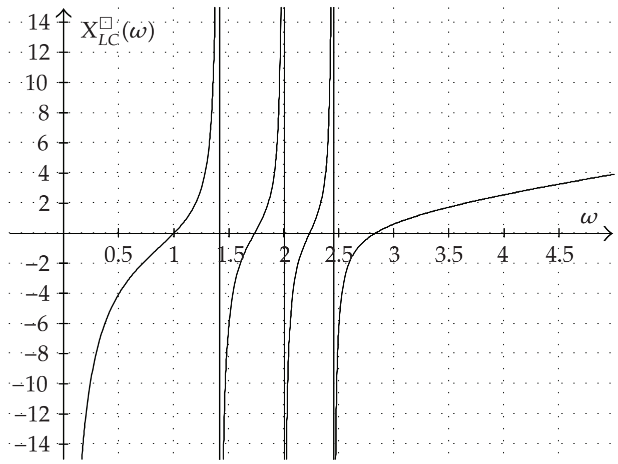

Figure 2: The reactance function of *LC* network from the example.

of that network, enumerated by 1, p $(p = 2, \ldots, n-1)$ and n, the common-ground capacitors with normalized capacitances

$$C_1 = \frac{A_2}{A_1(A_1+A_2)}, \qquad C_p = \frac{A_{p+1}}{\left(\sum_{q=1}^{n} A_q\right)\left(\sum_{r=1}^{p+1} A_r\right)} \quad (p = 2, \ldots, n-1), \qquad C_n = \frac{1}{\sum_{q=1}^{n} A_q}, \tag{5.7}$$

respectively. The nth port of the overall network realizes driving-point impedance $Z_{RC}(s)$, provided that all other ports are left open-circuited.

Algorithm 2. Realization of $Z_{RC}(s)$ with minimum number of common-ground capacitors and the reduced number of resistors

(1^0) The same as step (1^0) of Algorithm 1. Let $\mathbf{S} = \operatorname{diag}(s_1, s_2, \ldots, s_n)$. Recall that $s_n > s_{n-1} \cdots > s_1 > 0$.

(2^0) Select $\varepsilon_1 > 0$ and $\varepsilon_i = 0$ $(i = 2, \ldots, n-1)$. Thereafter, by using (2.21) and (5.4), calculate

$$C = \left[\varepsilon_1 (s_1^{-1} - s_2^{-1}) c_{n-1} c_{n-2} \cdots c_2 c_1 a_1\right]^2 \sum_{q=1}^{n} A_q, \qquad G_i = C s_i \quad (i = 1, \ldots, n). \tag{5.8}$$

Calculate $\operatorname{col}(\boldsymbol{\delta}) = \varepsilon_1 C^{-1}(\mathbf{U}\mathbf{S}^{-1}\mathbf{U}^{\mathrm{T}})\mathbf{e}_1$, where $\mathbf{e}_1 = [1 \;\; 0 \cdots 0]^{\mathrm{T}}$ is n-dimensional column vector.

(3^0) Calculate the entries of $\boldsymbol{\delta}(\mathbf{U}\mathbf{G}\mathbf{U}^{\mathrm{T}})\boldsymbol{\delta}$ and its hd margin $p_1 = \delta_1 \varepsilon_1$, where $\delta_1 = \varepsilon_1 C^{-1} \mathbf{e}_1^{\mathrm{T}} \mathbf{T}\mathbf{S}^{-1}\mathbf{T}^{\mathrm{T}} \mathbf{e}_1$. Set for other hd margins $p_i = 0$ $(i = 2, \ldots, n)$.

(4^0) Realize $\boldsymbol{\delta}(\mathbf{U}\mathbf{G}\mathbf{U}^{\mathrm{T}})\boldsymbol{\delta}$ by common-ground, transformerless, conductance network and attach to its ith port the common-ground capacitor with normalized capacitance $C\delta_i^2$ $(i = 1, \ldots, n)$. The nth port of the overall network realizes driving-point impedance $Z_{RC}(s)$, provided that all other ports are left open-circuited.

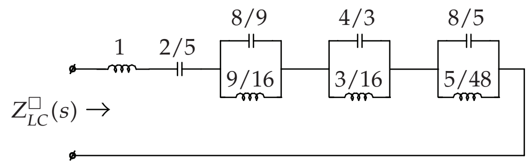

Figure 3: Driving-point impedance $Z_{LC}^{\square}(s)$ from example synthesized by using the first canonic Foster's form.

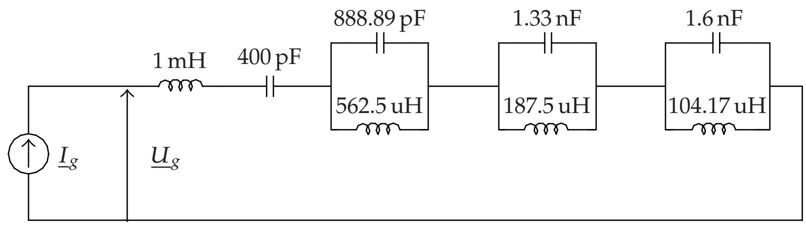

Figure 4: Denormalized realization of driving-point impedance $Z_{LC}^{\square}(s)$ from Figure 3.

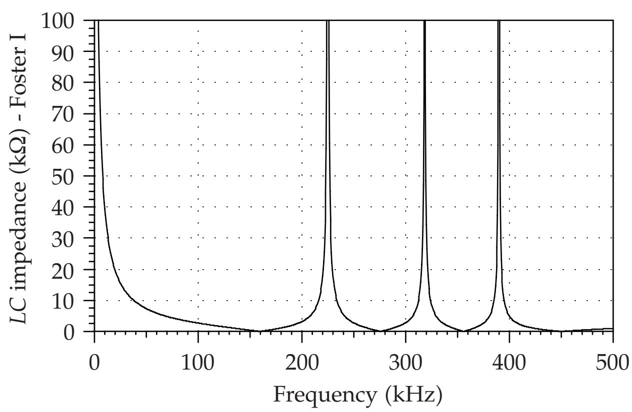

Figure 5: Driving-point impedance $|\underline{Z}_{LC}^{\square}|$ of *LC* network from Figure 4.

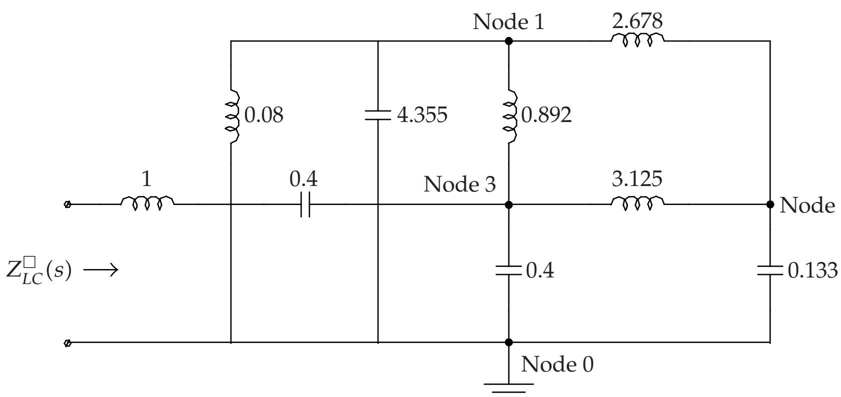

Figure 6: Driving-point impedance $Z_{LC}^{\square}(s)$ from the example synthesized by a noncanonic *LC* network.

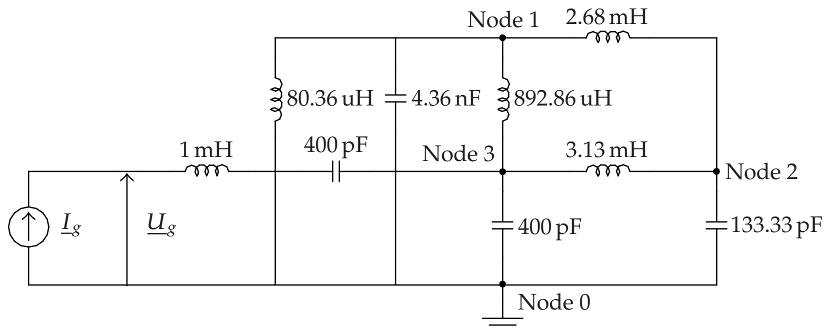

Figure 7: Denormalized realization of driving-point impedance $Z_{LC}^{\square}(s)$ from Figure 6.

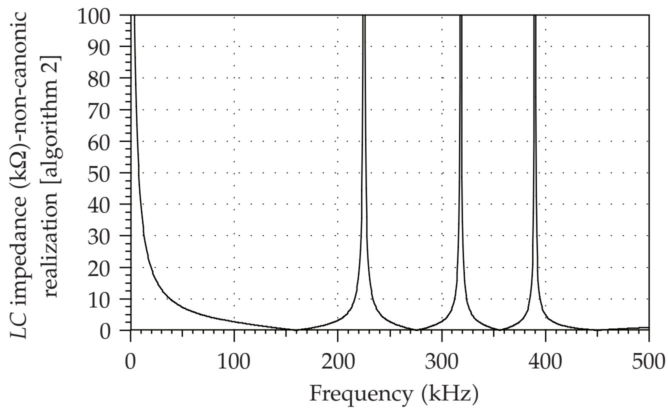

Figure 8: Driving-point impedance $|\underline{Z}_{LC}^{\square}|$ of LC network from Figure 7.

5.1. A numerical example

Consider realization of the real rational function $Z_{LC}^{\square}(s)$ as driving-point impedance of common-ground transformerless LC network with minimum number of capacitors and the reduced number of inductors

$$Z_{LC}^{\square}(s) = \frac{(s^2+1)(s^2+3)(s^2+5)(s^2+8)}{s(s^2+2)(s^2+4)(s^2+6)}. \tag{5.9}$$

This function satisfies the necessary and sufficient conditions for driving-point immittances of LC networks: (a) it is an odd real rational function in s; (b) it has only simple poles located strictly on imaginary axis; and (c) residues of those poles are real and positive. Therefore, $Z_{LC}^{\square}(s)$ can be realized both in two Foster's and in two Cauer's canonic forms [20]. The partial fraction expansion of $Z_{LC}^{\square}(s)$ reads

$$Z_{LC}^{\square}(s) = \frac{(s^2+1)(s^2+3)(s^2+5)(s^2+8)}{s(s^2+2)(s^2+4)(s^2+6)} = s + \frac{5}{2s} + \frac{9s/8}{s^2+2} + \frac{3s/4}{s^2+4} + \frac{5s/8}{s^2+6}. \tag{5.10}$$

The reactance function corresponding to $Z_{LC}^{\square}(s)$ is $X_{LC}^{\square}(\omega) := Z_{LC}^{\square}(j\omega)/j$ and it is depicted in Figure 2. The first canonic Foster's realization of $Z_{LC}^{\square}(s)$ with minimum number of nodes, noncommon-ground capacitors and inductors is depicted in Figure 3. Thereon are denoted the normalized values of LC parameters.

6. Conclusions

In Figure 4, it is depicted the first canonic Foster's realization of driving-point impedance $Z_{LC}^{\square}(s)$ from Figure 3 with selected normalization frequency $\Omega = 10^6$[rad/s] and selected normalization resistance $R_0 = 10^3$[kΩ]. The network is excited by a sinusoidal current generator having constant current amplitude and discretely varying frequency $f = \omega/(2\pi)$ within the range $f \in [0.1, 500]$ kHz. If the complex representative of generator current is $\underline{I}_g$ and the complex representative of the voltage across its terminals is $\underline{U}_g$, then the complex driving-point impedance of the overall LC network is $\underline{Z}_{LC}^{\square} = \underline{U}_g/\underline{I}_g$. The modulus of $\underline{Z}_{LC}^{\square}$, that is, $|\underline{Z}_{LC}^{\square}|$ (usually called LC impedance) obtained through PSPICE simulation within the range $f \in [0.1, 500]$ kHz is depicted in Figure 5.

Now, we will realize $Z_{LC}^{\square}(s)$ by using the proposed Algorithm 2. After $LC : RC$ transformation, we firstly produce the function $\overline{Z}_{RC}(s) = Z_{LC}^{\square}(\sqrt{s})/\sqrt{s} = 1 + 5/2s + Z_{RC}(s)$, where $Z_{RC}(s)$ is driving-point impedance of RC network which should be expanded into partial fractions as follows:

$$Z_{RC}(s) = \frac{A_1}{s+s_1} + \frac{A_2}{s+s_2} + \frac{A_3}{s+s_3} \left| \begin{array}{c} A_1 = \dfrac{9}{8}, \quad A_2 = \dfrac{3}{4}, \quad A_3 = \dfrac{5}{8}, \\ s_1 = 2, \quad s_2 = 4, \quad s_3 = 6, \\ s_3 > s_2 > s_1 > 0, \quad \mathbf{S} := \operatorname{diag}(s_1, s_2, s_3). \end{array} \right. \tag{6.1}$$

In step (1^0) of Algorithm 2 we determine the orthogonal matrix $\mathbf{U}$ by using A_1, A_2, and A_3 (see (2.26) and (5.4))

$$\mathbf{U} = \begin{bmatrix} \sqrt{\dfrac{2}{5}} & -\sqrt{\dfrac{3}{5}} & 0 \\ \sqrt{\dfrac{3}{20}} & \sqrt{\dfrac{1}{10}} & -\dfrac{\sqrt{3}}{2} \\ \sqrt{\dfrac{9}{20}} & \sqrt{\dfrac{3}{10}} & \dfrac{1}{2} \end{bmatrix}. \tag{6.2}$$

By assuming $\varepsilon_1 = 8\sqrt{5}/3$ in step (2^0), we further easily obtain $C = 1$, $G_1 = 2$, $G_2 = 4$, $G_3 = 6$ and $\operatorname{col}(\boldsymbol{\delta}) = [\ \delta_1 \ \ \delta_2 \ \ \delta_3\ ]^{\mathrm{T}} = [2.087 \ \ 0.365 \ \ 0.632]^{\mathrm{T}}$. In step ($3^0$) we firstly calculate $\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}$ and then hd margins of its rows (columns),

$$\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta} = \begin{bmatrix} 13.937 & -0.373 & -1.119 \\ -0.373 & 0.693 & -0.320 \\ -1.119 & -0.320 & 1.439 \end{bmatrix}, \qquad p_1 = 12.444, \qquad p_2 = p_3 = 0. \tag{6.3}$$

In step (4^0) we calculate the normalized capacitances of common-ground capacitors: $C_1 = 4.355$, $C_2 = 0.133$ and $C_3 = 0.400$. Realization of driving-point impedance $Z_{RC}(s)$ by transformerless, common-ground RC network with minimum number of nodes ($= n+1$), reduced number of inductors and minimum number of common-ground capacitors ($= n$), begins by realization of conductance matrix $\boldsymbol{\delta}(\mathbf{UGU}^{\mathrm{T}})\boldsymbol{\delta}$ which can be accomplished almost by inspection of that matrix [22]. Then, to the ith port of the realized conductance network, it should be connected to the capacitor $C_i = \delta_i^2$ ($i = 1, 2, 3$). The third port of the overall network realizes the *RC* driving-point impedance $Z_{RC}(s)$, provided that all other ports are left open-circuited. By embedding in the third port a series connection of resistor and capacitor with the normalized parameter values 1 and 2/5, respectively, and by applying RC : LC transformation thereafter, we finally produce noncanonic network which realizes $Z_{LC}^{\square}(s)$ with minimum number of nodes and capacitors and with reduced number of inductors. That network is depicted in Figure 6 whereon are denoted the normalized (dimensionless) values of *LC* parameters.

In Figure 7, is depicted the noncanonic realization of driving-point impedance $Z_{LC}^{\square}(s)$ from Figure 6 with selected normalization frequency $\Omega = 10^6[\mathrm{rad/s}]$ and selected normalization resistance $R_0 = 10^3[\mathrm{k}\Omega]$. The network is excited by a sinusoidal current generator with constant current amplitude and with discretely variable frequency $f = \omega/(2\pi)$ within the range $f \in [0.1, 500]$ kHz. If the complex representative of generator current is $\underline{I}_g$ and the complex representative of the voltage across its terminals is $\underline{U}_g$, then the complex driving-point impedance of the overall *LC* network is $\underline{Z}_{LC}^{\square} = \underline{U}_g/\underline{I}_g$. The modulus of $\underline{Z}_{LC}^{\square}$, that is, $|\underline{Z}_{LC}^{\square}|$ (usually called *LC* impedance) obtained through PSPICE simulation within the range $f \in [0.1, 500]$ kHz is depicted in Figure 8. Since *LC* networks in Figures 4 and 7 are intentionally designed to be equivalent, then their driving-point impedances $|\underline{Z}_{LC}^{\square}|$ must have the same variations in frequency, as can be verified from Figures 5 and 8 qualitatively and more precisely by using numerical results of simulation.

A novel procedure for explicit construction of entries of real symmetric matrices with assigned spectrum is developed by using a group of four types of canonic, second-order, orthogonal transformations. It has been also shown that the orthogonal modal matrices corresponding to the produced real symmetrix matrices, are either lower or upper Hessenberg with explicitly constructed entries too. Thereafter, the inverse eigenvalue problems of real symmetric matrices with twelve specific types of sign patterns (including hyperdominant one) are explicitly solved providing that the signs of eigenvalues are the same (zeros are permitted) and that they are enumerated such as to establish the increasing or decreasing sequence. It is proved to arise thereof a possibility of explicit solving the inverse eigenvalue problem of symmetric hyperdominant matrices having either uncommitted or assigned nonnegative spectrum. The results obtained are then applied in synthesis of driving-point immittance functions of transformerless, common-ground, two-element-kind *RLC* networks and in generation of their equivalent realizations with minimum number of nodes. The synthesis procedures proposed herein turn the synthesis problem of any immittance function of the two-element-kind *RLC* network to the synthesis problem of impedance function of a prototype *RC* network.

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