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Research Article

Large Deviations in Testing Squared Radial Ornstein-Uhleneck Model

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We study the large deviations and moderate deviations of hypothesis testing for squared radial Ornstein-Uhleneck model. Large deviation principles for the log-likelihood ratio are obtained, by which we give negative regions in testing squared radial Ornstein-Uhleneck model and get the decay rates of the error probabilities.

1. Introduction

Let us consider the hypothesis testing for the following squared radial Ornstein-Uhleneck model:

$$dX_t = (\delta + 2\alpha X_t)dt + 2\sqrt{X_t}dW_t, \quad X_0 = 0, \tag{1.1}$$

where $\alpha < 0$ is the unknown parameter to be tested on the basis of continuous observation of the process $\{X_t, t \geq 0\}$ on the time interval [0,T], W is a standard Brownian motion and, $\delta > 0$ is known. We denote the distribution of the solution (1.1) by P_α^δ.

We decide the two hypothesis:

$$H_0 : \alpha = \alpha_0, \qquad H_1 : \alpha = \alpha_1, \tag{1.2}$$

where $\alpha_0, \alpha_1 < 0$. The hypothesis testing is based on a partition of

$$\Omega_T = \left\{ \left.\frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta}\right|_{\mathcal{F}_{[0,T]}} \in \mathbb{R} \right\} \tag{1.3}$$

of the outcome process on $[0,T]$ into two (decision) regions B_T and its compliment B_T^c, and we decide that H_0 is true or false according to the outcome $X_\cdot \in B_T$ or $X_\cdot \in B_T^c$.

The probability $e_1(T)$ of accepting H_1 when H_0 is actually true is called the error probability of the first kind. The probability $e_2(T)$ of accepting H_0 when H_1 is actually true is called the error probability of the second kind. That is,

$$e_1(T) = P_{\alpha_0}^{\delta}(B_T), \qquad e_2(T) = P_{\alpha_1}^{\delta}(B_T^c). \tag{1.4}$$

By the Neyman-Pearson lemma (cf. [1]), the optional decision region B_T has the following form:

$$\left\{ \frac{1}{T} \log \frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}} \bigg|_{\mathcal{F}_{[0,T]}} \geq c \right\}, \tag{1.5}$$

where $\mathcal{F}_{[0,T]}$ is the σ-algebra generated by the outcome process on $[0,T]$.

The research of hypothesis testing problem has started in the 1930s (cf. [1]). Since the optional decision region B_T has the above form, we are interested in the calculation or approximation of the constant c, and the hypothesis testing problem can be studied by large deviations (cf. [2–6]). In those papers, some large deviation estimates of the error probabilities for some i.i.d. sequences, Markov chains, stationary Gaussian processes, stationary diffusion processes, Ornstein-Uhlenbeck processes are obtained. In this paper, we study the large deviations and moderate deviations for the hypothesis testing problem of squared radial Ornstein-Uhleneck model; by large deviation principle, we obtain that the decay of the error probability of the second kind approaches to 0 or 1 exponentially fast depending on the fixed exponent of the decay of the error probability of the first kind; we also give negative regions and get the decay rates of the error probabilities by moderate deviation principle. The large and moderate deviations for parameter estimators of squared radial Ornstein-Uhleneck model were studied in [7, 8].

2. Main Results

In this section, we state our main results.

Theorem 2.1. *Let $a(T)$ be a positive function satisfying*

$$\frac{a(T)}{T} \longrightarrow 0, \quad \frac{a(T)}{\sqrt{T}} \longrightarrow \infty \quad \text{as } T \longrightarrow \infty. \tag{2.1}$$

For any $a > 0$, set

$$B_T = \left\{ \frac{1}{a(T)} \log \frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}} \bigg|_{\mathcal{F}_{[0,T]}} \geq \frac{T}{4a(T)} \frac{(\alpha_1 - \alpha_0)^2 \delta}{\alpha_0} - \frac{\left|\alpha_1^2 - \alpha_0^2\right|}{2\alpha_0} \sqrt{\frac{a\delta}{-\alpha_0}} \right\}. \tag{2.2}$$

Then

$$\lim_{T\to\infty}\frac{T}{a^2(T)}\log P_{\alpha_0}^{\delta}(B_T)=-a,$$
$$\lim_{T\to\infty}\frac{T}{a^2(T)}\log P_{\alpha_1}^{\delta}(B_T^c)=-\infty. \tag{2.3}$$

Theorem 2.2. *If $\alpha_1<\alpha_0<0$, then for each $a>0$, there exists a $\xi(a)\in\mathbb{R}$, such that*

$$\lim_{T\to\infty}\frac{1}{T}\log P_{\alpha_0}^{\delta}\left(\frac{1}{T}\log\left.\frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\right|_{\mathcal{F}_{[0,T]}}\geq\xi(a)\right)=-a, \tag{2.4}$$

and when $a\in(0,z_{\alpha_1})$,

$$\liminf_{T\to\infty}\frac{1}{T}\log P_{\alpha_1}^{\delta}\left(\frac{1}{T}\log\left.\frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\right|_{\mathcal{F}_{[0,T]}}<\xi(a)\right)\leq-I_{\alpha_1}(\xi(a)), \tag{2.5}$$

when $a\in(z_{\alpha_1},+\infty)$,

$$\liminf_{T\to\infty}\frac{1}{T}\log\left(1-P_{\alpha_1}^{\delta}\left(\frac{1}{T}\log\left.\frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\right|_{\mathcal{F}_{[0,T]}}<\xi(a)\right)\right)\leq-I_{\alpha_1}(\xi(a)), \tag{2.6}$$

where

$$z_{\alpha_1}=-\frac{(\alpha_0-\alpha_1)^2\delta}{4\alpha_1},$$
$$I_{\alpha_1}(z)=\begin{cases}\dfrac{\alpha_0^2-\alpha_1^2}{z+(\alpha_1-\alpha_0)\delta/2}\left(\dfrac{\delta}{4}-\dfrac{\alpha_1(z+(\alpha_1-\alpha_0)\delta/2)}{\alpha_1^2-\alpha_0^2}\right)^2, & z<\dfrac{(\alpha_0-\alpha_1)\delta}{2},\\ +\infty, & \text{otherwise}.\end{cases} \tag{2.7}$$

Theorem 2.3. *If $\alpha_0<\alpha_1<0$, then for each $a>0$, there exists a $\tilde{\xi}(a)\in\mathbb{R}$, such that*

$$\lim_{T\to\infty}\frac{1}{T}\log P_{\alpha_0}^{\delta}\left(\frac{1}{T}\log\left.\frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\right|_{\mathcal{F}_{[0,T]}}\geq\tilde{\xi}(a)\right)=-a, \tag{2.8}$$

and when $a\in(0,\hat{z}_{\alpha_1})$,

$$\liminf_{T\to\infty}\frac{1}{T}\log P_{\alpha_1}^{\delta}\left(\frac{1}{T}\log\left.\frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\right|_{\mathcal{F}_{[0,T]}}<\tilde{\xi}(a)\right)\leq-\hat{I}_{\alpha_1}\left(\tilde{\xi}(a)\right), \tag{2.9}$$

when $a \in (\hat{z}_{\alpha_1}, +\infty)$,

$$\liminf_{T\to\infty}\frac{1}{T}\log\left(1-P^{\delta}_{\alpha_1}\left(\frac{1}{T}\log\frac{dP^{\delta}_{\alpha_1}}{dP^{\delta}_{\alpha_0}}\bigg|_{\mathcal{F}_{[0,T]}}<\tilde{\xi}(a)\right)\right)\le -\hat{I}_{\alpha_1}\left(\tilde{\xi}(a)\right), \tag{2.10}$$

where

$$\hat{z}_{\alpha_1} = -\frac{(\alpha_0-\alpha_1)^2\delta}{4\alpha_1},$$
$$\hat{I}_{\alpha_1}(z) = \begin{cases} \dfrac{\alpha_0^2-\alpha_1^2}{z+(\alpha_1-\alpha_0)\delta/2}\left(\dfrac{\delta}{4}-\dfrac{\alpha_1(z+(\alpha_1-\alpha_0)\delta/2)}{\alpha_1^2-\alpha_0^2}\right)^2, & z>\dfrac{(\alpha_0-\alpha_1)\delta}{2}, \\ +\infty, & \textit{otherwise}. \end{cases} \tag{2.11}$$

3. Moderate Deviations in Testing Squared Radial Ornstein-Uhleneck Model

In this section, we will prove Theorem 2.1. Let us introduce the log-likelihood ratio process of squared radial Ornstein-Uhleneck model and study the moderate deviations of the log-likelihood ratio process.

By [7], the log-likelihood ratio process has the representation

$$\log\frac{dP^{\delta}_{\alpha_1}}{dP^{\delta}_{\alpha_0}}\bigg|_{\mathcal{F}_{[0,T]}} = \frac{1}{2}(\alpha_1-\alpha_0)(X_t-\delta t)-\frac{\alpha_1^2-\alpha_0^2}{2}\int_0^t X_s^2\,ds. \tag{3.1}$$

The following Lemma (cf. [9]) plays an important role in this paper.

Lemma 3.1. *The law of* X_t *under* P^{δ}_{α} *is* $\gamma(\delta/2, \alpha/e^{2t\alpha-1})$, *where* $\gamma(a,b)$ *denotes the Gamma distribution:*

$$\gamma(a,b)(dx) = \frac{b^a x^{a-1}}{\Gamma(a)}e^{-bx}(dx), \quad x>0. \tag{3.2}$$

Moreover, for any $\theta\in\mathbb{R}$,

$$\mathbb{E}^{\delta}_{\alpha}\left(e^{\theta X_t}\right) = \left(1-\frac{\theta}{\alpha}\left(e^{2t\alpha}-1\right)\right)^{-\delta/2}. \tag{3.3}$$

Lemma 3.2. *For any closed subset* $F\subset\mathbb{R}$,

$$\limsup_{T\to\infty}\frac{T}{a^2(T)}\log P^{\delta}_{\alpha_0}\left(\frac{1}{a(T)}\left(\log\frac{dP^{\delta}_{\alpha_1}}{dP^{\delta}_{\alpha_0}}\bigg|_{\mathcal{F}_{[0,T]}}-\frac{(\alpha_1-\alpha_0)^2\delta T}{4\alpha_0}\right)\in F\right)\le -\inf_{z\in F}\frac{-4\alpha_0^3 z^2}{(\alpha_1^2-\alpha_0^2)^2\delta}, \tag{3.4}$$

and for any open subset $G \subset \mathbb{R}$*,*

$$\liminf_{T\to\infty} \frac{T}{a^2(T)} \log P_{\alpha_0}^{\delta}\left(\frac{1}{a(T)}\left(\log \frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\bigg|_{\mathcal{F}_{[0,T]}} - \frac{(\alpha_1-\alpha_0)^2\delta T}{4\alpha_0}\right) \in G\right) \geq -\inf_{z\in G} \frac{-4\alpha_0^3 z^2}{\left(\alpha_1^2-\alpha_0^2\right)^2\delta}. \tag{3.5}$$

Proof. Let

$$\Lambda_T(y) = \log \mathbb{E}_{\alpha_0}^{\delta} \exp\left\{\frac{a(T)y}{T}\left(\log \frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\bigg|_{\mathcal{F}_{[0,T]}} - \frac{(\alpha_1-\alpha_0)^2\delta T}{4\alpha_0}\right)\right\}. \tag{3.6}$$

By (3.1), for any $\varphi < 0$, we have

$$\begin{aligned}
\Lambda_T(y) &= -\frac{a(T)y\delta(\alpha_1-\alpha_0)^2}{4\alpha_0} + \log \mathbb{E}_{\alpha_0}^{\delta} \exp\left\{\lambda(X_T-\delta T) + u\int_0^T X_t^2 ds\right\} \\
&= -\frac{a(T)y\delta(\alpha_1-\alpha_0)^2}{4\alpha_0} + \log \mathbb{E}_{\varphi}^{\delta}\left[\frac{dP_{\alpha_0}^{\delta}}{dP_{\varphi}^{\delta}} \exp\left\{\lambda(X_T-\delta T) + u\int_0^T X_t^2 ds\right\}\right] \\
&= -\frac{a(T)y\delta\left(\alpha_1^2-\alpha_0^2\right)}{4\alpha_0} + \log \mathbb{E}_{\varphi}^{\delta}\left[\exp\left\{\left(\lambda + \frac{\alpha_0-\varphi}{2}\right)X_T - \frac{\alpha_0-\varphi}{2}\delta T\right.\right. \\
&\qquad \left.\left. + \left(u - \frac{1}{2}\alpha_0^2 + \frac{1}{2}\varphi^2\right)\int_0^T X_s^2 ds\right\}\right],
\end{aligned} \tag{3.7}$$

where

$$\lambda = \frac{a(T)y(\alpha_1-\alpha_0)}{2T}, \qquad u = -\frac{a(T)y\left(\alpha_1^2-\alpha_0^2\right)}{2T}. \tag{3.8}$$

For T large enough, $\alpha_0^2 - 2u > 0$, we can choose $\varphi = -\sqrt{\alpha_0^2 - 2u}$, then $\varphi < 0$ and

$$\Lambda_T(y) = -\frac{a(T)y\delta\left(\alpha_1^2-\alpha_0^2\right)}{4\alpha_0} - \frac{\alpha_0-\varphi}{2}\delta T + \log \mathbb{E}_{\varphi}^{\delta}\left[\exp\left\{\left(\lambda + \frac{\alpha_0-\varphi}{2}\right)X_T\right\}\right]. \tag{3.9}$$

By Lemma 3.1, we have

$$\begin{aligned}
&\lim_{T\to\infty} \frac{T}{a^2(T)} \log \mathbb{E}_{\varphi}^{\delta}\left[\exp\left\{\left(\lambda + \frac{\alpha_0-\varphi}{2}\right)X_T\right\}\right] \\
&\quad = \lim_{T\to\infty} -\frac{T\delta}{2a^2(T)} \log\left(1 - \frac{1}{\varphi}\left(\lambda + \frac{\alpha_0-\varphi}{2}\right)\left(e^{2T\varphi}-1\right)\right) = 0.
\end{aligned} \tag{3.10}$$

Therefore,

$$\begin{aligned}\Lambda(y) &:= \lim_{T\to\infty}\frac{T}{a^2(T)}\Lambda_T(y)\\ &= \lim_{T\to\infty}\frac{T}{a^2(T)}\left(-\frac{a(T)y\delta(\alpha_1^2-\alpha_0^2)}{4\alpha_0}-\frac{\alpha_0-\varphi}{2}\delta T\right)\\ &= \lim_{T\to\infty}\frac{T}{a^2(T)}\left(-\frac{a(T)y\delta(\alpha_1^2-\alpha_0^2)}{4\alpha_0}-\frac{\alpha_0\delta T}{2}\left(1-\sqrt{1+\frac{a(T)y(\alpha_1^2-\alpha_0^2)}{T\alpha_0^2}}\right)\right)\\ &= \frac{y^2}{16}\frac{(\alpha_1^2-\alpha_0^2)^2\delta}{-\alpha_0^3}.\end{aligned} \tag{3.11}$$

Finally, the Gärtner-Ellis theorem (cf. [10]) implies the conclusion of Lemma 3.2. □

Noting that

$$\log\left.\frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\right|_{\mathcal{F}_{[0,T]}} = -\log\left.\frac{dP_{\alpha_0}^{\delta}}{dP_{\alpha_1}^{\delta}}\right|_{\mathcal{F}_{[0,T]}}, \tag{3.12}$$

we also have the following result.

Lemma 3.3. *For any closed subset $F\subset\mathbb{R}$,*

$$\limsup_{T\to\infty}\frac{T}{a^2(T)}\log P_{\alpha_1}^{\delta}\left(\frac{1}{a(T)}\left(\log\left.\frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\right|_{\mathcal{F}_{[0,T]}}+\frac{(\alpha_1-\alpha_0)^2\delta T}{4\alpha_1}\right)\in F\right)\leq -\inf_{z\in F}\frac{-4\alpha_1^3z^2}{(\alpha_1^2-\alpha_0^2)^2\delta}, \tag{3.13}$$

and for any open subset $G\subset\mathbb{R}$,

$$\liminf_{T\to\infty}\frac{T}{a^2(T)}\log P_{\alpha_1}^{\delta}\left(\frac{1}{a(T)}\left(\log\left.\frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\right|_{\mathcal{F}_{[0,T]}}+\frac{(\alpha_1-\alpha_0)^2\delta T}{4\alpha_1}\right)\in G\right)\geq -\inf_{z\in G}\frac{-4\alpha_1^3z^2}{(\alpha_1^2-\alpha_0^2)^2\delta}. \tag{3.14}$$

Proof of Theorem 2.1. The first claim is a direct conclusion of Lemma 3.2. Since

$$\frac{T}{a(T)}\frac{(\alpha_1-\alpha_0)^2\delta}{\alpha_0}+\frac{T}{a(T)}\frac{(\alpha_1-\alpha_0)^2\delta}{\alpha_1}\longrightarrow -\infty,\quad \text{as } T\longrightarrow\infty, \tag{3.15}$$

by Lemma 3.3, we see that the second one also holds. □

4. Large Deviations in Testing Fractional Ornstein-Uhleneck Model

In this section, we will prove Theorems 2.2 and 2.3. We first study the large deviations of the log-likelihood ratio process.

Lemma 4.1. *Assume $\alpha_1 < \alpha_0 < 0$. Then for any closed subset $F \subset \mathbb{R}$,*

$$\limsup_{T\to\infty}\frac{1}{T}\log P^{\delta}_{\alpha_0}\left(\frac{1}{T}\log\left.\frac{dP^{\delta}_{\alpha_1}}{dP^{\delta}_{\alpha_0}}\right|_{\mathcal{F}_{[0,T]}} \in F\right) \le -\inf_{z\in F} I_{\alpha_0}(z), \tag{4.1}$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T\to\infty}\frac{1}{T}\log P^{\delta}_{\alpha_0}\left(\frac{1}{T}\log\left.\frac{dP^{\delta}_{\alpha_1}}{dP^{\delta}_{\alpha_0}}\right|_{\mathcal{F}_{[0,T]}} \in G\right) \ge -\inf_{z\in G} I_{\alpha_0}(z), \tag{4.2}$$

where

$$I_{\alpha_0}(z) = \begin{cases} \dfrac{\alpha_0^2-\alpha_1^2}{z+(\alpha_1-\alpha_0)\delta/2}\left(\dfrac{\delta}{4}-\dfrac{\alpha_0(z+(\alpha_1-\alpha_0)\delta/2)}{\alpha_1^2-\alpha_0^2}\right)^2, & z<\dfrac{(\alpha_0-\alpha_1)\delta}{2}, \\ +\infty, & \textit{otherwise}. \end{cases} \tag{4.3}$$

Proof. Let

$$\Lambda_T(y) = \log \mathbb{E}^{\delta}_{\alpha_0}\exp\left\{ y\log\left.\frac{dP^{\delta}_{\alpha_1}}{dP^{\delta}_{\alpha_0}}\right|_{\mathcal{F}_{[0,T]}}\right\}. \tag{4.4}$$

Then for $\varphi < 0$, we have

$$\begin{aligned} \Lambda_T(y) &= \log \mathbb{E}^{\delta}_{\alpha_0}\exp\left\{\lambda(X_T-\delta T)+u\int_0^T X_t^2 ds\right\} \\ &= \log \mathbb{E}^{\delta}_{\varphi}\left[\frac{dP^{\delta}_{\alpha_0}}{dP^{\delta}_{\varphi}}\exp\left\{\lambda(X_T-\delta T)+u\int_0^T X_t^2 ds\right\}\right] \\ &= \log \mathbb{E}^{\delta}_{\varphi}\left[\exp\left\{\left(\lambda+\frac{\alpha_0-\varphi}{2}\right)(X_T-\delta T)+\left(u-\frac{1}{2}\alpha_0^2+\frac{1}{2}\varphi^2\right)\int_0^T X_s^2 ds\right\}\right], \end{aligned} \tag{4.5}$$

where $\lambda = (\alpha_1-\alpha_0)y/2$, $u = y(\alpha_0^2-\alpha_1^2)/2$.

Since $\alpha_0^2 - 2u > 0$, for $y > (\alpha_0^2/\alpha_0^2-\alpha_1^2)$, we can choose $\varphi = -\sqrt{\alpha_0^2-2u}$, for each $y > (\alpha_0^2/\alpha_0^2-\alpha_1^2)$; then $\varphi < 0$ and

$$\Lambda_T(y) = -\delta T\left(\lambda+\frac{\alpha_0-\varphi}{2}\right)+\log \mathbb{E}^{\delta}_{\varphi}\left[e^{(\lambda+(\alpha_0-\varphi)/2)X_T}\right]. \tag{4.6}$$

By Lemma 3.1, we get

$$\lim_{T\to\infty}\frac{1}{T}\log \mathbb{E}_{\varphi}^{\delta}\left[\exp\left\{\left(\lambda+\frac{\alpha_0-\varphi}{2}\right)X_T\right\}\right]=0. \tag{4.7}$$

Therefore,

$$\Lambda(y)=:\lim_{T\to\infty}\frac{1}{T}\Lambda_T(y)=-\frac{\delta}{2}\left((\alpha_1-\alpha_0)y+\alpha_0+\sqrt{\alpha_0^2+(\alpha_1^2-\alpha_0^2)y}\right). \tag{4.8}$$

Since $\Lambda(y)$ is a strictly convex differentiable function on $\mathfrak{D}_\Lambda=(\alpha_-,+\infty)$ with

$$\alpha_-=\frac{\alpha_0^2}{\alpha_0^2-\alpha_1^2}<0,\qquad \lim_{y\to\alpha_-}\Lambda'(y)=+\infty, \tag{4.9}$$

where $\mathfrak{D}_\Lambda$ is the effective domain of Λ, we see that $\Lambda(y)$ is steep. Finally, by

$$\sup_{y\in\mathbb{R}}\{zy-\Lambda(y)\}=\begin{cases}\dfrac{\alpha_0^2-\alpha_1^2}{z+(\alpha_1-\alpha_0)\delta/2}\left(\dfrac{\delta}{4}-\dfrac{\alpha_0(z+(\alpha_1-\alpha_0)\delta/2)}{\alpha_1^2-\alpha_0^2}\right)^2, & z<\dfrac{(\alpha_0-\alpha_1)\delta}{2},\\ +\infty, & \text{otherwise,}\end{cases} \tag{4.10}$$

and Gärtner-Ellis theorem, we complete the proof of this lemma. □

Similarly, when $\alpha_0<\alpha_1<0$, we have

$$\Lambda(y)=:\lim_{T\to\infty}\frac{1}{T}\Lambda_T(y)=-\frac{\delta}{2}\left((\alpha_1-\alpha_0)y+\alpha_0+\sqrt{\alpha_0^2+(\alpha_1^2-\alpha_0^2)y}\right). \tag{4.11}$$

Since $\Lambda(y)$ is a strictly convex differentiable function on $\mathfrak{D}_\Lambda=(-\infty,\alpha_+)$ with

$$\alpha_+=\frac{\alpha_0^2}{\alpha_0^2-\alpha_1^2}>0, \tag{4.12}$$

and $\lim_{y\to\alpha_+}\Lambda'(y)=+\infty$, we can see that $\Lambda(y)$ is steep. By Gärtner-Ellis theorem, we also have the following result.

Lemma 4.2. *Assume $\alpha_0<\alpha_1<0$. Then for any closed subset $F\subset\mathbb{R}$,*

$$\limsup_{T\to\infty}\frac{1}{T}\log P_{\alpha_0}^{\delta}\left(\frac{1}{T}\log\left.\frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\right|_{\mathcal{F}_{[0,T]}}\in F\right)\leq-\inf_{z\in F}\widehat{I}_{\alpha_0}(z), \tag{4.13}$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T\to\infty} \frac{1}{T} \log P_{\alpha_0}^{\delta}\left(\frac{1}{T} \log \frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\bigg|_{\mathcal{F}_{[0,T]}} \in G \right) \geq -\inf_{z\in G} \widehat{I}_{\alpha_0}(z), \tag{4.14}$$

where

$$\widehat{I}_{\alpha_0}(z) = \begin{cases} \dfrac{\alpha_0^2-\alpha_1^2}{z+((\alpha_1-\alpha_0)\delta/2)}\left(\dfrac{\delta}{4} - \dfrac{\alpha_0(z+(\alpha_1-\alpha_0)\delta/2)}{\alpha_1^2-\alpha_0^2}\right)^2, & z > \dfrac{(\alpha_0-\alpha_1)\delta}{2}, \\ +\infty, & \text{otherwise}. \end{cases} \tag{4.15}$$

Note that

$$\log \frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\bigg|_{\mathcal{F}_{[0,T]}} = -\log \frac{dP_{\alpha_0}^{\delta}}{dP_{\alpha_1}^{\delta}}\bigg|_{\mathcal{F}_{[0,T]}}. \tag{4.16}$$

Then we have the following Lemma.

Lemma 4.3. *Assume $\alpha_1 < \alpha_0 < 0$. Then for any closed subset $F \subset \mathbb{R}$,*

$$\limsup_{T\to\infty} \frac{1}{T} \log P_{\alpha_1}^{\delta}\left(\frac{1}{T} \log \frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\bigg|_{\mathcal{F}_{[0,T]}} \in F \right) \leq -\inf_{z\in F} \widehat{I}_{\alpha_1}(z), \tag{4.17}$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T\to\infty} \frac{1}{T} \log P_{\alpha_1}^{\delta}\left(\frac{1}{T} \log \frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\bigg|_{\mathcal{F}_{[0,T]}} \in G \right) \geq -\inf_{z\in G} \widehat{I}_{\alpha_1}(z), \tag{4.18}$$

where

$$I_{\alpha_1}(z) = \begin{cases} \dfrac{\alpha_0^2-\alpha_1^2}{z+(\alpha_1-\alpha_0)\delta/2}\left(\dfrac{\delta}{4} - \dfrac{\alpha_1(z+(\alpha_1-\alpha_0)\delta/2)}{\alpha_1^2-\alpha_0^2}\right)^2, & z < \dfrac{(\alpha_0-\alpha_1)\delta}{2}, \\ +\infty, & \text{otherwise}. \end{cases} \tag{4.19}$$

Lemma 4.4. *Assume $\alpha_0 < \alpha_1 < 0$. Then for any closed subset $F \subset \mathbb{R}$,*

$$\limsup_{T\to\infty} \frac{1}{T} \log P_{\alpha_1}^{\delta}\left(\frac{1}{T} \log \frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\bigg|_{\mathcal{F}_{[0,T]}} \in F \right) \leq -\inf_{z\in F} I_{\alpha_1}(z), \tag{4.20}$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T\to\infty} \frac{1}{T} \log P_{\alpha_1}^{\delta}\left(\frac{1}{T} \log \frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\bigg|_{\mathcal{F}_{[0,T]}} \in G \right) \geq -\inf_{z\in G} I_{\alpha_1}(z), \tag{4.21}$$

where

$$\widehat{I}_{\alpha_1}(z) = \begin{cases} \dfrac{\alpha_0^2 - \alpha_1^2}{z + (\alpha_1 - \alpha_0)\delta/2}\left(\dfrac{\delta}{4} - \dfrac{\alpha_1 (z + (\alpha_1 - \alpha_0)\delta/2)}{\alpha_1^2 - \alpha_0^2} \right)^2, & z > \dfrac{(\alpha_0 - \alpha_1)\delta}{2}, \\ +\infty, & \textit{otherwise}. \end{cases} \tag{4.22}$$

By the expression of $I_{\alpha_0}(z)$, $I_{\alpha_1}(z)$, $\widehat{I}_{\alpha_0}(z)$, and $\widehat{I}_{\alpha_1}(z)$, the following lemma is.

Lemma 4.5. (i)

$$\begin{aligned} I_{\alpha_0}(z) = 0 \quad &\textit{iff } z_{\alpha_0} = \frac{(\alpha_0 - \alpha_1)^2 \delta}{4\alpha_0}, \\ I_{\alpha_1}(z) = 0 \quad &\textit{iff } z_{\alpha_1} = -\frac{(\alpha_1 - \alpha_0)^2 \delta}{4\alpha_1}, \end{aligned} \tag{4.23}$$

for all $z < (\alpha_0 - \alpha_1)\delta/2$,

$$I_{\alpha_0}(z) = I_{\alpha_1}(z) + z; \tag{4.24}$$

(ii)

$$\begin{aligned} \widehat{I}_{\alpha_0}(z) = 0 \quad &\textit{iff } \widehat{z}_{\alpha_0} = \frac{(\alpha_0 - \alpha_1)^2 \delta}{4\alpha_0}, \\ \widehat{I}_{\alpha_1}(z) = 0 \quad &\textit{iff } \widehat{z}_{\alpha_1} = -\frac{(\alpha_1 - \alpha_0)^2 \delta}{4\alpha_1}, \end{aligned} \tag{4.25}$$

for all $z > (\alpha_0 - \alpha_1)\delta/2$,

$$\widehat{I}_{\alpha_0}(z) = \widehat{I}_{\alpha_1}(z) + z. \tag{4.26}$$

Proof of Theorems 2.2 and 2.3. Since the proofs of the two theorems are similar, we only prove Theorem 2.2. Since $I_{\alpha_0}(z)$ is increasing on $(z_{\alpha_0}, (\alpha_0 - \alpha_1)\delta/2)$ and $I_{\alpha_0}(z_{\alpha_0}) = 0$, Therefore, for $a > 0$, by Lemma 4.1, we can choose a $\xi(a) \in (z_{\alpha_0}, (\alpha_0 - \alpha_1)\delta/2))$ such that

$$\lim_{T\to\infty} \frac{1}{T} \log P_{\alpha_0}^{\delta}\left(\frac{1}{T} \log \frac{dP_{\alpha_1}^{\delta}}{dP_{\alpha_0}^{\delta}}\bigg|_{\mathcal{F}_{[0,T]}} \geq \xi(a) \right) = -a. \tag{4.27}$$

It is clear that $\xi(a)$ is increasing for $a > 0$, and by Lemma 4.5, we get $I_{\alpha_0}(z_{\alpha_1}) = z_{\alpha_1}$, which implies $\xi(z_{\alpha_1}) = z_{\alpha_1}$. Hence for $a \in (0, z_{\alpha_1})$, we have $\xi(a) \in (0, z_{\alpha_1})$, and since $I_{\alpha_1}(z)$ is nonincreasing for $z \leq \xi(a)$, therefore we get

$$I_{\alpha_1}(\xi(a)) = \inf\{I_{\alpha_1}(z) : z \leq \xi(a)\}. \tag{4.28}$$

Similarly, for $a \in (z_{\alpha_1}, +\infty)$, we have $\xi(a) \in (z_{\alpha_1}, (\alpha_0 - \alpha_1)\delta/2)$, and since $I_{\alpha_1}(z)$ is non-decreasing for $z \geq \xi(a)$, therefore we get

$$I_{\alpha_1}(\xi(a)) = \inf\{I_{\alpha_1}(z) : z \geq \xi(a)\}, \tag{4.29}$$

which complete the proof of Theorem 2.2. □

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