Hindawi Publishing Corporation
Journal of Probability and Statistics
Volume 2009, Article ID 847830, 11 pages
doi:10.1155/2009/847830

Research Article

A Note on the Properties of Generalised Separable Spatial Autoregressive Process

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Received 27 February 2009; Accepted 20 July 2009

Recommended by Murray Clayton

Spatial modelling has its applications in many fields like geology, agriculture, meteorology, geography, and so forth. In time series a class of models known as Generalised Autoregressive (GAR) has been introduced by Peiris (2003) that includes an index parameter δ. It has been shown that the inclusion of this additional parameter aids in modelling and forecasting many real data sets. This paper studies the properties of a new class of spatial autoregressive process of order 1 with an index. We will call this a *Generalised Separable Spatial Autoregressive* (GENSSAR) Model. The spectral density function (SDF), the autocovariance function (ACVF), and the autocorrelation function (ACF) are derived. The theoretical ACF and SDF plots are presented as three-dimensional figures.

1. Introduction

Spatial modelling has its applications in many fields like geology, agriculture, meteorology, geography, and so forth. Spatial data can be classified as geostatistical data, lattice data, or point patterns. These differences are due to whether the spatial data has been observed on a continuous domain or at discrete locations. In point pattern analysis the domain is random and interest focuses on the location of events.

In this paper we concentrate on lattice data observed on a regular grid. Many models have been suggested in modelling spatial dependence like the Simultaneous Autoregression (SAR) [1], Conditional Autoregression (CAR) [2, 3], Moving Average (MA) [4], and Unilateral models [5].

For a two-dimensional *stationary process* we have the following definitions. Let $\{Y_{ij}, i, j = 0, \pm 1, \pm 2, \ldots\}$ be a sequence of spatial observations on a two-dimensional *regular* lattice. The mean function is $\mathrm{E}[Y_{ij}] = \mu$ (a constant). The autocovariance function is $\gamma_{h,k} = \mathrm{Cov}[Y_{i+h,j+k}, Y_{i,j}]$ and the autocorrelation function is given as $\rho_{h,k} = \gamma_{h,k}/\gamma_{0,0}$.

Now, there exists a class of models that are known as *separable* models which have the property of a reflection symmetric correlation structure (i.e., $\rho_{h,k} = \rho_{-h,-k} = \rho_{h,-k} = \rho_{-h,k}$). The linear by linear process X is defined as a stationary process where the autocovariance generating function of X is defined as proportional to the product of two one-dimensional processes, Y and Z (see [6]) and the relationship may be represented as $X = Y * Z$. As a consequence, its correlation structure can be expressed as a product of correlations (i.e., $\rho_{x,h,k} = \rho_{y,h}\rho_{z,k}$). Basawa et al. [7] have considered separable models on a k-dimensional lattice and have shown that the correlation structure is $\rho(\mathbf{h}) = \prod_i^k \rho_i(h_i)$, where $\mathbf{h}$ is the lag vector $(h_1, h_2, \ldots, h_k)'$.

On the other hand, in the area of time series a class of models known as generalised autoregressive (GAR) models has been introduced by Peiris [8] by including an additional index parameter, δ. This is a natural extension of the standard AR model. It has been shown in Peiris [8] and Peiris et al. [9] that the additional index parameter plays an important role in modelling and forecasting real data sets. Shitan and Peiris [10] have also studied the estimation problem of the GAR(1) model with a simulation study.

In this paper we will consider a special type of spatial model called a *Generalised Separable Spatial Autoregressive* (GENSSAR) Model. Some of its properties are discussed in Section 2. Finally in Section 3, some conclusions are drawn.

The GENSSAR Model

Let $\{Y_{ij}\}$ be a sequence of spatial observations on a two-dimensional *regular* lattice that satisfies

$$(1 - \phi_{10}B_1 - \phi_{01}B_2 + \phi_{10}\phi_{01}B_1B_2)^{\delta} Y_{ij} = Z_{ij}, \tag{1.1}$$

where B_1 is the usual backward shift operator acting in the ith direction, B_2 is the backward shift operator acting in the jth direction, and $\{Z_{i,j}\}$ is a two-dimensional white noise process with mean zero and variance.

The term $(1 - \phi_{10}B_1 - \phi_{01}B_2 + \phi_{10}\phi_{01}B_1B_2)$ can be factored out as $(1 - \phi_{10}B_1)(1 - \phi_{01}B_2)$ and hence (1.1) can be written as

$$(1 - \phi_{10}B_1)^{\delta}(1 - \phi_{01}B_2)^{\delta} Y_{ij} = Z_{ij}. \tag{1.2}$$

The inclusion of the extra index parameter δ generalises the standard separable spatial model. Hence, we call the model defined in (1.1) as the Generalised Separable Spatial Autoregressive model or GENSSAR(1,1) model.

The following section reports some of its properties in detail.

2. Some Properties of GENSSAR(1,1)

The solution of this process in (1.1) is given in Proposition 1.

Proposition 1. *For a process defined in* (1.1), *the solution is*

$$Y_{ij} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(k+\delta)\Gamma(l+\delta)}{\Gamma(k+1)\Gamma(\delta)\Gamma(l+1)\Gamma(\delta)}\phi_{10}^{k}\phi_{01}^{l}Z_{i-k,j-l}, \tag{2.1}$$

where $\Gamma(\cdot)$ *is the gamma function.*

Proof. From (1.2), we have $Y_{ij} = (1-\phi_{10}B_1)^{-\delta}(1-\phi_{01}B_2)^{-\delta}Z_{ij}$. Using binomial expansion, it follows that

$$Y_{ij} = \left[\sum_{k=0}^{\infty}(-1)^k\binom{-\delta}{k}(-\phi_{10}B_1)^k\right]\left[\sum_{l=0}^{\infty}(-1)^l\binom{-\delta}{l}(-\phi_{01}B_2)^l\right]Z_{ij}. \tag{2.2}$$

Note that

$$\begin{aligned}\binom{-\delta}{k} &= \frac{(-\delta)(-\delta-1)\cdots(-\delta-k+1)}{k!}\\ &= \frac{(-1)^k(\delta)(\delta+1)\cdots(\delta+k-1)}{k!}\\ &= \frac{(-1)^k\Gamma(k+\delta)}{\Gamma(k+1)\Gamma(\delta)}.\end{aligned} \tag{2.3}$$

Similarly

$$\binom{-d}{l} = \frac{(-1)^l\Gamma(l+d)}{\Gamma(l+1)\Gamma(d)}. \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2) and upon simplification completes the proof. □

Proposition 2. *For a process defined in* (1.1), *the spectral density is given as*

$$f(\lambda_1,\lambda_2) = \frac{\sigma^2}{4\pi^2\left(1-2\phi_{10}\cos\lambda_1+\phi_{10}^2\right)^{\delta}\left(1-2\phi_{01}\cos\lambda_2+\phi_{01}^2\right)^{\delta}}. \tag{2.5}$$

Proof. The proof is established by simplifying the following expression:

$$f(\lambda_1,\lambda_2) = \frac{\sigma^2}{4\pi^2}\times\frac{1}{\left|(1-\phi_{10}e^{-i\lambda_1})(1-\phi_{01}e^{-i\lambda_2})\right|^{2\delta}}. \tag{2.6}$$

□

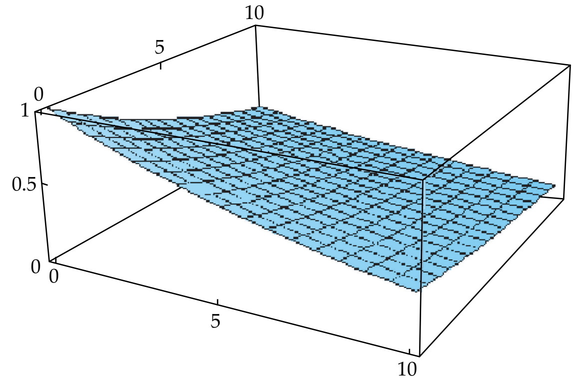

Figure 1: ACF plot for standard separable model ($\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 1.0$).

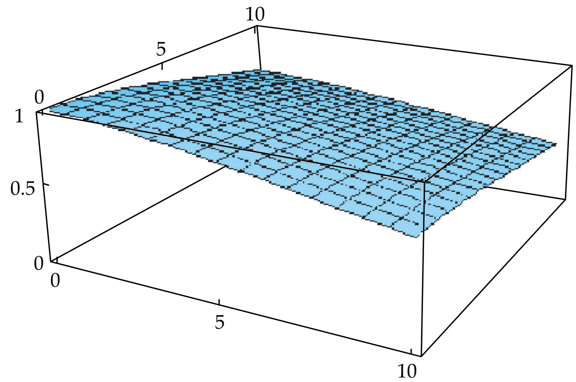

Figure 2: ACF plot for GENSSAR ($\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 1.8$).

The following proposition provides an expression for the autocovariance function of the GENSSAR(1,1) process.

Proposition 3. *For a process defined in* (1.1) *the autocovariance* $\gamma(k_1, k_2)$ *of the process* $\gamma(k_1, k_2)$ *is given as*

$$\gamma(k_1, k_2) = \sigma^2 \frac{\phi_{10}^{k_1}\Gamma(k_1+\delta)F\left(\delta, k_1+\delta; k_1+1; \phi_{10}^2\right)\phi_{01}^{k_2}\Gamma(k_2+\delta)F\left(\delta, k_2+\delta; k_2+1; \phi_{01}^2\right)}{\Gamma^2(\delta)\Gamma(k_1+1)\Gamma(k_2+1)}, \tag{2.7}$$

where $F(\cdot, \cdot; \cdot; \cdot)$ *is the hypergeometric function.*

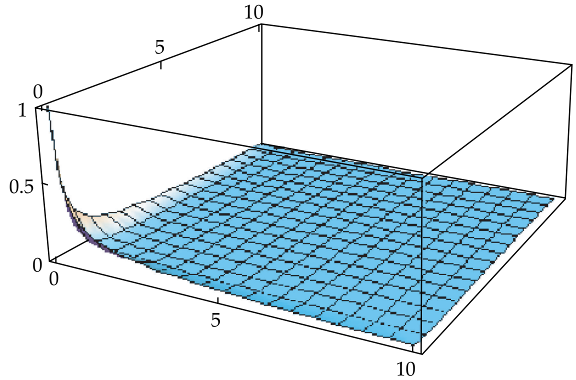

Figure 3: ACF plot for GENSSAR ($\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 0.2$).

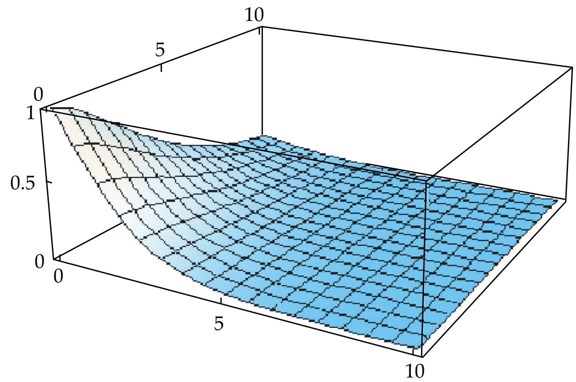

Figure 4: ACF plot for GENSSAR ($\phi_{10} = 0.4$, $\phi_{01} = 0.7$, $\delta = 1.8$).

Proof. We establish the previous proposition by integrating the spectral density as given in Proposition 2

$$\begin{aligned}
\gamma(k_1, k_2) &= \iint_{-\pi}^{\pi} e^{k_1\lambda_1 + k_2\lambda_2} f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \\
&= \frac{\sigma^2}{4\pi^2} \iint_{-\pi}^{\pi} \frac{e^{k_1\lambda_1 + k_2\lambda_2} d\lambda_1 d\lambda_2}{\left(1 - 2\phi_{10}\cos\lambda_1 + \phi_{10}^2\right)^{\delta} \left(1 - 2\phi_{01}\cos\lambda_2 + \phi_{01}^2\right)^{\delta}} \\
&= \frac{\sigma^2}{4\pi^2} \int_{-\pi}^{\pi} \frac{e^{k_1\lambda_1} d\lambda_1}{\left(1 - 2\phi_{10}\cos\lambda_1 + \phi_{10}^2\right)^{\delta}} \int_{-\pi}^{\pi} \frac{e^{k_2\lambda_2} d\lambda_2}{\left(1 - 2\phi_{01}\cos\lambda_2 + \phi_{01}^2\right)^{\delta}} \\
&= \frac{\sigma^2}{\pi^2} \int_{0}^{\pi} \frac{\cos k_1\lambda_1 d\lambda_1}{\left(1 - 2\phi_{10}\cos\lambda_1 + \phi_{10}^2\right)^{\delta}} \int_{0}^{\pi} \frac{\cos k_2\lambda_2 d\lambda_2}{\left(1 - 2\phi_{01}\cos\lambda_2 + \phi_{01}^2\right)^{\delta}}.
\end{aligned} \tag{2.8}$$

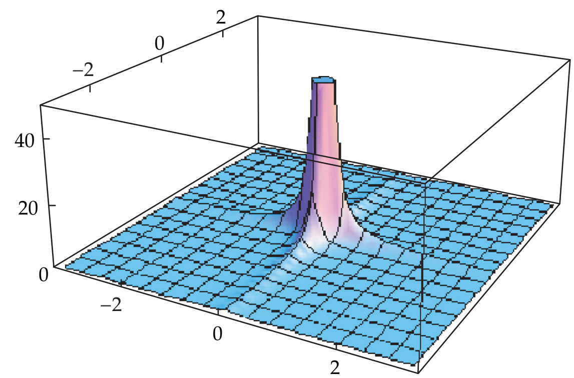

Figure 5: Spectral density plot of standard separable model ($\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 1.0$).

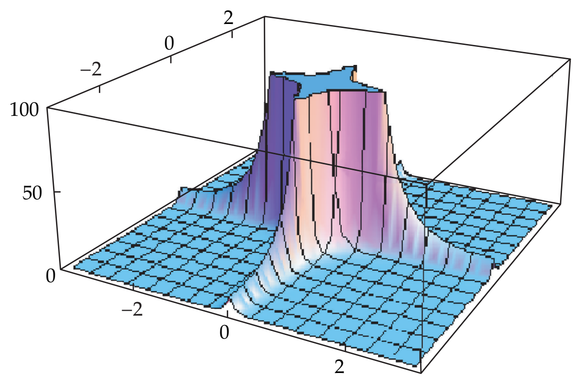

Figure 6: Spectral density plot of GENSSAR model ($\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 1.8$).

Now by making use of the identity (see [8]),

$$\int_0^{\pi} \frac{\cos kx \, dx}{(1 - 2\alpha \cos x + \alpha^2)^{\delta}} = \frac{\pi \alpha^k \Gamma(k+\delta) F(\delta, k+\delta; k+1; \alpha^2)}{\Gamma(\delta)\Gamma(k+1)}, \tag{2.9}$$

we obtain

$$\gamma(k_1, k_2) = \sigma^2 \frac{\phi_{10}^{k_1} \Gamma(k_1+\delta) F(\delta, k_1+\delta; k_1+1; \phi_{10}^2) \phi_{01}^{k_2} \Gamma(k_2+\delta) F(\delta, k_2+\delta; k_2+1; \phi_{01}^2)}{\Gamma^2(\delta)\Gamma(k_1+1)\Gamma(k_2+1)}, \tag{2.10}$$

which completes the proof. □

Corollary 4. *For a process defined in* (1.1) *the variance of the process* $\gamma(0,0)$ *is given as*

$$\gamma(0,0) = \sigma^2 F\left(\delta, \delta; 1; \phi_{10}^2\right) F\left(\delta, \delta; 1; \phi_{01}^2\right). \tag{2.11}$$

Proof. This result is directly from Proposition 3 by letting $k_1 = k_2 = 0$. □

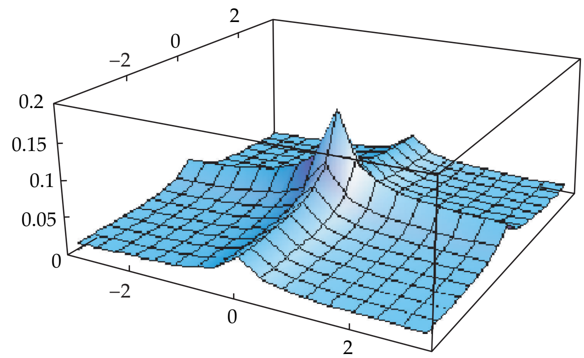

Figure 7: Spectral density plot of GENSSAR model ($\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 0.2$).

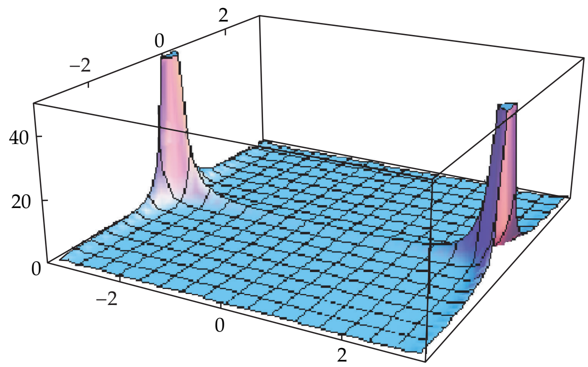

Figure 8: Spectral density plot of standard separable model ($\phi_{10} = -0.9$, $\phi_{01} = 0.9$, $\delta = 1.0$).

The autocorrelation function (ACF) of the model in (1.1) is given as

$$\begin{aligned}\rho(k_1,k_2) &= \frac{\gamma(k_1,k_2)}{\gamma(0,0)}\\ &= \frac{\phi_{10}^{k_1}\Gamma(k_1+\delta)F(\delta,k_1+\delta;k_1+1;\phi_{10}^2)\phi_{01}^{k_2}\Gamma(k_2+\delta)F(\delta,k_2+\delta;k_2+1;\phi_{01}^2)}{\Gamma^2(\delta)\Gamma(k_1+1)\Gamma(k_2+1)F(\delta,\delta;1;\phi_{10}^2)F(\delta,\delta;1;\phi_{01}^2)}.\end{aligned} \tag{2.12}$$

Remark 5. Note when $\delta = 1$, we have the standard separable spatial model. Substituting $\delta = 1$ in Proposition 3, we obtain

$$\begin{aligned}\gamma(k_1,k_2) &= \sigma^2\frac{\phi_{10}^{k_1}\Gamma(k_1+1)F(1,k_1+1;k_1+1;\phi_{10}^2)\phi_{01}^{k_2}\Gamma(k_2+1)F(1,k_2+1;k_2+1;\phi_{01}^2)}{\Gamma^2(1)\Gamma(k_1+1)\Gamma(k_2+1)}\\ &= \sigma^2\phi_{10}^{k_1}F\left(1,k_1+1;k_1+1;\phi_{10}^2\right)\phi_{01}^{k_2}F\left(1,k_2+1;k_2+1;\phi_{01}^2\right).\end{aligned} \tag{2.13}$$

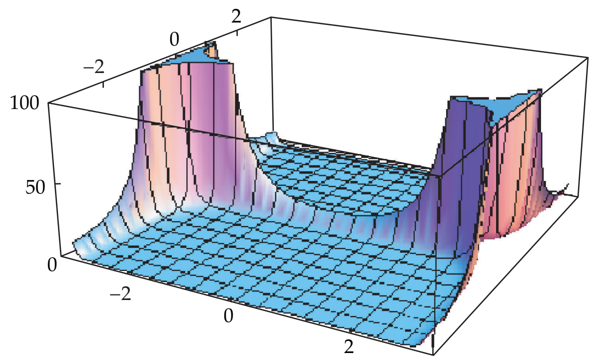

Figure 9: Spectral density plot of GENSSAR model ($\phi_{10} = -0.9$, $\phi_{01} = 0.9$, $\delta = 1.8$).

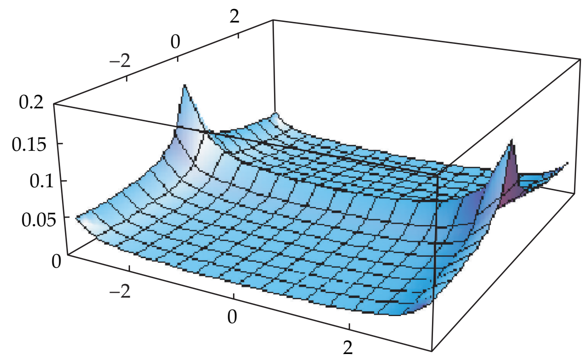

Figure 10: Spectral density plot of GENSSAR model ($\phi_{10} = -0.9$, $\phi_{01} = 0.9$, $\delta = 0.2$).

Using the following identity (see Abramowitz and Stegun [11, Page 556, Identity No. 15.1.8]):

$$F(a,b,b,z) = \frac{1}{(1-z)^a}, \tag{2.14}$$

(2.13) reduces to

$$\gamma(k_1,k_2) = \frac{\sigma^2 \phi_{10}^{k_1} \phi_{01}^{k_2}}{(1-\phi_{10}^2)(1-\phi_{01}^2)}. \tag{2.15}$$

Hence, Proposition 3 reduces to the autocovariance function of the standard separable spatial model when $\delta = 1$.

In Table 1, we have tabulated (to three decimal places) the ACF, $\rho(k_1,k_2)$ computed by using (2.12) with $\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 1.0$. This is the standard separable model. Clearly, we can see that the numerical values computed by using (2.12) agree with the ACF of the standard separable model which is $\rho(k_1,k_2) = \phi_{10}^{k_1}\phi_{01}^{k_2}$. Hence, this verifies (2.12).

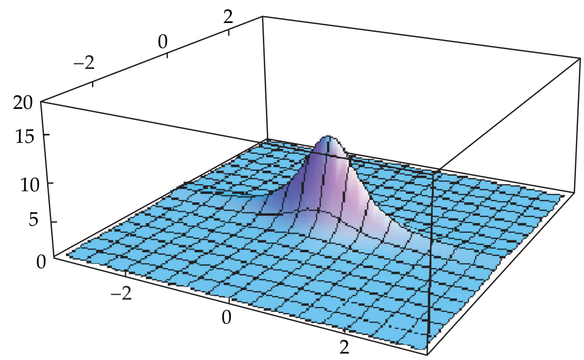

Figure 11: Spectral density plot of GENSSAR model ($\phi_{10} = 0.4$, $\phi_{01} = 0.7$, $\delta = 1.8$).

Table 1: ACF, $\rho(k_1, k_2)$ computed by using (2.12), ($\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 1.0$, i.e., standard separable model).

$k_1 : k_2$	0	1	2	3	4	5	6	7	8	9	10
0	1.000	0.900	0.810	0.729	0.656	0.590	0.531	0.478	0.430	0.387	0.349
1	0.900	0.810	0.729	0.656	0.590	0.531	0.478	0.430	0.387	0.349	0.314
2	0.810	0.729	0.656	0.590	0.531	0.478	0.430	0.387	0.349	0.314	0.282
3	0.729	0.656	0.590	0.531	0.478	0.430	0.387	0.349	0.314	0.282	0.254
4	0.656	0.590	0.531	0.478	0.430	0.387	0.349	0.314	0.282	0.254	0.229
5	0.590	0.531	0.478	0.430	0.387	0.349	0.314	0.282	0.254	0.229	0.206
6	0.531	0.478	0.430	0.387	0.349	0.314	0.282	0.254	0.229	0.206	0.185
7	0.478	0.430	0.387	0.349	0.314	0.282	0.254	0.229	0.206	0.185	0.167
8	0.430	0.387	0.349	0.314	0.282	0.254	0.229	0.206	0.185	0.167	0.150
9	0.387	0.349	0.314	0.282	0.254	0.229	0.206	0.185	0.167	0.150	0.135
10	0.349	0.314	0.282	0.254	0.229	0.206	0.185	0.167	0.150	0.135	0.122

In Table 2, we have tabulated (to three decimal places) the ACF, $\rho(k_1, k_2)$ of the GENSSAR model ($\delta = 1.8$) computed by using (2.12) with $\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 1.8$. While Table 3 shows the ACF values of the GENSSAR model ($\delta = 0.2$) computed by (2.12) with $\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 0.2$.

Figures 1, 2, and 3 show the ACF for the three models considered in this paper.

From the tables and figures we can clearly see that the behaviour of ACF depends on the index parameter δ. When $\delta > 1.0$, the ACF decays slower than that of the standard separable model. On the other hand, when $\delta < 1.0$ the ACF decays faster than the standard model. Hence, the GENSSAR model can be used to model many types of autocorrelation structure.

We also considered a further illustrative example when ϕ_{10} and ϕ_{01} were not equal to each other. That is, we chose the parameter values to be $\phi_{10} = 0.4$, $\phi_{01} = 0.7$, $\delta = 1.8$. In Table 4, we have tabulated (to three decimal places) the ACF, $\rho(k_1, k_2)$ of the GENSSAR model ($\delta = 1.8$) computed by using (2.12) with $\phi_{10} = 0.4$, $\phi_{01} = 0.7$, $\delta = 1.8$, and Figure 4 shows a plot of the ACF. Clearly we can see that the decay in the autocorrelation is more rapid along the k_1 axis as compared to the k_2 axis. Hence, we can model data whose autocorrelations decay at different rates in different directions.

For the models considered in this paper, the two-dimensional spectral densities for various parameter values are shown in Figures 5, 6, 7, 8, 9, 10, and 11.

Table 2: ACF, $\rho(k_1, k_2)$ computed for GENSSAR model by using (2.12), ($\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 1.8$).

$k_1 : k_2$	0	1	2	3	4	5	6	7	8	9	10
0	1.000	0.992	0.972	0.945	0.916	0.874	0.835	0.794	0.752	0.710	0.668
1	0.992	0.983	0.964	0.937	0.904	0.867	0.828	0.787	0.745	0.704	0.662
2	0.972	0.964	0.945	0.918	0.886	0.850	0.812	0.771	0.731	0.690	0.650
3	0.945	0.937	0.918	0.892	0.861	0.826	0.789	0.750	0.710	0.671	0.631
4	0.912	0.904	0.886	0.861	0.831	0.797	0.761	0.723	0.685	0.647	0.609
5	0.874	0.867	0.850	0.826	0.797	0.765	0.730	0.694	0.657	0.621	0.584
6	0.835	0.828	0.812	0.789	0.761	0.730	0.697	0.662	0.627	0.592	0.558
7	0.794	0.787	0.771	0.750	0.723	0.694	0.662	0.630	0.596	0.563	0.530
8	0.752	0.745	0.731	0.710	0.685	0.657	0.627	0.596	0.565	0.533	0.502
9	0.710	0.704	0.690	0.671	0.647	0.621	0.592	0.563	0.533	0.504	0.474
10	0.668	0.663	0.650	0.631	0.609	0.584	0.558	0.530	0.502	0.474	0.447

Table 3: ACF, $\rho(k_1, k_2)$ computed for GENSSAR model by using (2.12), ($\phi_{10} = 0.9$, $\phi_{01} = 0.9$, $\delta = 0.2$).

$k_1 : k_2$	0	1	2	3	4	5	6	7	8	9	10
0	1.000	0.201	0.113	0.077	0.056	0.043	0.034	0.027	0.022	0.018	0.015
1	0.201	0.040	0.028	0.015	0.011	0.009	0.007	0.005	0.004	0.004	0.003
2	0.113	0.023	0.013	0.009	0.006	0.005	0.004	0.003	0.002	0.002	0.002
3	0.077	0.015	0.008	0.006	0.004	0.003	0.003	0.002	0.002	0.001	0.001
4	0.056	0.011	0.006	0.004	0.003	0.002	0.002	0.002	0.001	0.001	0.001
5	0.043	0.009	0.005	0.003	0.002	0.002	0.001	0.001	0.001	0.001	0.001
6	0.034	0.007	0.004	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001
7	0.027	0.005	0.003	0.002	0.002	0.001	0.001	0.001	0.001	0.000	0.000
8	0.022	0.004	0.002	0.002	0.001	0.001	0.001	0.001	0.000	0.000	0.000
9	0.018	0.004	0.002	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000
10	0.015	0.003	0.002	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000

Table 4: ACF, $\rho(k_1, k_2)$ computed for GENSSAR model by using (2.12), ($\phi_{10} = 0.4$, $\phi_{01} = 0.7$, $\delta = 1.8$).

$k_1 : k_2$	0	1	2	3	4	5	6	7	8	9	10
0	1.000	0.920	0.787	0.646	0.517	0.405	0.313	0.240	0.182	0.137	0.102
1	0.645	0.593	0.507	0.417	0.333	0.261	0.202	0.154	0.117	0.088	0.066
2	0.347	0.319	0.273	0.224	0.179	0.141	0.109	0.083	0.063	0.047	0.035
3	0.172	0.158	0.135	0.111	0.089	0.070	0.054	0.041	0.031	0.024	0.018
4	0.082	0.075	0.064	0.053	0.042	0.033	0.026	0.020	0.015	0.011	0.008
5	0.038	0.035	0.030	0.024	0.019	0.015	0.012	0.009	0.007	0.005	0.004
6	0.017	0.016	0.013	0.011	0.009	0.007	0.005	0.004	0.003	0.002	0.002
7	0.007	0.007	0.006	0.005	0.004	0.003	0.002	0.002	0.001	0.001	0.001
8	0.003	0.003	0.003	0.002	0.002	0.001	0.001	0.001	0.001	0.000	0.000
9	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000
10	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

3. Conclusion

The objective of this research is to introduce a new class of models called *GENSSAR* models by including an additional index parameter δ and to establish some of its properties. We have established the autocovariance function. The GENSSAR(1,1) model is a more general model than the standard separable spatial AR(1,1) process. Due to the generality of this model, it is a useful model.

The authors are working on the other aspects of this model with applications and will be reported in a future paper.

Acknowledgments

The authors are very grateful to the reviewer and the editor for their valuable comments and suggestions to improve the quality of this paper. They express their thanks to the Department of Mathematics and the Institute of Mathematical Research, University Putra Malaysia for their support. They also wish to thank the School of Mathematics and Statistics, The University of Sydney, for their support during the first author's visit in 2008.

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