Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2007, Article ID 28430, 11 pages
doi:10.1155/2007/28430

Research Article

Nonlinear Equations with a Retarded Argument in Discrete-Continuous Systems

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Received 18 April 2006; Accepted 20 March 2007

Recommended by Katica R. (Stevanović) Hedrih

The paper deals with dynamic problems of discrete-continuous systems with local nonlinearities, the analysis of which is reduced to solving nonlinear differential equations with a retarded argument. This concerns the discrete-continuous systems subject to torsional, longitudinal, or shear deformations, where the equations of motion for elastic elements are classical wave equations. It is assumed that the characteristics of local nonlinearities are of a soft-type and in the paper they are described by four nonlinear functions. After a short general description of the approach used, the detailed considerations and numerical results are presented for a multimass discrete-continuous system with a local nonlinearity having the characteristics of a soft-type subject to shear deformations.

1. Introduction and nonlinear forces

Equations with a retarded argument have a wide application in the theory of mathematical control, in theory of optimal control, in mathematical biology, in mathematical economics, and so forth (Bainov and Mischev [1]; Hale [2]; Muszyñski and Myszkis [3]). In such equations, the unknown functions and their derivatives have different arguments. It appears that certain discrete-continuous systems can also be described by ordinary differential equations with a retarded argument. Such systems consist of rigid bodies connected with elastic elements. Among these systems, one can distinguish those where elastic elements undergo torsional, longitudinal, or shear deformations. The equations of motion for these elements are classical wave equations. In torsionally deformed systems, one may consider shafts which can be found, for example, in branched systems, gear transmissions, internal combustion engines, and transport drive systems (Nadolski and Pielorz [4, 5]; Pielorz [6–9], Szolc [10]). For longitudinally deformed systems, one may consider certain machine elements, truss members, railway cars, and river barges (Pielorz [6, 11, 12]). Among systems subject to shear deformations, one may consider, for example, string systems and low structures subject to transversal excitations (Bogacz and Szolc [13], Pielorz [6, 14–16], Szolc [10]).

The paper deals with discrete-continuous systems having local nonlinearities. The characteristics of these nonlinearities can be of a hard or of a soft-type. In the paper by Pielorz [16], the discrete-continuous system with a local nonlinearity described by the polynomial of a third degree is considered. There it was shown that such a nonlinear function can be used in both cases of the characteristic. Here the discussion is concentrated only on the systems having nonlinearities with the characteristic of a soft-type; however, they are described by the following four nonlinear functions:

$$F(y_i) = k_1 y_i + k_3 y_i^3 \quad \text{with } k_3 \le 0, \tag{1.1}$$

$$F(y_i) = A\sin(By_i), \tag{1.2}$$

$$F(y_i) = A\tanh(By_i), \tag{1.3}$$

$$\begin{aligned} F(y_i) &= A[-1+\exp(By_i)] \quad \text{for } y_i \le 0,\\ F(y_i) &= A[1-\exp(-By_i)] \quad \text{for } y_i \ge 0, \end{aligned} \tag{1.4}$$

where y_i is the displacement of a cross-section where a local nonlinearity is located. It is assumed that all functions describe the same linear case, (1.1) is the expansion of the sinusoidal function (1.2), and that the polynomial function and the functions (1.2)–(1.4) have close maximum values. Thus, the constants A and B are connected with the constants k_1 and k_3 by relations (Pielorz [15])

$$AB = k_1, \qquad AB^3 = -6k_3. \tag{1.5}$$

Four nonlinear functions are proposed for the description of the local nonlinearities in order to avoid some inconveniences which appear when the function (1.1) is used, that is, escape phenomena (Stewart et al. [17]).

The aim of the paper is to show that using a wave approach, the dynamical analysis of the discrete-continuous systems with a local nonlinearity leads to solving nonlinear ordinary differential equations with a retarded argument of the neutral type and that four proposed nonlinear functions can be incorporated in the description of the local nonlinearity. After a short general introduction to the approach used, the detailed considerations are presented for a multimass discrete-continuous system subject to shear deformations. Some numerical results are given for a four-mass system.

2. Wave approach

Consider the systems described by discrete-continuous models consisting of an arbitrary number of homogeneous elastic elements connected by means of a suitable number of rigid bodies. The cross-sections of elastic elements remain flat during the motion. Besides, the elastic elements have finite lengths and constant cross-sections, so their motion is described by the classical wave equation. The displacements y_i and velocities of all cross-sections of the elastic elements are assumed to be equal to zero at the time instant

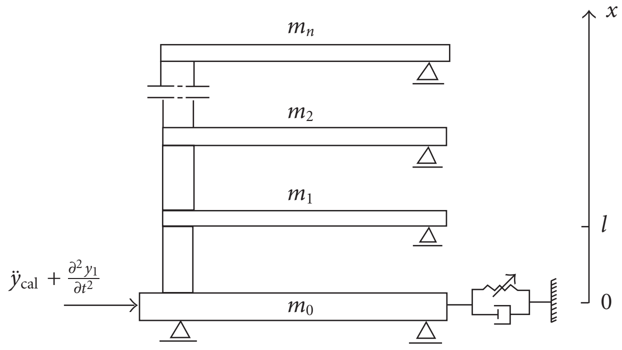

Figure 3.1. Nonlinear discrete-continuous system subject to shear deformations.

$t = 0$, and the system is loaded by the forces or is subjected to external excitation. In these models, local nonlinearities represented by nonlinear discrete elements are introduced. The inclusion of such types of nonlinearities is suggested by many engineering solutions and here they are described by means of four nonlinear functions (1.1)–(1.4).

The equation of motion for the ith elastic element is a classical wave equation. Searching solutions for specific problems, we must add to equations of motion the zero initial conditions and appropriate boundary conditions. These are the conditions for the displacements of the ith and the $i+1$th elastic elements of the system in the cross-sections of the contact of these elements, or the conditions for forces acting either in the cross-sections of the contact of the neighboring elastic elements or in the cross-sections in which rigid bodies are attached.

The solutions of the equations of motion are sought in the form of the sum of two functions (the solution of the d'Alembert type), representing waves propagating in the elastic element of the systems under consideration. The more detailed considerations are performed below for the multimass system subject to shear deformations.

3. The system subject to shear deformations

As an example, the system shown in Figure 3.1 is considered. The elastic elements of this system have the transverse dimension, alongside of which shear forces act close to the length of the element, that is, they have the low slenderness ratio. To such structures belong, for example, machine supports, bridge piers, and low columns in buildings, Pielorz [14, 15].

The studied model consists of n elastic elements connected by rigid bodies. It is assumed that all the elements are characterized by shear modulus G, the cross-sectional area A, shear coefficient k, density ρ, and the length l. To the rigid body m_0, a discrete element with a nonlinear spring can be attached. Such an element may represent various parts of considered structures which ought to be described by local nonlinearities. For example, it may represent an elastic segment of an isolation type.

The rigid body m_0 is subject to the absolute acceleration $(\partial^2 y_1/\partial t^2)(0,t) + \ddot{y}_{\text{cal}}(t)$, where $y_1(0,t)$ is the displacement of the rigid body m_0 in relation to the ground and $y_{\text{cal}}(t)$ is the ground displacement in relation to the fixed spatial system. Damping in the model is described by means of an equivalent external and internal damping expressed by

$$R_{di} = d_i \frac{\partial y_i}{\partial t}, \qquad R_{Vi} = D_i \frac{\partial^2 y_i}{\partial x \partial t}, \quad i = 0, 1, \ldots, n, \tag{3.1}$$

where constants d_i and D_i are coefficients of external and internal damping. The x-axis direction is normal to the direction of displacements y_i, its origin coincides with the location of the rigid body m_0 in an undisturbed state, and velocities and displacements of the cross-sections of all the elastic elements are equal to zero at time instant $t = 0$.

Upon the introduction of the appropriate nondimensional quantities (Pielorz [14, 15])

$$\begin{gathered} \bar{x} = \frac{x}{l}, \qquad \bar{t} = \frac{ct}{l}, \qquad K_r = \frac{A\rho l}{m_r}, \qquad \bar{D}_i = \frac{D_i c}{l}, \qquad \bar{l} = 1, \\ \bar{d}_i = \frac{d_i l}{m_r c}, \qquad \bar{y}_i = \frac{y_i}{y_r}, \qquad R_i = \frac{m_i}{m_r}, \qquad \bar{F} = \frac{F l^2}{c^2 y_r m_r}, \end{gathered} \tag{3.2}$$

where $c^2 = kG/\rho$ is a wave speed, m_r and y_r are a fixed mass and displacement, respectively. The problem of determining the displacements, strains, and velocities in the cross-sections of the elastic elements for the analyzed model is reduced to solving n classical wave equations

$$\frac{\partial^2 y_i}{\partial t^2} - \frac{\partial^2 y_i}{\partial x^2} = 0 \quad \text{for } i = 1, 2, \ldots, n \tag{3.3}$$

with the zero initial conditions

$$y_i(x, 0) = \frac{\partial y_i}{\partial t}(x, 0) = 0 \quad \text{for } i = 1, 2, \ldots, n \tag{3.4}$$

and with the following nonlinear boundary conditions:

$$\begin{gathered} R_0 \ddot{y}_{\text{cal}}(t) + R_0 \frac{\partial^2 y_1}{\partial t^2} + d_0 \frac{\partial y_1}{\partial t} - K_r \left(D_1 \frac{\partial^2 y_1}{\partial x \partial t} + \frac{\partial y_1}{\partial x} \right) + F(y_1) = 0 \quad \text{for } x = 0, \\ y_i = y_{i+1} \quad \text{for } x = i,\ i = 1, 2, \ldots, n-1, \\ K_r \left(D_i \frac{\partial^2 y_i}{\partial x \partial t} + \frac{\partial y_i}{\partial x} \right) - K_r \left(D_{i+1} \frac{\partial^2 y_{i+1}}{\partial x \partial t} + \frac{\partial y_{i+1}}{\partial x} \right) + R_i \frac{\partial^2 y_{i+1}}{\partial t^2} + d_i \frac{\partial y_{i+1}}{\partial t} = 0 \\ \text{for } x = i,\ i = 1, 2, \ldots, n-1, \\ K_r \left(D_n \frac{\partial^2 y_n}{\partial x \partial t} + \frac{\partial y_n}{\partial x} \right) + R_n \frac{\partial^2 y_n}{\partial t^2} + d_n \frac{\partial y_n}{\partial t} = 0 \quad \text{for } x = n. \end{gathered} \tag{3.5}$$

In (3.3)–(3.5), bars denoting dimensionless quantities are omitted for convenience.

The solutions of (3.3) taking into account the initial conditions (3.4) are sought in the form

$$y_i(x, t) = f_i(t - x) + g_i(t + x - 2(i - 1)), \quad i = 1, 2, \ldots, n, \tag{3.6}$$

where the unknown functions f_i and g_i represent the waves, caused by the kinematic excitation, propagating in the ith elastic element of the discrete-continuous model. In the sought solutions (3.6), it is taken into account that the first disturbance occurs in the ith element at time $t = i - 1$ in the cross-section $x = i - 1$ for $i = 1, 2, \ldots, n$. The functions f_i and g_i are continuous and identical to zero for negative arguments.

Upon substituting the solutions (3.6) into the boundary conditions (3.5) and denoting the largest argument of functions appearing in each equality by z, the following nonlinear equations are obtained for the functions f_i and g_i:

$$
\begin{gathered}
g_i(z) = f_{i+1}(z-2) + g_{i+1}(z-2) - f_i(z-2), \quad i = 1, 2, \ldots, n-1, \\
r_{n+1,1}\, g_n''(z) + r_{n+1,2}\, g_n'(z) = r_{n+1,3}\, f_n''(z-2) + r_{n+1,4}\, f_n'(z-2), \\
r_{11}\, f_1''(z) = -R_0 \ddot{y}_{\mathrm{cal}}(z) + r_{12}\, g_1''(z) + r_{13}\, f_1'(z) + r_{14}\, g_1'(z) + F\big(g_1(z) + f_1(z)\big), \\
r_{i1}\, f_i''(z) + r_{i2}\, f_i'(z) = r_{i3}\, g_i''(z) + r_{i4}\, g_i'(z) + r_{i5}\, f_{i-1}''(z) + r_{i6}\, f_{i-1}'(z), \quad i = 2, 3, \ldots, n,
\end{gathered}
\tag{3.7}
$$

where

$$
\begin{aligned}
& r_{11} = K_r D_1 + R_0, \qquad r_{12} = K_r D_1 - R_0, \\
& r_{13} = -K_r - d_0, \qquad r_{14} = K_r - d_0, \\
& r_{i1} = K_r D_i + K_r D_{i-1} + R_{i-1}, \qquad r_{i2} = 2K_r + d_{i-1}, \\
& r_{i3} = K_r D_i - K_r D_{i-1} - R_{i-1}, \qquad r_{i4} = -d_{i-1}, \\
& r_{i5} = 2K_r D_{i-1}, \qquad r_{i6} = 2K_r, \quad i = 2, 3, \ldots, n \\
& r_{n+1,1} = K_r D_n + R_0, \qquad r_{n+1,2} = K_r + d_n, \\
& r_{n+1,3} = K_r D_n - R_0, \qquad r_{n+1,4} = K_r - d_n.
\end{aligned}
\tag{3.8}
$$

Equations (3.7) are nonlinear differential equations with a retarded argument of the neutral type. They can be solved only numerically using, for example, the Runge-Kutta method. Having obtained from (3.7) the functions $f_i(z)$ and $g_i(z)$ and their derivatives, one can determine displacements, strains, and velocities in an arbitrary cross-section of the elastic elements in the considered model at an arbitrary time instant. Equations (3.7) differ from appropriate equations in (Pielorz [16]) by the possibility to use four nonlinear functions (1.1)–(1.4) for the description of the local nonlinearity.

The properties of the solutions of equations with a retarded argument are discussed in the literature (Bainov and Mischev [1]; Hale [2]; Muszyñski and Myszkis [3]). Here we are interested only in the numerical investigations of some properties of the system described by (3.7). Especially, we are interested in the effect of the local nonlinearity with the characteristic of a soft-type described by four nonlinear functions on the behavior of the discrete-continuous system shown in Figure 3.1.

4. Numerical results

The numerical analysis is performed for the model presented in Figure 3.1 with four rigid bodies, $n = 3$. The function of the external excitation $y_{\mathrm{cal}}(t)$ is arbitrary: irregular or regular, periodic or nonperiodic. In the paper, in the analogy to the nonlinear discrete problems, it is assumed in the form

$$\ddot{y}_{\mathrm{cal}}(t) = a_0 \sin(pt) \tag{4.1}$$

and the considerations are focused on the determination of displacements in the steady states. By means of the function (4.1) various direct and indirect external excitations can be described, where p is the dimensionless frequency of the external excitation. Under the external excitation (4.1) changing in time harmonically, harmonic, superharmonic, and subharmonic vibrations can appear. The paper concerns only harmonic vibrations.

The considered discrete-continuous systems represent low structures and are described by the dimensionless parameters R_i, K_r; see (3.2). These parameters can have various values. The constants R_i are the ratios of the masses m_i and the mass of the foundation m_0 while the constant K_r is the ratio of the mass of columns and m_0. For real structures such parameters are usually smaller than 1. In the presented calculations, they are assumed to be equal 0.5 and 0.3, similarly as in the paper by Pielorz [14].

In numerical calculations, we concentrate on the presentation of the influence of the local nonlinearity with the characteristic of a soft-type on displacements and forces in selected cross-sections. The local nonlinearity is described by the functions (1.1)–(1.4). The dimensionless parameter k_1 is fixed and is equal to 0.3 while the dimensionless parameters k_3 and a_0 can be varied. The parameter k_3 is connected directly with the local nonlinearity, and a_0 is the amplitude of the external excitation (4.1). The remaining dimensionless parameters appearing in equations (3.8) are fixed and equal to $R_0 = 1.0$, $K_r = 0.3$, $k_1 = 0.3$, $m_r = m_0$, $R_i = 0.5$, $i = 1, 2, \ldots, n$.

Equations (3.7) enable us to determine the numerical solution in an arbitrary cross-section of the discrete-continuous systems. Below, however, the effect of the local nonlinearity on displacements and forces in the cross-section $x = 0$ is investigated because in this cross-section the local nonlinearity is introduced.

Diagrams in Figures 4.1–4.3 show amplitude-frequency curves for displacements and dynamic forces, correspondingly, of the four-mass system with $n = 3$ in $x = 0$. One would expect that with the increase of the amplitude of the external excitation, the amplitudes of displacements and forces should increase for each p. When the function (1.1) is used, it is true up to the frequency p for which the function (1.1) approaches the maximum value postulated by the constant k_3. For $k_3 = -0.05$, this maximum value is equal to 0.2828. Then the solutions begin to diverge to infinity, and that is connected with the properties of the potential of the function (1.1). The escape phenomenon is known in nonlinear discrete systems (Stewart et al. [17]). Thus, this phenomenon is also noticed in the study of nonlinear discrete-continuous systems with the local nonlinearity described by the polynomial of the third degree. Extreme values for p for which harmonic vibrations with the period equal to the period of the external excitation can be obtained are marked by dots. The dots determine the intervals of p where the polynomial (1.1) is rather not

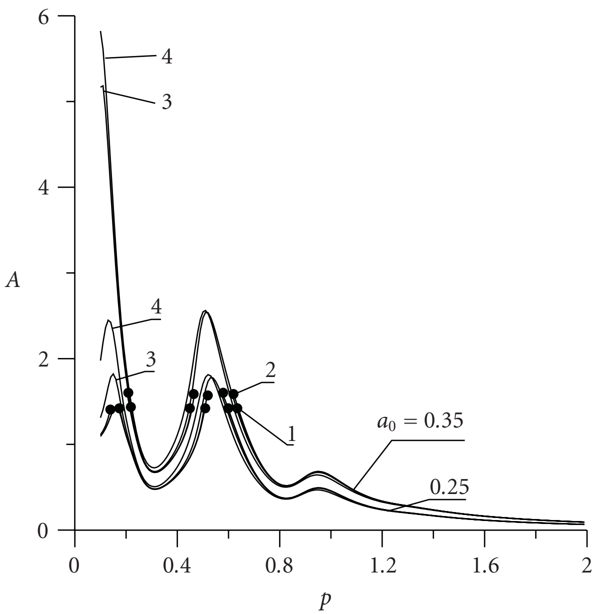

Figure 4.1. Amplitude-frequency curves for displacements in $x = 0$.

useful for the description of the nonlinear characteristics of a soft-type. Similar effects are noticed when the sinusoidal function (1.2) is used. In these cases, there also exist intervals where the solutions stop to behave as a sinusoidal function. These intervals are also marked by dots.

In Figure 4.1, the displacement amplitudes A are presented, including 4 resonant regions ($\omega_1 = 0.22$, $\omega_2 = 0.594$, $\omega_3 = 0.949$, $\omega_4 = 1.278$) for $k_3 = -0.05$, $d_0 = d_i = D_i = 0.1$ and for the amplitude of the external excitation $a_0 = 0.25, 0.35$. Only three resonant regions are distinct. From Figure 4.1, it follows that in the first resonant region the maximal displacement amplitudes occur for nonlinear function (1.4) for both values of a_0. In the second resonant region, functions (1.3) and (1.4) give similar results. In further resonant regions, the results for all of the four functions are similar. In the case of function (1.1), solutions may diverge to infinity for $a_0 = 0.25$ as well as for $a_0 = 0.35$. For function (1.2) with $a_0 = 0.25$ one can expect solutions not behaving as harmonic vibrations only in the second resonance, while with $a_0 = 0.35$ in both first resonant regions.

In Figure 4.2, amplitudes F_A of forces described by four nonlinear functions (1.1)–(1.4) for the amplitude of the external excitation $a_0 = 0.25$ are presented. Only three resonant regions are distinct. In the first and the second resonant regions, the maximal amplitudes are obtained for the sinusoidal function (1.2), and the smallest ones for the exponential function (1.4). After the fourth resonant region the results for all assumed functions are difficult to distinguish. In the first resonant region, one can notice the interval of p, where the solution diverges to infinity when the polynomial function is used.

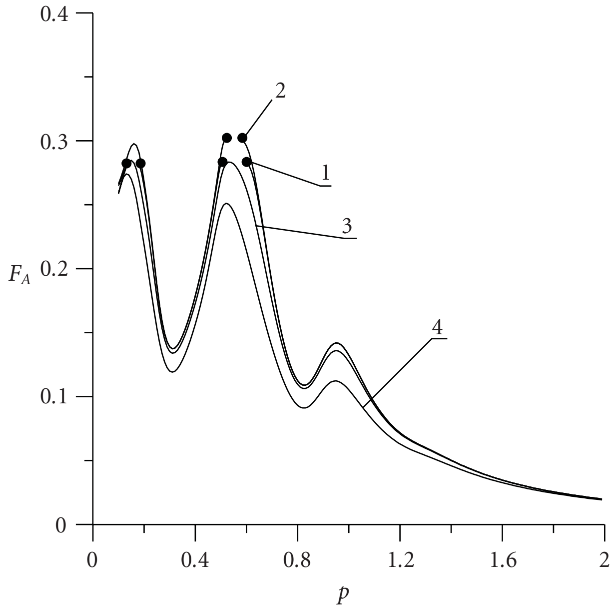

Figure 4.2. Amplitude-frequency curves for forces for $k_3 = -0.05$, $a_0 = 0.25$ with nonlinear functions (1.1)–(1.4).

In the second resonant region, the function (1.1) as well as the sinusoidal function (1.2) can give solutions losing their physical meaning.

In Figure 4.3, the diagrams of the force amplitudes F_A for $k_3 = -0.05$ and $a_0 = 0.35$ are plotted. In the third resonant region, the highest amplitudes are obtained for functions (1.1) and (1.2) while the smallest ones when the exponential function is applied for the description of the local nonlinearity. In the case of functions (1.1), (1.2), one can observe the intervals of the solutions not behaving as harmonic vibrations in the first as well as in the second resonant regions. For the remaining functions, the diagrams of the force amplitudes in the first resonant region form the plateau. It means that in this region the solutions corresponding to these functions approach the maximum value postulated by the parameters k_1, k_3. The plateau is wider for the hyperbolic tangent function. The maximum of the function (1.1) for the assumed parameters is equal to 0.2828, while for the functions (1.2)–(1.4) it is equal to 0.3.

Figures 4.2 and 4.3 show exemplary diagrams of amplitudes of forces and the diagrams in Figure 4.1 for amplitudes of displacements. These diagrams indicate that the polynomial (1.1) and the sinusoidal function (1.2) may have some restrictions for their application in the discussion of the nonlinear vibrations of discrete-continuous systems with the local nonlinearities having the characteristics of a soft-type. Though these functions have some limitations for their application, it seems to be interesting to find admissible values of the excitation amplitude a_0 for which solutions are harmonic vibrations with the frequency p.

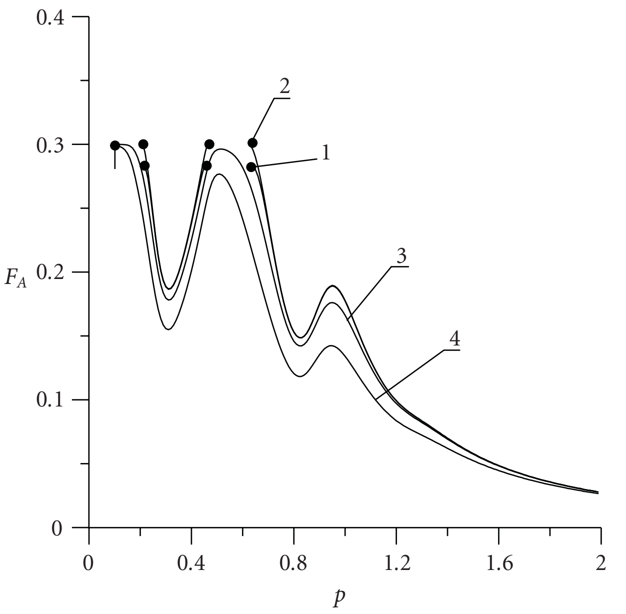

Figure 4.3. Amplitude-frequency curves for forces for $k_3 = -0.05$, $a_0 = 0.35$ with nonlinear functions (1.1)–(1.4).

The application ranges of the polynomial (1.1) (dashed lines) and the sinusoidal function (1.2) (continuous lines) for the considered system are shown in Figure 4.4 for $k_3 = -0.025, -0.05, -0.1$. They include 3 resonant regions. The appropriate diagrams inform how the admissible values of the amplitude a_0 decrease with the decrease of the parameter k_3 representing the local nonlinearity. The strongest restrictions occur in the neighborhood of the resonances and the minimal admissible values of a_0 increase with the increase of the frequency p. The application ranges are wider for sinusoidal function (1.2), and the acceptable a_0 for the polynomial function can increase in a linear manner between the first and the second resonant regions. Besides, one can notice that the differences between the diagrams for the functions (1.1) and (1.2) increase slowly with the increase of the frequency of the external excitation p.

5. Final remarks

In the paper, it is shown that there exist discrete-continuous systems with local nonlinearities having the characteristics of a soft-type which can be reduced to solving nonlinear ordinary differential equations with a retarded argument. It was also shown that various nonlinear functions can be incorporated for the description of local nonlinearities. Detailed investigations are presented for the nonlinear discrete-continuous system subjected to shear deformations and numerical results are given for the system having four rigid bodies and three elastic elements. The research concerns the effect of the local nonlinearity on the solutions in several resonant regions using four nonlinear functions. This effect is seen in the first two resonant regions. It is noticed that some limitations there

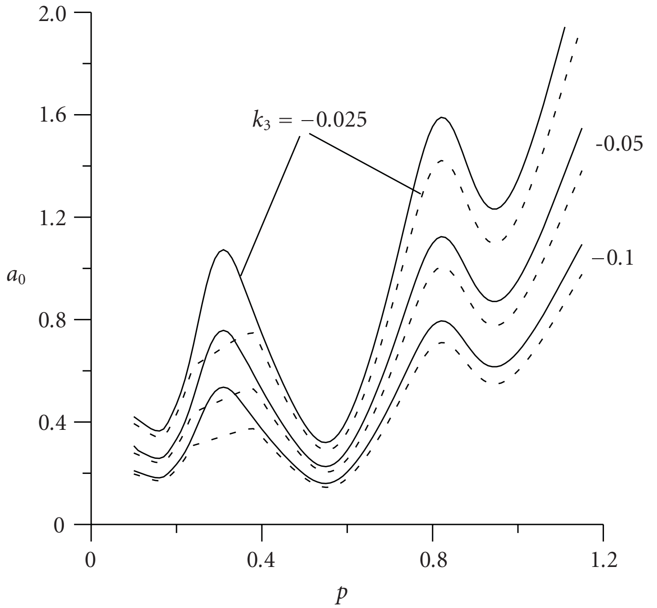

Figure 4.4. Application ranges for $k_3 = -0.025, -0.05, -0.1$ for polynomial function (dashed lines) and sinusoidal function (continuous lines).

exist for the application of the polynomial function and of the sinusoidal function. Analogous considerations can be done for the system with other number rigid bodies and for discrete-continuous systems with local nonlinearities subjected to torsional or longitudinal deformations. The local nonlinearities can have the characteristics of a soft as well as of a hard type (Nadolski and Pielorz [5]; Pielorz [7, 8, 11, 12, 14, 15]).

Acknowledgment

The paper was presented at the *Workshop on Nonlinear Phenomena*: *Modeling and Their Applications*, held in Rio Claro, 4–6 May, 2005, SP, Brazil.

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