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Research Article

On Motion of Robot End-Effector Using the Curvature Theory of Timelike Ruled Surfaces with Timelike Rulings

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The trajectory of a robot end-effector is described by a ruled surface and a spin angle about the ruling of the ruled surface. In this way, the differential properties of motion of the end-effector are obtained from the well-known curvature theory of a ruled surface. The curvature theory of a ruled surface generated by a line fixed in the end-effector referred to as the tool line is used for more accurate motion of a robot end-effector. In the present paper, we first defined tool trihedron in which tool line is contained for timelike ruled surface with timelike ruling, and transition relations among surface trihedron: tool trihedron, generator trihedron, natural trihedron, and Darboux vectors for each trihedron, were found. Then differential properties of robot end-effector's motion were obtained by using the curvature theory of timelike ruled surfaces with timelike ruling.

1. Introduction

The methods of robot trajectory control currently used are based on PTP (point to point) and CP (continuous path) methods. These methods are basically interpolation techniques and, therefore, are approximations of the real path trajectory [1]. In such cases, when a precise trajectory is needed, or we need to trace a free formed or analytical surface accurately, the precision is only proportional to the number of intermediate data points for teach-playback or offline programming.

For accurate robot trajectory, the most important aspect is the continuous representation of orientation whereas the position representation is relatively easy. There are methods such as homogeneous transformation, Quaternion, and Euler Angle representation to describe the orientation of a body in a three-dimensional space [2]. These methods are easy in concept but have high redundancy in parameters and are discrete representation in nature rather than continuous. Therefore, a method based on the curvature theory of a ruled surface has been proposed as an alternative [3].

Ruled surfaces were first investigated by G. Monge who established the partial differential equation satisfied by all ruled surfaces. Ruled surfaces have been widely applied in designing cars, ships, manufacturing (e.g., CAD/CAM) of products and many other areas such as motion analysis and simulation of rigid body, as well as model-based object recognition system. After the period of 1960–1970, Coons, Ferguson, Gordon, Bezier, and others developed new surface definitions that were investigated in many areas. However, ruled surfaces are still widely used in many areas in modern surface modelling systems.

The ruled surfaces are surfaces swept out by a straight line moving along a curve. The study of ruled surfaces is an interesting research area in the theory of surfaces in Euclidean geometry. Theory of ruled surfaces is developed by using both surface theory and E. Study map which enables one to investigate the geometry of ruled surfaces by means of one-real parameter. It is also known that theory of ruled surfaces is applicable to theoretical kinematics.

The positional variation and the angular variation of the rigid body are determined from the curvature theory of a ruled surface. These properties are very important to determine the properties of the robot end-effector motion. A brief mathematical background of the curvature theory of a ruled surface may be found in [4–10].

Minkowski space is named after the German mathematician Hermann Minkowski, who around 1907 realized that the theory of special relativity (previously developed by Einstein) could be elegantly described by using a four-dimensional space-time model which combines the dimension of time with the three dimensions of space.

Four-dimensional Euclidean space with a Lorentz metric, which is the space of special relativity, is called Minkowski space-time, and a Lorentz space is a hypersurface of Minkowski space-time. There are lots of papers dealing with ruled surfaces in Lorentz space (see [11–16]).

2. Preliminaries

Let $\mathbb{R}^3_1$ be a 3-dimensional Lorentzian space with Lorentz metric $ds^2 = dx_1^2 + dx_2^2 - dx_3^2$. If $\langle X, Y \rangle = 0$, X and Y are called perpendicular in the sense of Lorentz, where $\langle \cdot, \cdot \rangle$ is the induced inner product in $\mathbb{R}^3_1$. The norm of $X \in \mathbb{R}^3_1$ is, as usual, $\|X\| = \sqrt{|\langle X, X \rangle|}$.

Let $X \in \mathbb{R}^3_1$ be a vector. If $\langle X, X \rangle < 0$, then X is called timelike; if $\langle X, X \rangle > 0$ and $X = 0$, then X is called spacelike and if $\langle X, X \rangle = 0$, $X \neq 0$, then X is called lightlike (null) vector [17]. We can observe that a timelike curve corresponds to the path of an observer moving at less than the speed of light while the spacelike curves faster and the null curves are equal to the speed of light.

A smooth regular curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3_1$ is said to be a timelike, spacelike, or lightlike curve if the velocity vector $\alpha'(s)$ is a timelike, spacelike, or lightlike vector, respectively.

A surface in the Minkowski 3-space is called a timelike surface if the induced metric on the surface is a Lorentz metric, that is, the normal on the surface is a spacelike vector. A timelike ruled surface in $\mathbb{R}^3_1$ is obtained by a timelike straight line moving along a spacelike curve or by a spacelike straight line moving along a timelike curve. The timelike ruled surface

M is given by the parameterization

$$\varphi : I \times \mathbb{R} \longrightarrow \mathbb{R}_1^3, \qquad \varphi(s,u) = \alpha(s) + uX(s) \tag{2.1}$$

in $\mathbb{R}_1^3$ (see [11–13, 18, 19]).

The arc-length of a spacelike curve α, measured from $\alpha(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} \|\alpha'(u)\| du, \tag{2.2}$$

The parameter, s, is determined such that $\|\alpha'(u)\| = 1$, where $\alpha'(s) = d\alpha/ds$. Let us denote $V_1 = \alpha'$ and call $V_1(s)$ a unit tangent vector of α at the point s. We define the curvature by

$$k_1(s) = \sqrt{|\langle \alpha''(s), \alpha''(s) \rangle|}. \tag{2.3}$$

If $k_1(s) \neq 0$, then the unit principal normal vector, $V_2(s)$, of a timelike curve α at the point s is given by

$$\alpha''(s) = k_1(s) V_2(s). \tag{2.4}$$

For any $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}_1^3$, the Lorentzian vector product of X and Y is defined as

$$X \wedge Y = \left(x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_2 y_1 - x_1 y_2 \right). \tag{2.5}$$

The unit vector $V_3(s) = V_1(s) \wedge V_2(s)$ is called a unit binormal vector of the timelike curve α at the point s. Now, we give some properties of $\wedge$ without proof: let $A, B, C \in \mathbb{R}_1^3$; it is straightforward to see the following: $A \wedge B = -B \wedge A$; $\langle A \wedge B, A \rangle = \langle A \wedge B, B \rangle = 0$; A or B is timelike $\Rightarrow$ $A \wedge B$ is spacelike; $\langle A \wedge B, C \rangle = \langle B \wedge C, A \rangle$; A and B are spacelike $\Rightarrow$ $A \wedge B$ is timelike; $\langle A \wedge B, A \wedge B \rangle = \langle A, B \rangle^2 - \langle A, A \rangle \langle B, B \rangle$ and $A \wedge B = 0 \Leftrightarrow A$ and B are linearly dependent (see [13, 17]).

A timelike curve $\alpha = \alpha(s) \in \mathbb{R}_1^3$, parameterized by natural parameterization, is a frame field $\{e1, e2, e3\}$ having the following properties:

$$e_1 \wedge e_2 = -e_3, \qquad e_1 \wedge e_3 = e_2, \qquad e_2 \wedge e_3 = e_1. \tag{2.6}$$

Lorentzian vector product $u \wedge v$ of u and v is defined by

$$u \wedge v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{2.7}$$

Let u and v be vectors in the Minkowski 3-space.

(i) If u and v are future pointing (or past pointing) timelike vectors, then $u \wedge v$ is a spacelike vector. $\langle u, v\rangle = -\|u\|\,\|v\|\cosh\theta$ and $\|u \wedge v\| = \|u\|\,\|v\|\sinh\theta$ where θ is the hyperbolic angle between u and v.

(ii) If u and v are spacelike vectors satisfying the inequality $|\langle u, v\rangle| < \|u\|\,\|v\|$, then $u\wedge v$ is timelike, $\langle u, v\rangle = \|u\|\,\|v\|\cos\theta$ and $\|u \wedge v\| = \|u\|\,\|v\|\sinh\theta$ where θ is the angle between u and v (see [12, 16, 20]).

3. Representation of robot trajectory by a ruled surface

The motion of a robot end-effector is referred to as the robot trajectory. A robot trajectory consists of: (i) a sequence of positions, velocities, and accelerations of a point fixed in the end-effector; and (ii) a sequence of orientations, angular velocities, and angular accelerations of the end-effector. The point fixed in the end-effector will be referred to as the Tool Center Point and denoted as the TCP.

Path of a robot may be represented by a tool center point and tool frame of end-effector. In Figure 1, the tool frame is represented by three mutually perpendicular unit vectors $[O, A, N]$, defined as where O is the orientation vector (timelike), A is the approach vector (spacelike), and N is the normal vector (spacelike), as shown in Figure 1. In this paper, the ruled surface generated by O is chosen for further analysis without loss of generality.

The motion of a robot end-effector in space has six degrees of freedom, in general, and six independent parameters are required to describe the position and orientation of the end-effector. Since α and $\vec{R}$ are vectors in three-dimensional space, there are three independent parameters to represent each vector. However, the ruling has a constant magnitude which gives one constraint, therefore, it takes five independent parameters to represent a timelike ruled surface.

The path of tool center point is referred to as directrix and vector O, ruling. The directrix and ruling represent five parameters for the 6 degrees of freedom spatial motion. The final parameter is the spin angle, η, which represents the rotation from the surface normal vector, S_n, about O.

4. Frames of reference

Each vector of tool frame in end-effector defines its own timelike ruled surface while the robot moves. The path of tool center point is directrix and O is the ruling. As $\alpha(s)$ is a spacelike curve and $\vec{R}(s)$ is timelike straight line, let us take the following timelike ruled surface as

$$X(s, u) = \alpha(s) + u\vec{R}(s), \tag{4.1}$$

where the space curve $\alpha(s)$ is the specified path of the TCP (called the directrix of the timelike ruled surface), u is a real-valued parameter, and $\vec{R}(s)$ is the vector generating the timelike ruled surface (called the ruling). The shape of the ruled surface $X(\psi, u)$ is independent of the parameter ψ chosen to identify the sequence of lines along it therefore we choose a standard

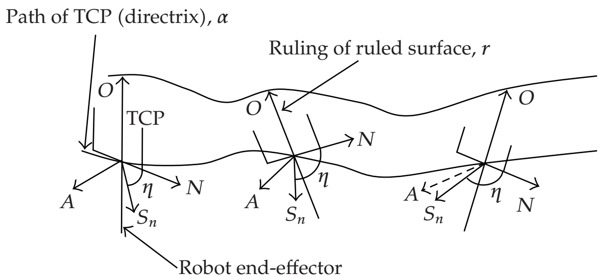

Figure 1: Ruled surface generated by O of tool frame.

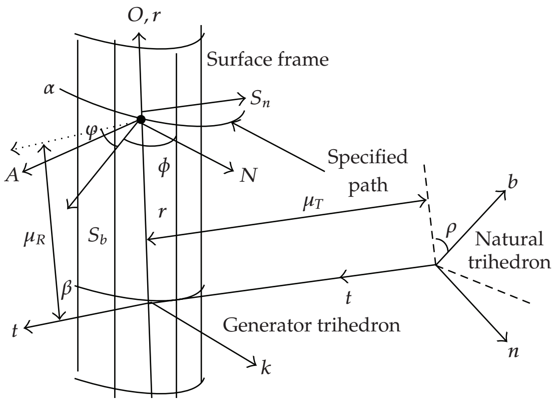

Figure 2: Frames of reference.

parametrization. It convenient to use the arc-length of the spherical indicatrix $\vec{R}(\varphi)$ as this standard parameter. The arc-length parameter s is defined by

$$s(\varphi) = \int \left| \frac{d\vec{R}(\varphi)}{d\varphi} \right| d\varphi. \tag{4.2}$$

Here $\delta = |d\vec{R}(\varphi)/d\varphi|$ is called the speed of $\vec{R}(\varphi)$. If $\delta \neq 0$, then above equation can be inverted to yield $\varphi(s)$ allowing the definition of $\vec{R}(\varphi(s)) = \vec{R}(s)$. $\vec{R}(\varphi)$ has unit speed, that is, its tangent vector is of unit magnitude (see, [2, 6, 8, 21]).

To describe the orientation of tool frame relative to the timelike ruled surface, we define a surface frame at the TCP as shown in Figures 1 and 2. The surface frame, $[O, S_n, S_b]$, may be determined as follows:

$$S_n = \frac{X_u \wedge X_s}{\|X_u \wedge X_s\|}, \tag{4.3}$$

which is the unit normal of timelike ruled surface in TCP;

$$S_b = O \wedge S_n \tag{4.4}$$

is the unit binormal vector of the surface.

Generator trihedron in Figure 2 is used to study the positional and angular variation of timelike ruled surface. Let us take $r = (1/\mathfrak{R})\vec{R}(s)$ timelike generator vector, $t = \vec{R}(s)$ spacelike central normal vector, $k = r \wedge t$ spacelike tangent vector, where $\mathfrak{R}$ is $\|\vec{R}(s)\|$.

The striction curve of timelike ruled surface is

$$\beta(s) = \alpha(s) - \mu(s)\vec{R}(s), \tag{4.5}$$

where the parameter

$$\mu(s) = \frac{|\langle \alpha'(s), \vec{R}'(s)\rangle|}{\|\vec{R}'\|} \tag{4.6}$$

indicates the position of the TCP relative to the striction point of the timelike ruled surface. Since the ruling $\vec{R}$ is not necessarily a unit vector, the distance from the striction point of ruled surface to the TCP is $\mu\mathfrak{R}$ in the positive direction of the generator vector. The generator trihedron and the striction curve of the ruled surface are unique in the sense that they do not depend on the choice of the directrix of the ruled surface. Therefore, by studying the motion of generator trihedron and the striction curve, we can obtain the differential motion of the end-effector in a simple and systematic manner.

The first-order angular variation of the generator trihedron may be expressed in the matrix form as

$$\frac{d}{ds}\begin{bmatrix} r \\ t \\ k \end{bmatrix} = \frac{1}{\mathfrak{R}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -\gamma \\ 0 & \gamma & 0 \end{bmatrix}\begin{bmatrix} r \\ t \\ k \end{bmatrix} = U_r \wedge \begin{bmatrix} r \\ t \\ k \end{bmatrix}, \tag{4.7}$$

where γ is defined as

$$\gamma = \langle \vec{R}'', \vec{R} \wedge \vec{R}'\rangle, \tag{4.8}$$

$$U_r = \frac{\gamma}{\mathfrak{R}}r - \frac{1}{\mathfrak{R}}k \tag{4.9}$$

is the Darboux vector of the generator trihedron.

Differentiating (4.5) gives first order positional variation of the striction point of the timelike ruled surface expressed in the generator trihedron, with the aid of (4.6), (4.7) and generator trihedron,

$$\beta'(s) = r\Gamma + \Delta k, \tag{4.10}$$

where

$$\begin{aligned} \Gamma &= -\frac{1}{\mathfrak{R}}\langle \alpha'(s), \vec{R}(s)\rangle - \frac{1}{\mathfrak{R}}\mu'(s), \\ \Delta &= -\frac{1}{\mathfrak{R}}\langle \alpha'(s), \vec{R}(s) \wedge \vec{R}'(s)\rangle. \end{aligned} \tag{4.11}$$

5. Central normal surface

The natural trihedron used to study the angular and positional variation of the normalia is defined by the following three orthonormal vectors: the central normal vector t (spacelike), principal normal vector n (spacelike), and binormal vector b (timelike), as shown in Figure 2.

As the generator trihedron moves along the striction curve, the central normal vector generates another ruled surface called the normalia or central normal surface. The normalia, which is important in the study of the higher order properties of the ruled surface, is defined as

$$X_T(s,u) = \beta(s) + ut(s). \tag{5.1}$$

Let the hyperbolic angle, ρ, be between the timelike vectors r and b. Then, we have

$$r = \sinh\rho n + \cosh\rho b, \tag{5.2}$$

$$k = t \wedge r = \sinh\rho b + \cosh\rho n. \tag{5.3}$$

So (5.2) can be written in matrix form as

$$\begin{bmatrix} r \\ t \\ k \end{bmatrix} = \begin{bmatrix} 0 & \sinh\rho & \cosh\rho \\ 1 & 0 & 0 \\ 0 & \cosh\rho & \sinh\rho \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}. \tag{5.4}$$

The solution of (5.4) is

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\sinh\rho & 0 & \cosh\rho \\ \cosh\rho & 0 & -\sinh\rho \end{bmatrix} \begin{bmatrix} r \\ t \\ k \end{bmatrix}. \tag{5.5}$$

Substituting (4.7) into (5.5) and using $t' = \kappa n$, it follows that

$$t' = \frac{1}{\mathfrak{R}}(r - \gamma k) = \kappa(-\sinh\rho r + \cosh\rho k). \tag{5.6}$$

Hence

$$\cosh\rho = -\frac{\gamma}{\mathfrak{R}\kappa}, \qquad \sinh\rho = -\frac{1}{\mathfrak{R}\kappa}. \tag{5.7}$$

Then the geodesic curvature may also be written as

$$\coth\rho = \gamma. \tag{5.8}$$

Substituting this into (4.7) we get

$$\frac{d}{ds}\begin{bmatrix} r \\ t \\ k \end{bmatrix} = \frac{1}{\mathfrak{R}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -\coth\rho \\ 0 & \coth\rho & 0 \end{bmatrix}\begin{bmatrix} r \\ t \\ k \end{bmatrix} \tag{5.9}$$

and from the following equality

$$U_r = \frac{\Upsilon}{\mathfrak{R}}r - \frac{1}{\mathfrak{R}}k = \frac{1}{\mathfrak{R}}(\coth\rho r - k). \tag{5.10}$$

The natural trihedron consists of the following vectors:

$$\begin{aligned} t &= \vec{R}', \\ n &= \frac{1}{\kappa}t', \\ b &= t \wedge n, \end{aligned} \tag{5.11}$$

where $\kappa = \|t'\|$ is the curvature. The origin of the natural trihedron is a striction point of the normalia. The striction curve is defined as

$$\beta_T(s) = \beta(s) - \mu_T(s)t(s), \tag{5.12}$$

where

$$\mu_T(s) = \left|\frac{\langle \beta'(s), t'(s)\rangle}{\langle t'(s), t'(s)\rangle}\right|, \tag{5.13}$$

which is the distance from the striction point of the normalia to the striction point of the timelike ruled surface in the positive direction of the central normal vector. Substituting (5.9) and (4.10) into (5.13), we obtain

$$\mu_T(s) = \mathfrak{R}^2\cosh^2\rho(-\Gamma + \Delta\coth\rho). \tag{5.14}$$

The first-order angular variation of natural trihedron may be expressed in the matrix form as

$$\frac{d}{ds}\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} t \\ n \\ b \end{bmatrix} = U_T \wedge \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{5.15}$$

where $\tau = \langle n', b\rangle$ is torsion.

As in the case of the generator trihedron, (5.15) may also be written as

$$U_T = -\tau t - \kappa b \tag{5.16}$$

is referred to as the Darboux vector of the natural trihedron.

Observe that both the Darboux vector of the generator trihedron and the Darboux vector of the natural trihedron describe the angular motion of the ruled surface and the central normal surface. The curvature κ is defined by (5.9) as follows:

$$\kappa = \frac{1}{\mathfrak{R}}\sqrt{|-1+\cosh^2\rho|} = \frac{1}{\mathfrak{R}\sinh\rho}. \tag{5.17}$$

Differentiating (5.12) and substituting (4.7) and (4.10) into the result, we obtain

$$\beta_T' = \Gamma_T t + \Delta_T b, \tag{5.18}$$

where

$$\Gamma_T = \eta_T' t, \qquad \Delta_T = -\Gamma\cosh\rho + \Delta\sinh\rho. \tag{5.19}$$

The four functions, given by (5.17) and (5.19), characterize the normalia in the same way as (4.7) and (4.11) characterize the timelike ruled surface.

6. Relationship between the frames of reference

To utilize the curvature theory of timelike ruled surfaces in the accurate motion of a robot end-effector, we must first determine the position of (i) the TCP relative to the striction point of the timelike ruled surface, and (ii) the striction point of the timelike ruled surface relative to the striction point of the normalia; and the orientation of (i) the tool frame relative to the generator trihedron, and (ii) the generator trihedron relative to the natural trihedron. The position solutions may be obtained from (4.6) and (5.13), the orientation solutions will be presented in this section.

The orientation of the surface frame relative to the tool frame and the generator trihedron is shown in Figure 2. Let angle between S_b and A spacelike vectors be defined by φ, referred to as spin angle, we have

$$\begin{gathered}\langle S_b, A\rangle = \cos\varphi,\\ A = \sin\varphi S_n + \cos\varphi S_b, \qquad A\wedge O = N = \sin\varphi S_b - \cos\varphi S_n.\end{gathered} \tag{6.1}$$

We may express the results in matrix form as

$$\begin{bmatrix} O\\ A\\ N\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\\ 0 & \sin\varphi & \cos\varphi\\ 0 & -\cos\varphi & \sin\varphi\end{bmatrix}\begin{bmatrix} O\\ S_n\\ S_b\end{bmatrix}. \tag{6.2}$$

Equation (6.2) may also be rewritten as

$$\begin{bmatrix} O \\ S_n \\ S_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\varphi & -\cos\varphi \\ 0 & \cos\varphi & \sin\varphi \end{bmatrix} \begin{bmatrix} O \\ A \\ N \end{bmatrix}. \tag{6.3}$$

Let the angle between the spacelike vectors S_b and k be defined as ϕ. We have

$$\begin{aligned} S_n &= \sin\phi t + \cos\phi k, \\ S_b &= \cos\phi t - \sin\phi k. \end{aligned} \tag{6.4}$$

We may express the results in matrix form as

$$\begin{bmatrix} O \\ S_n \\ S_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} r \\ t \\ k \end{bmatrix}. \tag{6.5}$$

With the aid of (6.2) and (6.5), we have

$$\begin{bmatrix} O \\ A \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\sum & \cos\sum \\ 0 & -\cos\sum & \sin\sum \end{bmatrix} \begin{bmatrix} r \\ t \\ k \end{bmatrix}, \tag{6.6}$$

where $\varphi + \phi = \sum$. The solution of (6.6) is

$$\begin{bmatrix} r \\ t \\ k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\sum & -\cos\sum \\ 0 & \cos\sum & \sin\sum \end{bmatrix} \begin{bmatrix} O \\ A \\ N \end{bmatrix}, \tag{6.7}$$

where $\sum$ describes the orientation of the end-effector. Substituting the partial derivatives of (4.1) into (4.3).

Because the surface normal vector is determined at the TCP which is on the directrix, u is zero:

$$X_u \wedge X_s = \vec{R} \wedge \alpha', \tag{6.8}$$

and substituting

$$\alpha' = \beta' + \mu\vec{R}' + \mu'\vec{R} \tag{6.9}$$

and with the aid of (4.10),

$$X_u \wedge X_s = \Delta t - \mu k, \qquad \|X_u \wedge X_s\| = \sqrt{\Delta^2 + \mu^2}. \tag{6.10}$$

Finally

$$S_n = \frac{\Delta t - \mu k}{\sqrt{\Delta^2 + \mu^2}}, \tag{6.11}$$

$$S_b = \frac{\Delta k + \mu t}{\sqrt{\Delta^2 + \mu^2}}. \tag{6.12}$$

Comparing (6.11) with (6.5), we observe that

$$\cos\phi = \frac{\Delta}{\sqrt{\Delta^2 + \mu^2}}, \qquad \sin\phi = \frac{-\mu}{\sqrt{\Delta^2 + \mu^2}}. \tag{6.13}$$

Substituting (5.5) and (5.19) into (5.10) gives

$$U_r = \frac{1}{\mathfrak{R}}(\coth\rho r - k) = \kappa b, \tag{6.14}$$

which shows that the binormal vector plays the role of the instantaneous axis of rotation for the generator trihedron.

7. Differential motion of the tool frame

In this section, we obtain expressions for the first and second-order positional variation of the TCP. The space curve generated by TCP from (4.5) is

$$\alpha(s) = \beta(s) + \mu\vec{R}(s). \tag{7.1}$$

Differentiating (7.1) with respect to the arc length, using (4.7) and (4.10), the first-order positional variation of the TCP, expressed in the generator trihedron is

$$\alpha'(s) = r(\Gamma + \mu'\mathfrak{R}) + \Delta k + \mu t. \tag{7.2}$$

Substituting (6.7) into (7.2), it gives

$$\begin{aligned} \alpha'(s) &= (\Gamma + \mu'\mathfrak{R})O + (\mu\sin\Sigma + \Delta\cos\Sigma)A + (-\mu\cos\Sigma + \Delta\sin\Sigma)N, \\ \alpha''(s) &= \left(\Gamma' + \mu''\mathfrak{R} + \frac{\mu}{\mathfrak{R}}\right)r + \left(\frac{\Gamma}{\mathfrak{R}} + 2\mu' + \frac{\Delta}{\mathfrak{R}}\coth\sigma\right)t + \left(\Delta' - \frac{\mu}{\mathfrak{R}}\coth\sigma\right)k. \end{aligned} \tag{7.3}$$

With the aid of (6.7) gives

$$\alpha''(s) = \left(\Gamma' + \mu''\mathfrak{R} + \frac{\mu}{\mathfrak{R}}\right)O + \left[\sin\Sigma\left(\frac{\Gamma}{\mathfrak{R}} + 2\mu' + \frac{\Delta}{\mathfrak{R}}\coth\sigma\right) + \cos\Sigma\left(\Delta' - \frac{\mu}{\mathfrak{R}}\coth\sigma\right)\right]A$$
$$+ \left[-\cos\Sigma\left(\frac{\Gamma}{\mathfrak{R}} + 2\mu' + \frac{\Delta}{\mathfrak{R}}\coth\sigma\right) + \sin\Sigma\left(\Delta' - \frac{\mu}{\mathfrak{R}}\coth\sigma\right)\right]N. \tag{7.4}$$

Differentiating (6.6) and substituting (5.9) into the result to determine the first order angular variation of the tool frame and substituting (6.7) into result gives

$$\frac{d}{ds}\begin{bmatrix} O \\ A \\ N \end{bmatrix} = \frac{1}{\mathfrak{R}}\begin{bmatrix} 0 & \sin\sum & -\cos\sum \\ \sin\sum & 0 & -\Omega\mathfrak{R} \\ -\cos\sum & \Omega\mathfrak{R} & 0 \end{bmatrix}\begin{bmatrix} O \\ A \\ N \end{bmatrix} = U_O \wedge \begin{bmatrix} O \\ A \\ N \end{bmatrix}, \tag{7.5}$$

where

$$\Omega = \sum' + \frac{\coth\rho}{\mathfrak{R}}. \tag{7.6}$$

Thus,

$$U_O = \Omega O - \frac{1}{\mathfrak{R}}\cos\sum A - \frac{1}{\mathfrak{R}}\sin\sum N \tag{7.7}$$

is the Darboux vector of the tool frame. Substituting (6.7) into (7.7) gives

$$U_O = \Omega r - \frac{1}{\mathfrak{R}}k \tag{7.8}$$

and with the aid of (7.6) we have

$$U_O = \Sigma' r + \frac{1}{\mathfrak{R}}\coth\rho r - \frac{1}{\mathfrak{R}}k \tag{7.9}$$

and using (5.10), it becomes

$$U_O = \Sigma' r + U_r. \tag{7.10}$$

Substituting (6.14) into the result, it gives

$$U_O = \Sigma' r + \kappa b. \tag{7.11}$$

The second-order angular variation of the frames may now be obtained by differentiating the Darboux vectors. Differentiating (5.16) gives

$$U_t' = -\kappa' t - \tau' b. \tag{7.12}$$

Differentiating (6.14) gives

$$U_r' = -\kappa' b - \kappa b' \tag{7.13}$$

with the aid of (5.15) the first order derivatives of the generator trihedron may be written as

$$U_r' = -\kappa' b - \kappa\tau n. \tag{7.14}$$

Differentiating (7.8) gives

$$U_O' = \Omega' r + \Omega r' b - \frac{1}{\mathfrak{R}}\kappa'. \tag{7.15}$$

With the aid of (4.7), it is rewritten as

$$U_O' = \Omega' r + t\left(\Omega\frac{1}{\mathfrak{R}} - \frac{1}{\mathfrak{R}}\coth\rho\right) \tag{7.16}$$

or the first order derivatives of the tool frame may be written as

$$U_O' = \Omega' r + \frac{\Sigma'}{\mathfrak{R}} t. \tag{7.17}$$

From the chain rule, the linear velocity and the linear acceleration of the TCP, respectively, are

$$V = \alpha' \dot{S}, \qquad a = \alpha' \ddot{S} + \alpha'' \dot{S}^2 . \tag{7.18}$$

Also, the angular acceleration of the end-effector, respectively, are

$$\overline{w} = \overline{U}_O \dot{S}, \qquad \dot{\overline{w}} = \overline{U}_O \ddot{S} + \overline{U}' \dot{S}^2 . \tag{7.19}$$

Example 7.1. For the timelike ruled surface, shown in Figure 3,

$$\varphi(s,u) = \left(\sqrt{2}\cos s - 2\sqrt{2}u\sin s, \sqrt{2}\sin s + 2\sqrt{2}u\cos s, s + 3u\right) \tag{7.20}$$

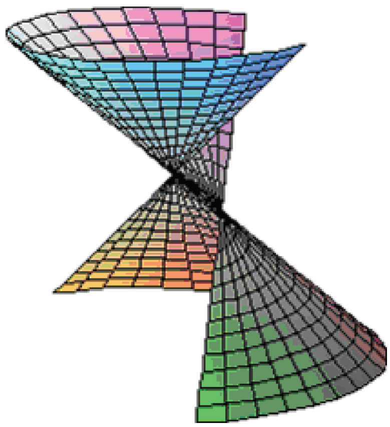

Figure 3: The timelike ruled surface with timelike rulings.

it is easy to see that $\alpha(s) = (\sqrt{2}\cos s, \sqrt{2}\sin s, s)$ is the base curve (spacelike) and $\vec{R}(s) = (-2\sqrt{2}\sin s, 2\sqrt{2}\cos s, 3)$ is the generator (timelike). The striction curve is the base curve. This surface is a timelike ruled surface. Differentiating $\alpha(s)$gives

$$\alpha'(s) = (-\sqrt{2}\sin s, \sqrt{2}\cos s, 1), \tag{7.21}$$

where $\langle \alpha'(s), \alpha'(s) \rangle = 2\,\sin^2 s + 2\cos^2 s - 1 = 1$, so $\alpha(s)$ is spacelike vector. However, $\langle \vec{R}, \vec{R} \rangle = 8\,\sin^2 s + 8\,\cos^2 s - 9 = -1$, therefore $\mathfrak{R} = \|\vec{R}(s)\| = 1$. Hence, generator trihedron is defined as

$$\begin{aligned} r &= \frac{\vec{R}}{\mathfrak{R}} = (-2\sqrt{2}\sin s, 2\sqrt{2}\cos s, 3), \\ t &= (-\cos s, -\sin s, 0), \end{aligned} \tag{7.22}$$

where $\langle t, t \rangle = \sin^2 s + \cos^2 s = 1$ then t is spacelike vector and $\|\vec{R}'(s)\| = 2\sqrt{2}$. Also, $k = t \wedge r = (3\sin s, -3\cos s, -2\sqrt{2})$ where $\langle k, k \rangle = 1 > 0$. So k is spacelike vector. Therefore, we get $\mu = 0$, $\mu'(s) = 0$, $\Gamma = -1$, $\Delta = -\sqrt{2}$ and $\gamma = -3$.

The natural trihedron is defined by

$$t = (-\cos s, -\sin s, 0), \qquad t' = (\sin s, -\cos s, 0), \tag{7.23}$$

where $\kappa = \sqrt{|\langle t', t' \rangle|} = 1$,

$$n = \frac{t'}{\kappa} = (\sin s, -\cos s, 0) \tag{7.24}$$

and $\langle n, n \rangle = \cos^2 s + \sin^2 s = 1$. So n is a spacelike vector. Hence $b = t \wedge n = (0, 0, 1)$.

The Darboux vector of generator trihedron is

$$U_r = (0, 0, 2\sqrt{2}), \tag{7.25}$$

and since $\tau = \langle n', b\rangle = 0$, the Darboux vector of natural trihedron is $U_T = (0, 0, -2\sqrt{2})$.

Substituting the equation $\Delta = -\sqrt{2}$ into (6.13), we have

$$\cos\phi = \frac{-\sqrt{2}}{\sqrt{2+\mu^2}}, \qquad \sin\phi = \frac{-\mu}{\sqrt{2+\mu^2}}. \tag{7.26}$$

So $\tan\phi = \mu/\sqrt{2}$. Differentiating equation gives

$$\left(1+\tan^2\phi\right)\phi' = \frac{\mu'}{\sqrt{2}}. \tag{7.27}$$

Thus

$$\phi' = \frac{\sqrt{2}\mu'}{2+\mu^2}, \qquad \phi'' = \frac{\sqrt{2}}{2+\mu^2}\left(\mu'' - \frac{2\mu(\mu')^2}{2+\mu^2}\right). \tag{7.28}$$

Since the spin angle is zero, $\varphi = 0$ so $\varphi' = 0$, $\Sigma = \phi$, $\Sigma' = \phi'$ and $\Omega = \phi' - 3$. The approach vector and the normal vector, respectively, are

$$\begin{aligned} A &= \frac{1}{\sqrt{2+\mu^2}}\left(-3\sin s + \mu\cos s, 3\sqrt{2}\cos s + \mu\sin s, 4\right), \\ N &= \frac{1}{\sqrt{2+\mu^2}}\left(-\sqrt{2}\cos s - 3\mu\sin s, -\sqrt{2}\sin s + 3\mu\cos s, 2\sqrt{2}\mu\right). \end{aligned} \tag{7.29}$$

The first order positional variation of the TCP may be expressed in the tool frame as

$$\alpha' = (-1+\mu')O + \frac{2-\mu^2}{\sqrt{2+\mu^2}}A + \frac{2\sqrt{2}\mu}{\sqrt{2+\mu^2}}N \tag{7.30}$$

and the Darboux vector of the tool frame is

$$U_O = \Omega O + \frac{\sqrt{2}}{\sqrt{2+\mu^2}}A + \frac{\mu}{\sqrt{2+\mu^2}}N. \tag{7.31}$$

The first order derivative of the Darboux vector of the tool frame

$$U_O' = \phi'' r + \phi' t. \tag{7.32}$$

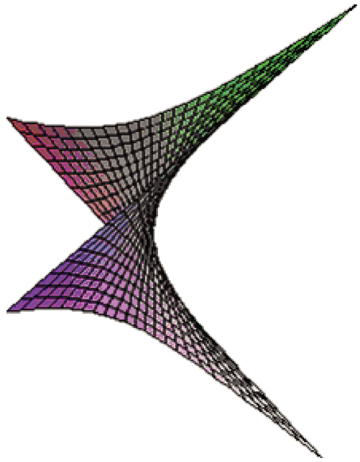

Figure 4: The timelike ruled surface with timelike rulings.

Example 7.2. Let us now consider the timelike ruled surface,

$$\varphi(s,u)=\left(\frac{2\sqrt{3}}{3}s-u\frac{\sqrt{3}}{3},\cosh\left(\frac{\sqrt{3}}{3}s\right)-u\frac{2\sqrt{3}}{3}\sinh\left(\frac{\sqrt{3}}{3}s\right),\sinh\left(\frac{\sqrt{3}}{3}s\right)\right.$$
$$\left.-u\frac{2\sqrt{3}}{3}\cosh\left(\frac{\sqrt{3}}{3}s\right)\right). \tag{7.33}$$

It is easy to see that $\alpha(s)=((2\sqrt{3}/3)s,\cosh((\sqrt{3}/3)s),\sinh((\sqrt{3}/3)s))$ is the base curve (spacelike) and $\vec{R}(s)=(-(\sqrt{3}/3),-(2\sqrt{3}/3)\sinh((\sqrt{3}/3)s),-(2\sqrt{3}/2)\cosh((\sqrt{3}/3)s))$ is the generator (timelike). The striction curve is the base curve. It is clear that this surface is a timelike ruled surface with timelike rulings. Differentiating $\alpha(s)$ gives

$$\alpha'(s)=\left(\frac{2\sqrt{3}}{3},\frac{\sqrt{3}}{3}\sinh\left(\frac{\sqrt{3}}{3}s\right),\frac{\sqrt{3}}{3}\cosh\left(\frac{\sqrt{3}}{3}s\right)\right), \tag{7.34}$$

where $\langle\alpha'(s),\alpha'(s)\rangle = 4/3+(1/3)\sinh^2((\sqrt{3}/3)s)-(1/3)\cosh^2((\sqrt{3}/3)s) = 1$, so $\alpha(s)$ is spacelike vector. $\langle\vec{R},\vec{R}\rangle=-1$, therefore $\mathfrak{R}=\|\vec{R}(s)\|=1$. Hence, generator trihedron is defined as

$$r=\frac{\vec{R}}{\mathfrak{R}}=\left(-\frac{\sqrt{3}}{3},-\frac{2\sqrt{3}}{3}\sinh\left(\frac{\sqrt{3}}{3}s\right),-\frac{2\sqrt{3}}{3}\cosh\left(\frac{\sqrt{3}}{3}s\right)\right),$$
$$t=\left(0,-\cosh\left(\frac{\sqrt{3}}{3}s\right),-\sinh\left(\frac{\sqrt{3}}{3}s\right)\right), \tag{7.35}$$

where $\langle t,t\rangle=\cosh^2((\sqrt{3}/3)s)-\sinh^2((\sqrt{3}/3)s)=1$, then t is spacelike vector and $\|\vec{R}'(s)\|=2/3$.

Also, $k = t \wedge r = (-(2\sqrt{3}/3), -(\sqrt{3}/3)\sinh((\sqrt{3}/3)s), -(\sqrt{3}/3)\cosh((\sqrt{3}/3)s))$ where $\langle k, k\rangle=1 > 0$. So k is spacelike vector. Therefore, we obtain $\mu = 0$, $\mu'(s) = 0$, $\Gamma = 0$, $\Delta = -(2/3)$ and $\gamma = 4/27$.

The natural trihedron is defined by

$$t' = \left(0, -\frac{\sqrt{3}}{3}\sinh\left(\frac{\sqrt{3}}{3}s\right), -\frac{\sqrt{3}}{3}\cosh\left(\frac{\sqrt{3}}{3}s\right)\right). \tag{7.36}$$

Hence

$$\begin{aligned} \kappa &= \sqrt{|\langle t', t'\rangle|} = \frac{1}{\sqrt{3}}, \\ n &= \frac{t'}{\kappa} = \left(0, -\sinh\left(\frac{\sqrt{3}}{3}s\right), -\cosh\left(\frac{\sqrt{3}}{3}s\right)\right), \\ b &= t \wedge n = (-1, 0, 0). \end{aligned} \tag{7.37}$$

Here $\langle n, n\rangle = \sinh^2((\sqrt{3}/3)s) - \cosh^2((\sqrt{3}/3)s) = -1$. So n is a timelike vector.

The Darboux vector of generator trihedron is

$$U_r = \left(-\frac{1}{\sqrt{3}}, 0, 0\right), \tag{7.38}$$

and since $\tau = \langle n', b\rangle = 0$, the Darboux vector of natural trihedron is $U_T = (1/\sqrt{3}, 0, 0)$.

Substituting the equation $\Delta = -(2/3)$ into (6.13), we have

$$\cos\phi = \frac{-2}{\sqrt{4 + 9\mu^2}}, \qquad \sin\phi = \frac{-3\mu}{\sqrt{4 + 9\mu^2}}. \tag{7.39}$$

Thus

$$\phi' = \frac{6\mu'}{4 + 9\mu^2}, \qquad \phi'' = \frac{6}{4 + 9\mu^2}\left(\mu'' - \frac{18\mu(\mu')^2}{4 + 9\mu^2}\right).$$

Since the spin-angle is zero, $\varphi = 0$ so $\varphi' = 0$, $\Sigma = \phi$, $\Sigma' = \phi'$, and $\Omega = \phi' + 4/27$. The approach vector and the normal vector, respectively, are

$$\begin{aligned} A &= \frac{1}{\sqrt{12 + 27\mu^2}}\left(4, 3\sqrt{3}\mu\cosh\left(\frac{\sqrt{3}}{3}s\right) + 2\sinh\left(\frac{\sqrt{3}}{3}s\right), 3\sqrt{3}\mu\sinh\left(\frac{\sqrt{3}}{3}s\right) - 2\cosh\left(\frac{\sqrt{3}}{3}s\right)\right), \\ N &= \frac{1}{\sqrt{12 + 27\mu^2}}\left(6\mu, 2\sqrt{3}\cosh\left(\frac{\sqrt{3}}{3}s\right) + 3\mu\sinh\left(\frac{\sqrt{3}}{3}s\right),\right. \\ &\qquad \left. -2\sqrt{3}\sinh\left(\frac{\sqrt{3}}{3}s\right) + 3\mu\cosh\left(\frac{\sqrt{3}}{3}s\right)\right). \end{aligned} \tag{7.40}$$

The first order positional variation of the TCP may be expressed in the tool frame as

$$\alpha' = \mu' O + \frac{4 - 9\mu^2}{3\sqrt{4 + 9\mu^2}} A + \frac{4\mu}{\sqrt{4 + 9\mu^2}} N \tag{7.41}$$

and the Darboux vector of the tool frame is

$$U_O = \Omega O + \frac{2}{\sqrt{4 + 9\mu^2}} A + \frac{3\mu}{\sqrt{4 + 9\mu^2}} N. \tag{7.42}$$

Finally, the first order derivative of the Darboux vector of the tool frame

$$U_O' = \phi'' r + \phi' t. \tag{7.43}$$

8. Conclusion

The results of this paper will make the accurate motion of the end-effector of a robotic device possible. The paper presents the curvature theory of a general timelike ruled surface. The curvature theory of timelike ruled surfaces is used to determine the differential properties of the motion of a robot end-effector. This provides the properties of the robot end-effector motion in analytical form. The trajectory of a robot end-effector is described by a ruled surface and a spin angle about the ruling of the ruled surface. We consider the curvature theory of timelike ruled surface in Lorentz three-space. This follows three-space series of works in Euclidean space, mainly of [4, 6, 7, 9, 10]. In [2, 3], Ryuh have studied motion of robot by using the curvature theory of ruled surfaces. In [1, 14], the authors have studied of Manipulators. In [3, 16], the authors studied the motion and curvatures in Minkowski space. In [8, 11–13, 18, 19, 22], the authors have studied timelike ruled surfaces in Minkowski space.

This paper has presented the study of the motion of a robot end-effector based on the curvature theory of timelike ruled surfaces with timelike rulings in the Minkowski space. Special types of timelike ruled surfaces such as developable surfaces, which have many practical applications, are not included in this paper. The different studies on the timelike ruled surfaces may be presented in a future publication.

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