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Research Article

Weak Solutions in Elasticity of Dipolar Porous Materials

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The main aim of our study is to use some general results from the general theory of elliptic equations in order to obtain some qualitative results in a concrete and very applicative situation. In fact, we will prove the existence and uniqueness of the generalized solutions for the boundary value problems in elasticity of initially stressed bodies with voids (porous materials).

1. Introduction

The theories of porous materials represent a material length scale and are quite sufficient for a large number of the solid mechanics applications.

In the following, we restrict our attention to the behavior of the porous solids in which the matrix material is elastic and the interstices are voids of material. The intended applications of this theory are to the geological materials, like rocks and soils and to the manufactured porous materials.

The plane of the paper is the following one. In the beginning, we write down the basic equations and conditions of the mixed boundary value problem within context of linear theory of initially stressed bodies with voids, as in the papers of [1, 2]. Then, we accommodate some general results from the paper [3], and the book [4], in order to obtain the existence and uniqueness of a weak solution of the formulated problem. For convenience, the notations chosen are almost identical to those of [2, 5].

2. Basic equations

Let B be an open region of three-dimensional Euclidean space R^3 occupied by our porous material at time $t = 0$. We assume that the boundary of the domain B, denoted by ∂B, is a closed, bounded and pice-wise smooth surface which allows us the application of the divergence theorem. A fixed system of rectangular Cartesian axes is used and we adopt the Cartesian tensor notations. The points in B are denoted by (x_i) or (x). The variable t is the time and $t \in [0, t_0)$. We will employ the usual summation over repeated subscripts while subscripts preceded by a comma denote the partial differentiation with respect to the spatial argument. We also use a superposed dot to denote the partial differentiation with respect to t. The Latin indices are understood to range over the integers (1, 2, 3).

In the following, we designate by n_i the components of the outward unit normal to the surface ∂B. The closure of domain B, denoted by $\overline{B}$, means $\overline{B} = B \cup \partial B$.

Also, the spatial argument and the time argument of a function will be omitted when there is no likelihood of confusion.

The behavior of initially stressed bodies with voids is characterized by the following kinematic variables:

$$u_i = u_i(x,t), \qquad \varphi_{jk} = \varphi_{jk}(x,t), \qquad \sigma = \sigma(x,t), \quad (x,t) \in B \times [0, t_0). \tag{2.1}$$

In our study, we analyze an anisotropic and homogeneous initially stressed elastic solid with voids. We restrict our considerations to the Elastostatics, so that the basic equations become as follows.

(i) The equations of equilibrium is as follows:

$$\begin{gathered} (\tau_{ij} + \eta_{ij})_{,j} + \rho F_i = 0, \\ \mu_{ijk,i} + \eta_{jk} + u_{j,i} M_{ik} + \varphi_{ki} M_{ji} - \varphi_{kr,i} N_{ijr} + \rho G_{jk} = 0; \end{gathered} \tag{2.2}$$

(ii) the balance of the equilibrated forces is as follows:

$$h_{i,i} + g + \rho L = 0; \tag{2.3}$$

(iii) the constitutive equations are as follows:

$$\begin{aligned} \tau_{ij} &= u_{j,k} P_{ki} + C_{ijmn} \varepsilon_{mn} + G_{mnij} \kappa_{mn} + F_{mnrij} \chi_{mnr} + a_{ij} \sigma + d_{ijk} \sigma_{,k}, \\ \eta_{ij} &= -\varphi_{jk} M_{ik} + \varphi_{jk,r} N_{rik} + G_{ijmn} \varepsilon_{mn} + B_{ijmn} \kappa_{mn} + D_{ijmnr} \chi_{mnr} + b_{ij} \sigma + e_{ijk} \sigma_{,k}, \\ \mu_{ijk} &= u_{j,r} N_{irk} + F_{ijkmn} \varepsilon_{mn} + D_{mnijk} \kappa_{mn} + A_{ijkmnr} \chi_{mnr} + c_{ijk} \sigma + f_{ijkm} \sigma_{,m}, \\ h_i &= d_{mni} \varepsilon_{mn} + e_{mni} \kappa_{mn} + f_{mnri} \chi_{mnr} + d_i \sigma + g_{ij} \sigma_{,j}, \\ g &= -a_{mn} \varepsilon_{mn} - b_{mn} \kappa_{mn} - c_{mnr} \chi_{mnr} - \xi \sigma + d_i \sigma_{,i}; \end{aligned} \tag{2.4}$$

(iv) the geometric equations are

$$\begin{gathered} \varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}), \qquad \kappa_{ij} = u_{j,i} - \varphi_{ij}, \\ \chi_{ijk} = \varphi_{jk,i}, \qquad \sigma = \nu - \nu_0. \end{gathered} \tag{2.5}$$

In the above equations we have used the following notations:

(i) ρ—the constant mass density;

(ii) u_i—the components of the displacement field;

(iii) φ_{jk}—the components of the dipolar displacement field;

(iv) ν—the volume distribution function which in the reference state is ν_0;

(v) σ—a measure of volume change of the bulk material resulting from void compaction or distension;

(vi) τ_{ij}, η_{ij}, μ_{ij}—the components of the stress tensors;

(vii) h_i—the components of the equilibrated stress;

(viii) F_i—the components of body force per unit mass;

(ix) G_{jk}—the components of dipolar body force per unit mass;

(x) L—the extrinsic equilibrated body force;

(xi) g—the intrinsic equilibrated body force;

(xii) ε_{ij}, κ_{ij}, χ_{ijk}—the kinematic characteristics of the strain tensors;

(xiii) $C_{ijmn}, B_{ijmn}, \ldots, D_{ijm}, E_{ijm}, \ldots, a_{ij}, b_{ij}, c_{ijk}, d_i, \xi$ represent the characteristic functions of the material (the constitutive coefficients) and they obey to the following symmetry relations

$$\begin{gathered}C_{ijmn} = C_{mnij} = C_{ijnm}, \qquad B_{ijmn} = B_{mnij},\\ G_{ijmn} = G_{ijnm}, \qquad F_{ijkmn} = F_{ijknm}, \qquad A_{ijkmnr} = A_{mnrijk},\\ a_{ij} = a_{ji}, \qquad P_{ij} = P_{ji}, \qquad g_{ij} = g_{ji}.\end{gathered} \tag{2.6}$$

The physical significances of the functions L and h_i are presented in the works [6, 7].

The prescribed functions P_{ij}, M_{ij} and N_{ijk} from (2.2) and (2.3) satisfy the following equations:

$$\left(P_{ij} + M_{ij}\right)_{,j} = 0, \qquad N_{ijk,i} + P_{jk} = 0. \tag{2.7}$$

3. Existence and uniqueness theorems

In the main section of our paper, we will accommodate some theoretical results from the theory of elliptic equations in order to derive the existence and the uniqueness of a generalized solution of the mixed boundary-value problem in the context of initially stressed bodies with voids.

Throughout this section, we assume that B is a Lipschitz region of the Euclidian three-dimensional space R^3. We use the following notations:

$$\mathbf{W} = \left[W^{1,2}(B)\right]^{13}, \qquad \mathbf{W}_0 = \left[W_0^{1,2}(B)\right]^{13}, \tag{3.1}$$

with the convention that $A^{13} = A \times A \times \cdots \times A$, the Cartesian product is considered to be of 13-times. Also, $W^{k,m}$ is the familiar Sobolev space. With other words, $\mathbf{W}$ is defined as the space of all $\mathbf{u} = (u_i, \varphi_{ij}, \sigma)$, where $u_i, \varphi_{ij}, \sigma \in W^{1,2}(B)$ with the norm

$$|\mathbf{u}|_W^2 = |\sigma|^2_{W^{1,2}(B)} + \sum_{i=1}^{3}|u_i|^2_{W^{1,2}(B)} + \sum_{j=1}^{3}\left(\sum_{i=1}^{3}|\varphi_{ij}|^2_{W^{1,2}(B)}\right). \tag{3.2}$$

For clarity and simplification in presentation, we consider the following regularity hypotheses on the considered functions:

(i) all the constitutive coefficients are functions of class C^2 on B;

(ii) the body loads F_i, G_{jk}, and H are continuous functions on B.

The ordered array $(u_i, \varphi_{jk}, \sigma)$ is an admissible process on $\overline{B} = B \cup \partial B$ provided u_i, φ_{jk}, $\sigma \in C^1(\overline{B}) \cap C^2(B)$. Also, the ordered array of functions $(\tau_{ij}, \eta_{ij}, \mu_{ijk}, h_i)$ is an admissible system of stress on $\overline{B}$ if $\tau_{ij}, \eta_{ij}, \mu_{ijk}, h_i \in C^1(B) \cap C^0(\overline{B})$ and $\tau_{ij,i}, \eta_{ij,i}, \mu_{ijk,k}, h_{k,k}, h \in C^0(\overline{B})$.

Let $\partial B = S_u \cup S_t \cup C$ be a disjunct decomposition of ∂B, where C is a set of surface measure and S_u and S_t are either empty or open in ∂B. Assume the following boundary conditions:

$$
\begin{gathered}
u_i = \tilde{u}_i, \quad \varphi_{jk} = \tilde{\varphi}_{jk} \quad \sigma = \tilde{\sigma} \quad \text{on } S_u, \\
t_i \equiv (\tau_{ij} + \eta_{ij}) n_j = \tilde{t}_i, \quad \mu_{jk} \equiv \mu_{ijk} n_i = \tilde{\mu}_{jk}, \quad h \equiv h_i n_i = \tilde{h} \quad \text{on } S_t,
\end{gathered}
\tag{3.3}
$$

where the functions $\tilde{u}_i$, $\tilde{\varphi}_{jk}$, $\tilde{\sigma}$, $\tilde{t}_i$, $\tilde{\mu}_{jk}$, and $\tilde{h}$ are prescribed, $\tilde{u}_i, \tilde{\varphi}_{jk}, \tilde{\sigma} \in W^{1,2}(S_u)$, and $\tilde{t}_i$, $\tilde{\mu}_{jk}$, $\tilde{h} \in L_2(S_t)$. Also, we define $\mathbf{V}$ as a subspace of the space $\mathbf{W}$ of all functions $\mathbf{u} = (u_i, \varphi_{ij}, \sigma)$ which satisfy the boundary conditions:

$$
u_i = 0, \quad \varphi_{ij} = 0, \quad \sigma = 0 \quad \text{on } S_u. \tag{3.4}
$$

On the product space $\mathbf{W} \times \mathbf{W}$, we consider a bilinear form $A(\mathbf{v}, \mathbf{u})$, defined by

$$
\begin{aligned}
A(\mathbf{v}, \mathbf{u}) = \int_B \{ & C_{ijmn} \varepsilon_{mn}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v}) + G_{mnij} [\varepsilon_{ij}(\mathbf{v}) \kappa_{mn}(\mathbf{u}) + \varepsilon_{ij}(\mathbf{u}) \kappa_{mn}(\mathbf{v})] \\
& + F_{mnrij} [\varepsilon_{ij}(\mathbf{v}) \chi_{mnr}(\mathbf{u}) + \varepsilon_{ij}(\mathbf{u}) \chi_{mn}(\mathbf{v})] + B_{ijmn} \kappa_{ij}(\mathbf{v}) \kappa_{mn}(\mathbf{u}) \\
& + D_{ijmnr} [\kappa_{ij}(\mathbf{v}) \chi_{mnr}(\mathbf{u}) + \kappa_{ij}(\mathbf{u}) \chi_{mn}(\mathbf{v})] + A_{ijkmnr} \chi_{ijk}(\mathbf{v}) \chi_{mnr}(\mathbf{u}) \\
& + P_{ki} u_{j,k} v_{j,i} - M_{ik} (u_{j,i} \varphi_{jk} + v_{j,i} \varphi_{jk}) + N_{rik} (u_{j,k} \varphi_{jk,r} + v_{j,k} \varphi_{jk,r}) \\
& + a_{ij} [\varepsilon_{ij}(\mathbf{v}) \sigma + \varepsilon_{ij}(\mathbf{u}) \gamma)] + b_{ij} [\kappa_{ij}(\mathbf{v}) \sigma + \kappa_{ij}(\mathbf{u}) \gamma)] \\
& + c_{ijk} [\chi_{ijk}(\mathbf{v}) \sigma + \chi_{ij}(\mathbf{u}) \gamma)] + d_{ijk} [\varepsilon_{ij}(\mathbf{v}) \sigma_{,k} + \varepsilon_{ij}(\mathbf{u}) \gamma_{,k})] \\
& + e_{ijk} [\kappa_{ij}(\mathbf{v}) \sigma_{,k} + \kappa_{ij}(\mathbf{u}) \gamma_{,k})] + f_{ijkm} [\chi_{ijk}(\mathbf{v}) \sigma_{,m} + \chi_{ijk}(\mathbf{u}) \gamma_{,m})] \\
& + d_i [\sigma \gamma_{,i} + \gamma \sigma_{,i}] + g_{ij} \sigma_{,i} \gamma_{,j} + \xi \sigma \gamma \} dV,
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{gathered}
\mathbf{u} = (u_i, \varphi_{ij}, \sigma), \qquad \mathbf{v} = (v_i, \psi_{ij}, \gamma), \qquad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{j,i} + u_{i,j}), \\
\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}(v_{j,i} + v_{i,j}), \qquad \kappa_{ij}(\mathbf{u}) = u_{j,i} - \varphi_{ij}, \qquad \kappa_{ij}(\mathbf{v}) = v_{j,i} - \psi_{ij}, \\
\chi_{ijk}(\mathbf{u}) = \varphi_{jk,i}, \qquad \chi_{ijk}(\mathbf{v}) = \psi_{jk,i}.
\end{gathered}
\tag{3.6}
$$

We assume that the constitutive coefficients are bounded measurable functions in B which satisfy the symmetries (2.6). Then, by using relations (3.5) and (2.6) it is easy to deduce that

$$
A(\mathbf{v}, \mathbf{u}) = A(\mathbf{u}, \mathbf{v}). \tag{3.7}
$$

Also, by using symmetries (2.6) into (3.5), it results in

$$
\begin{aligned}
A(\mathbf{u},\mathbf{u}) = \int_B \big[& C_{ijmn}\varepsilon_{mn}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}) + 2G_{mnij}\varepsilon_{ij}(\mathbf{u})\kappa_{mn}(\mathbf{u}) \\
& + B_{ijmn}\kappa_{ij}(\mathbf{u})\kappa_{mn}(\mathbf{u}) + 2F_{mnrij}\varepsilon_{ij}(\mathbf{u})\chi_{mnr}(\mathbf{u}) + 2D_{ijmnr}\kappa_{ij}(\mathbf{u})\chi_{mnr}(\mathbf{u}) \\
& + A_{ijkmnr}\chi_{ijk}(\mathbf{u})\chi_{mnr}(\mathbf{u}) + P_{ki}u_{j,k}u_{j,i} - 2M_{ik}u_{j,i}\varphi_{jk} \\
& + 2N_{rik}u_{j,i}\varphi_{jk,r} + 2a_{ij}\varepsilon_{ij}(\mathbf{u})\sigma + 2b_{ij}\kappa_{ij}(\mathbf{u})\sigma + 2c_{ijk}\chi_{ijk}(\mathbf{u})\sigma \\
& + 2d_{ijk}\varepsilon_{ij}(\mathbf{u})\sigma_{,k} + 2e_{ijk}\kappa_{ij}(\mathbf{u})\sigma_{,k} + 2f_{ijkm}\chi_{ijk}(\mathbf{u})\sigma_{,m} \\
& + 2d_i\sigma\gamma_{,i} + g_{ij}\sigma_{,i}\sigma_{,j} + \xi\sigma^2 \big] dV,
\end{aligned}
\tag{3.8}
$$

and thus

$$
A(\mathbf{u},\mathbf{u}) = 2\int_B U dV, \tag{3.9}
$$

where $U = \rho e$ is the internal energy density associated to $\mathbf{u}$.

We suppose that U is a positive definite quadratic form, that is, there exists a positive constant c such that

$$
\begin{aligned}
& C_{ijmn}x_{ij}x_{mn} + 2G_{ijmn}x_{ij}y_{mn} + 2F_{ijmnr}x_{ij}z_{mnr} \\
& \quad + B_{ijmn}y_{ij}y_{mn} + 2D_{ijmnr}y_{ij}z_{mnr} + A_{ijkmnr}z_{ijk}z_{mn} \\
& \quad + P_{ki}x_{ji}x_{jk} - 2M_{ik}x_{ji}y_{jk} + 2N_{rik}x_{ji}z_{jkr} + 2a_{ij}x_{ij}w \\
& \quad + 2b_{ij}y_{ij}w + 2c_{ijk}z_{ij}w + 2d_{ijk}x_{ij}\omega_k + 2e_{ijk}y_{ij}\omega_k \\
& \quad + 2f_{ijkm}z_{ijk}\omega_m + 2d_i w\omega_i + g_{ijk}\omega_i\omega_j + \xi w^2 \\
& \geq c\left(x_{ij}x_{ij} + y_{ij}x_{ij} + z_{ijk}z_{ijk} + \omega_i\omega_i + w^2\right),
\end{aligned}
\tag{3.10}
$$

for all x_{ij}, y_{ij}, z_{ijk}, ω_i, and w.

Now, we introduce the functionals $f(\mathbf{v})$ and $g(\mathbf{v})$ by

$$
\begin{aligned}
f(\mathbf{v}) &= \int_B \rho\left(F_i v_i + G_{jk}\psi_{jk} + L\gamma\right) dV, \quad \mathbf{v} \in \mathbf{W}, \\
g(\mathbf{v}) &= \int_{S_t} \left(\tilde{t}_i v_i + \tilde{\mu}_{jk}\psi_{jk} + \tilde{h}\gamma\right) dA, \quad \mathbf{v} \in \mathbf{W},
\end{aligned}
\tag{3.11}
$$

where $\mathbf{v} = (v_i, \psi_{jk}, \gamma) \in \mathbf{W}$ and ρ, F_i, G_{jk}, $L \in L_2(B)$.

Let $\mathbf{v} = (\tilde{u}_i, \tilde{\varphi}_{jk}, \tilde{\gamma}) \in \mathbf{W}$ be such that $\tilde{u}_i$, $\tilde{\varphi}_{jk}$, $\tilde{\gamma}$ on S_u may by obtained by means of embedding the space $W^{1,2}$ into the space $L_2(S_u)$.

The element $\mathbf{v} = (u_i, \varphi_{jk}, \sigma) \in \mathbf{W}$ is called *weak (or generalized) solution* of the boundary value problem, if

$$
\begin{aligned}
\mathbf{u} - \tilde{\mathbf{u}} &\in \mathbf{V}, \\
A(\mathbf{u},\mathbf{u}) &= f(\mathbf{u}) + g(\mathbf{v})
\end{aligned}
\tag{3.12}
$$

hold for each $\mathbf{v} \in \mathbf{V}$. In the above relations, we used the spaces $L_2(B)$ and $L_2(S_u)$ which represent, as it is well known, the space of real functions which are square-integrable on B, respectively, on $S_u \subset \partial B$.

It follows from (3.10) and (3.8) that

$$A(\mathbf{v},\mathbf{v}) \geq 2c\int_B \left[\varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) + \kappa_{ij}(\mathbf{v})\kappa_{ij}(\mathbf{v}) + \chi_{ijk}(\mathbf{v})\chi_{ijk}(\mathbf{v}) + \gamma_i\gamma_i + \gamma^2\right]dV, \tag{3.13}$$

for any $\mathbf{v} = (v_i, \psi_{jk}, \gamma) \in \mathbf{W}$.

Let us consider the operators $N_k\mathbf{v}$, $k = 1,2,\ldots,49$, mapping the space $\mathbf{W}$ into the space $L_2(B)$, defined by

$$\begin{gathered} N_i\mathbf{v} = \varepsilon_{1i}(\mathbf{v}), \quad N_{3+i}\mathbf{v} = \varepsilon_{2i}(\mathbf{v}), \quad N_{6+i}\mathbf{v} = \varepsilon_{2i}(\mathbf{v}), \\ N_{9+i}\mathbf{v} = \kappa_{1i}(\mathbf{v}), \quad N_{12+i}\mathbf{v} = \kappa_{1i}(\mathbf{v}), \quad N_{15+i}\mathbf{v} = \kappa_{1i}(\mathbf{v}), \\ N_{18+i}\mathbf{v} = \chi_{11i}(\mathbf{v}), \quad N_{21+i}\mathbf{v} = \chi_{12i}(\mathbf{v}), \quad N_{24+i}\mathbf{v} = \chi_{13i}(\mathbf{v}), \\ N_{27+i}\mathbf{v} = \chi_{21i}(\mathbf{v}), \quad N_{30+i}\mathbf{v} = \chi_{22i}(\mathbf{v}), \quad N_{33+i}\mathbf{v} = \chi_{23i}(\mathbf{v}), \\ N_{36+i}\mathbf{v} = \chi_{31i}(\mathbf{v}), \quad N_{39+i}\mathbf{v} = \chi_{32i}(\mathbf{v}), \quad N_{42+i}\mathbf{v} = \chi_{33i}(\mathbf{v}), \\ N_{45+i}\mathbf{v} = \sigma_{,i}(\mathbf{v}), \quad N_{49}\mathbf{v} = \sigma(\mathbf{v}) \quad (i = 1,\ 2,\ 3). \end{gathered} \tag{3.14}$$

It is easy to see that, in fact, the operators $N_k\mathbf{v}$, $k = 1,2,\ldots,49$, defined above, have the following general form:

$$N_k\mathbf{v} = \sum_{r=1}^{m}\sum_{|\alpha|\leq k_r} n_{k_r\alpha} D^\alpha v_r, \quad p = |\alpha|, \tag{3.15}$$

where $n_{k_r\alpha}$ are bounded and measurable functions on B. Also, we have used the notation D^α for the multi-indices derivative, that is,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}. \tag{3.16}$$

By definition, the operators $N_k\mathbf{v}$, $(k = 1,2,\ldots,49)$ form a coercive system of operators on $\mathbf{W}$if for each $\mathbf{v} \in \mathbf{W}$ the following inequality takes place:

$$\sum_{k=1}^{49}\left|N_k\mathbf{v}\right|^2_{L_2(B)} + \sum_{r=1}^{13}\left|v_r\right|^2_{L_2(B)} \geq c_1|\mathbf{v}|^2_{\mathbf{W}}, \quad c_1 > 0. \tag{3.17}$$

In this inequality, the constant c_1 does not depend on $\mathbf{v}$ and the norms $|\cdot|_{L_2}$ and $|\cdot|_{\mathbf{W}}$ represent the usual norms in the spaces $L_2(B)$ and $\mathbf{W}$, respectively.

In the following theorem, we indicate a necessary and sufficient condition for a system of operators to be a coercive system.

Theorem 3.1. *Let $n_{ps\alpha}$ be constant for $|\alpha| = k_s$. Then the system of operators $N_p\mathbf{v}$ is coercive on $\mathbf{W}$ if and only if the rank of the matrix*

$$\left(N_{ps}\xi\right) = \left(\sum_{|\alpha|=k_s} n_{ps\alpha}\xi_\alpha\right) \tag{3.18}$$

is equal to m for each $\xi \in \mathbf{C}_3$, $\xi \neq 0$, where $\mathbf{C}_3$ is the notation for the complex three-dimensional space, and

$$\xi_\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}. \tag{3.19}$$

The demonstration of this result can be find in [4].

In the following, we assume that for each $\mathbf{v} \in \mathbf{W}$, we have

$$A(\mathbf{v},\mathbf{v}) \geq c_2 \sum_{k=1}^{49} \left| N_k \mathbf{v} \right|_{L_2}^2, \quad c_2 > 0, \tag{3.20}$$

where the constant c_2 does not depend on $\mathbf{v}$.

We denote by $\mathcal{P}$ the following set:

$$\mathcal{P} = \left\{ \mathbf{v} \in \mathbf{V} : \sum_{k=1}^{49} \left| N_k \mathbf{v} \right|_{L_2}^2 = 0 \right\}, \tag{3.21}$$

and by $\mathbf{V}/\mathcal{P}$ the factor-space of classes $\tilde{\mathbf{v}}$, where

$$\tilde{\mathbf{v}} = \{ \mathbf{v} + \mathbf{p},\ \mathbf{v} \in \mathbf{V},\ \mathbf{p} \in \mathcal{P} \}, \tag{3.22}$$

having the norm

$$\left| \tilde{\mathbf{v}} \right|_{\mathbf{V}/\mathcal{P}} = \inf_{\mathbf{p} \in \mathcal{P}} |\mathbf{v} + \mathbf{p}|_{\mathbf{W}}. \tag{3.23}$$

In the following theorem, it is indicated a necessary and sufficient condition for the existence of a weak solution of the boundary-value problem.

Theorem 3.2. *Let $A(\mathbf{v},\mathbf{u}) = [\tilde{\mathbf{v}}, \tilde{\mathbf{u}}]$ define a bilinear form for each $\tilde{\mathbf{v}}, \tilde{\mathbf{u}} \in \mathbf{W}/\mathcal{P}$, where $\mathbf{u} \in \tilde{\mathbf{u}}$ and $\mathbf{v} \in \tilde{\mathbf{v}}$. If it is supposed that the inequalities* (3.17) *and* (3.20) *hold, then a necessary and sufficient condition for the existence of a weak solution of the boundary value problem is*

$$\mathbf{p} \in \mathcal{P} \Longrightarrow f(\mathbf{p}) + g(\mathbf{p}) = 0. \tag{3.24}$$

Moreover, the weak solution, $\mathbf{u} \in \mathbf{W}$, satisfies the following inequality:

$$|\mathbf{u}|_{\mathbf{W}/\mathcal{P}} \leq c_3 \left[\left| \tilde{\mathbf{u}} \right|_{\mathbf{W}} + \left(\sum_{i=1}^{m} \left| f_i \right|_{L_2(B)} \right)^{1/2} + \left(\sum_{i=1}^{m} \left| g_i \right|_{L_2(S)} \right)^{1/2} \right], \tag{3.25}$$

where c_3 is a real positive constant.

Further, one has

$$A(\tilde{\mathbf{v}}, \tilde{\mathbf{v}}) \geq c_4 \left| \tilde{\mathbf{v}} \right|_{\mathbf{W}/\mathcal{P}}, \quad c_4 > 0, \tag{3.26}$$

for each $\tilde{\mathbf{v}} \in \mathbf{W}/\mathcal{P}$.

For the prove of this result, see [3].

In the following, we intend to apply the above two results in order to obtain the existence of a weak solution for the boundary value problem formulated in the context of theory of initially stressed elastic solids with voids.

Theorem 3.3. *Let $\mathcal{P} = \{0\}$. Then there exists one and only one weak solution $\mathbf{u} \in \mathbf{W}$ of our boundary-value problem.*

Proof. Clearly, from (3.13) and (3.14) we immediately obtain (3.20). The matrix (3.20) has the rank 13 for each $\xi \in C_3$, $\xi \neq 0$. Thus by Theorem 3.1 we conclude that the system of N_k operators, defined in (3.14), is coercive on the space **W**.

According to definition (3.21) of $\mathcal{P}$, we have that $\varepsilon_{ij}(\mathbf{v}) = 0, \kappa_{ij}(\mathbf{v}) = 0$, $\chi_{ijk}(\mathbf{v}) = 0$, $\gamma_i(\mathbf{v}) = 0$, $\psi = 0$ for each $\mathbf{v} \in \mathcal{P}$, $\mathbf{v} = (v_i, \varphi_{jk}, \psi)$.

So, we deduce that $\mathcal{P}$ reduces to

$$\mathcal{P} = \{v = (v_i, \varphi_{jk}, \gamma) \in \mathbf{V} : v_i = a_i + \varepsilon_{ijk} b_j x_k, \varphi_{jk} = \varepsilon_{jks} b_s, \gamma = c\}, \tag{3.27}$$

where a_i and b_i and c are arbitrary constants and ε_{ijk} is the alternating symbol.

We will consider two distinct cases. First, we suppose that the set S_u is nonempty. Then the set $\mathcal{P}$ reduces to $\mathcal{P} = \{0\}$, and therefore, condition (3.24) is satisfied. By using Theorem 3.2, we immediately obtain the desired result. □

In the second case, we assume that S_u is an empty set. Then we have the following result.

Theorem 3.4. *The necessary and sufficient conditions for the existence of a weak solution* $\mathbf{u} \in \mathbf{W}$ *of the boundary-value problem for elastic dipolar bodies with stretch, are given by*

$$\begin{gathered}\int_B \rho F_i dV + \int_{\partial B} \tilde{t}_i \, dA = 0, \\ \int_B \rho \varepsilon_{ijk} (x_j F_k + G_{jk}) \, dV + \int_{\partial B} \varepsilon_{ijk} (x_j \tilde{t}_k + \tilde{\mu}_{jk}) \, dA = 0,\end{gathered} \tag{3.28}$$

where ε_{ijk} *is the alternating symbol.*

Proof. In this case, the boundary value problem $\mathcal{P}$ is given by (3.27), where a_i, b_i, and c are arbitrary constants such that we can apply, once again, Theorem 3.2 to obtain the above result. □

4. Conclusion

For the considered initial-boundary value problem the basic results still valid. Now, for different particular cases, the solution can be found because it exists and is unique.

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