

# Generalized Nonlinear Mixed Implicit Quasi-Variational Inclusions with Set-Valued Mappings

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In this paper, we introduce and study a new class of implicit quasi-variational inclusions, which is called the generalized nonlinear mixed implicit quasi-variational inclusion with set-valued mappings. Using the resolvent operator technique for maximal monotone mapping, we construct some new iterative algorithms for solving this class of generalized nonlinear mixed implicit quasi-variational inclusions with non-compact set-valued mappings. We prove the existence of solution for this kind of generalized nonlinear mixed implicit quasi-variational inclusions with non-compact set-valued mappings and the convergence of iterative sequences generated by the algorithms. We also discuss the convergence and stability of perturbed iterative algorithm with errors for solving a class of generalized nonlinear mixed implicit quasi-variational inclusions with single-valued mappings.

*Keywords:* Mixed implicit quasi-variational inclusion; Set-valued mapping; Iterative algorithm; Perturbed algorithm with errors; Stability

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## 1 INTRODUCTION

Variational inequality theory and complementarity problem theory are very powerful tool of the current mathematical technology. In recent years, classical variational inequality and complementarity problem

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have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional, structural, transportation, elasticity, and applied sciences, etc., see [1], [3–7], [9–16], [19–29], [31], [33–38], [41–43], [45–50] and the references therein. A useful and an important generation of variational inequalities is a mixed variational inequality containing nonlinear term. Due to the presence of the nonlinear term, the projection method cannot be used to study the existence of a solution for the mixed variational inequalities. In 1994, Hassouni and Moudafi [20] used the resolvent operator technique for maximal monotone mapping to study a new class of mixed variational inequalities for single-valued mappings. In 1996, Huang [21] extended this technique for a new class of general mixed variational inequalities (inclusions) with non-compact set-valued mappings and Adly [1] modified this technique for another new class of general mixed variational inequalities (inclusions) for single-valued mappings, which includes the mixed variational inequality considered by Hassouni and Moudafi [20] as special cases. Recently, Huang [22–24] and Huang *et al.* [25–27] introduced and studied some new classes of variational inequalities and inclusions with non-compact set-valued mappings in Hilbert spaces.

On the other hand, Huang [23] introduced and studied the Mann and Ishikawa type perturbed iterative algorithms with errors for the generalized implicit quasi-variational inequality (inclusion) in Hilbert spaces. Very recently, Huang *et al.* [26] constructed a new perturbed iterative algorithm for solving a class of generalized nonlinear mixed quasi-variational inequalities (inclusions) and proved the convergence and stability of the iterative sequences generated by the perturbed iterative algorithm with errors.

Inspired and motivated by recent research works, in this paper, we introduce and study a new class of implicit quasi-variational inclusions, which is called the generalized nonlinear mixed implicit quasi-variational inclusion with set-valued mappings. We establish the equivalence between generalized nonlinear mixed implicit quasi-variational inclusion and fixed point problems by employing the resolvent operator technique for maximal monotone mapping. Using this equivalence, we construct some new iterative algorithms for solving this class of generalized nonlinear mixed implicit quasi-variational inclusions with set-valued mappings. We prove the existence of solution for this kind of

generalized nonlinear mixed implicit quasi-variational inclusions with non-compact set-valued mappings and the convergence of iterative sequences generated by the algorithms. We also discuss the convergence and stability of perturbed iterative algorithm with errors for solving a class of generalized nonlinear mixed implicit quasi-variational inclusions with single-valued mappings. The results shown in this paper improve and extend the previously known results in this area.

## 2 PRELIMINARIES

Let  $H$  be a real Hilbert space endowed with a norm  $\| \cdot \|$  and inner product  $\langle \cdot, \cdot \rangle$ . Let  $G, S, T, P: H \rightarrow 2^H$  be set-valued mappings, where  $2^H$  denotes the family of all nonempty subsets of  $H$ , and  $N: H \times H \rightarrow H$  be a single-valued mapping. Suppose that  $M: H \times H \rightarrow 2^H$  is a set-valued mapping such that, for each fixed  $t \in H, M(\cdot, t): H \rightarrow 2^H$  is a maximal monotone mapping and  $\text{range}(P) \cap \text{dom}(M(\cdot, t)) \neq \emptyset$  for each  $t \in H$ . We consider the following problem:

Find  $u \in H, x \in Su, y \in Tu, z \in Gu, w \in Pu$  such that  $w \in \text{dom}(M(\cdot, z))$  and

$$0 \in N(x, y) + M(w, z). \tag{2.1}$$

The problem (2.1) is called the *generalized nonlinear mixed implicit quasi-variational inclusion with set-valued mappings*.

A well known example [30] of a maximal monotone mapping is the subdifferential of a proper lower semicontinuous convex function. Therefore, we can get some special cases of the problem (2.1) as follows:

(I) If  $M(\cdot, t) = \partial\varphi(\cdot, t)$  for each  $t \in H$ , where  $\varphi: H \times H \rightarrow R \cup \{+\infty\}$  such that for each fixed  $t \in H, \varphi(\cdot, t): H \rightarrow R \cup \{+\infty\}$  is a proper convex lower semicontinuous function on  $H$  and  $P(H) \cap \text{dom}(\partial\varphi(\cdot, t)) \neq \emptyset$  for each  $t \in H$  and  $\partial\varphi(\cdot, t)$  denotes the subdifferential of function  $\varphi(\cdot, t)$ , then the problem (2.1) is equivalent to finding  $u \in H, x \in Su, y \in Tu, z \in Gu$  and  $w \in Pu$  such that

$$\begin{cases} w \in \text{dom}(\partial\varphi(\cdot, z)), \\ \langle N(x, y), v - w \rangle \geq \varphi(w, z) - \varphi(v, z) \end{cases} \tag{2.2}$$

for all  $v \in H$ .

(II) If  $G$  is the identity mapping, then the problem (2.1) reduces to the problem of finding  $u \in H$ ,  $x \in Su$ ,  $y \in Tu$ ,  $w \in Pu$  such that  $w \in \text{dom}(M(\cdot, z)) \neq \emptyset$  and

$$0 \in N(x, y) + M(w, u). \quad (2.3)$$

(III) If  $G$  is the identity mapping, then the problem (2.2) reduces to the problem of finding  $u \in H$ ,  $x \in Su$ ,  $y \in Tu$  and  $w \in Pu$  such that

$$\begin{cases} w \in \text{dom}(\partial\varphi(\cdot, z)), \\ \langle N(x, y), v - w \rangle \geq \varphi(w, z) - \varphi(v, z) \end{cases} \quad (2.4)$$

for all  $v \in H$ .

(IV) If  $P$  is a single-valued mapping, then the problem (2.1) reduces to the problem of finding  $u \in H$ ,  $x \in Su$ ,  $y \in Tu$ ,  $z \in Gu$  such that  $Pu \in \text{dom}(M(\cdot, z))$  and

$$0 \in N(x, y) + M(Pu, z). \quad (2.5)$$

The problem (2.5) is called the *generalized nonlinear set-valued mixed quasi-variational inequality*, which was introduced and studied by Huang *et al.* [26].

(V) If  $G$  is the identity mapping,  $P$  is a single-valued mapping, and  $M(s, t) = M(s)$  for all  $t \in H$ , where  $M : H \rightarrow 2^H$  is a maximal monotone mapping, then the problem (2.1) is equivalent to finding  $u \in H$ ,  $x \in Su$ ,  $y \in Tu$  such that  $Pu \in \text{dom}(M)$  and

$$0 \in N(x, y) + M(Pu). \quad (2.6)$$

This problem (2.6) is called the *generalized set-valued mixed variational inclusion*, which was introduced and studied by Huang [24].

(VI) If  $G$  is the identity mapping,  $P$  is a single-valued mapping, and  $M(\cdot, t) = \partial\varphi$  for each  $t \in H$ , where  $\varphi : H \rightarrow R \cup \{+\infty\}$  is a proper convex lower semicontinuous function on  $H$  and  $P(H) \cap \text{dom}(\partial\varphi) \neq \emptyset$  and  $\partial\varphi$  denotes the subdifferential of function  $\varphi$ , then the problem (2.1) is equivalent to finding  $u \in H$ ,  $x \in Su$ ,  $y \in Tu$  such that

$$\begin{cases} Pu \in \text{dom}(\partial\varphi), \\ \langle N(x, y), v - Pu \rangle \geq \varphi(Pu) - \varphi(v) \end{cases} \quad (2.7)$$

for all  $v \in H$ . The problem (2.7) is called the *generalized set-valued mixed variational inequality*, which was studied by Noor, Noor and Rassias [37]. It is known that a number of problems involving the non-monotone, nonconvex and nonsmooth mappings arising in structural engineering, mechanics, economics, and optimization theory can be studied via the problem (2.7), see, for example, [12], [16] and the references therein.

(VII) If  $G$  is the identity mapping,  $P$ ,  $S$  and  $T$  are all single-valued mappings, then the problem (2.1) is equivalent to finding  $u \in H$  such that  $Pu \in \text{dom}(M(\cdot, u))$  and

$$0 \in N(Su, Tu) + M(Pu, u), \quad (2.8)$$

which is called the *generalized nonlinear mixed implicit quasi-variational inclusion*.

It is well known [44], [49] that there exist maximal monotone mappings which are not subdifferentials of lower semicontinuous proper convex functions. Therefore the problem (2.1) is more general than the problems (2.2)–(2.8).

For a suitable choice of the mappings  $S$ ,  $T$ ,  $G$ ,  $N$ ,  $P$ ,  $M$  and the space  $H$ , a number of known classes mixed variational inequalities, variational inequalities, quasi-variational inequalities, complementarity problems, and quasi-(implicit) complementarity problems in [1], [3], [5], [7], [10], [13], [14], [20–26], [29], [34–38], [43], [45–49] can be obtained as special cases of the generalized nonlinear mixed implicit quasi-variational inclusion (2.1). Further, these type of implicit quasi-variational inclusions enable us to study many important problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional, structural, transportation, elasticity, and applied sciences in a general and unified framework.

### 3 ITERATIVE ALGORITHMS

It is well known (cf. [8], [30]) that, if  $M$  is a maximal monotone mapping from  $H$  to  $2^H$ , then, for every  $\mu > 0$ , the resolvent  $(I + \mu M)^{-1}$  is a well-defined single-valued non-expansive operator mapping  $H$  into itself. By using the resolvent operator technique, it is possible to convert

the generalized nonlinear mixed implicit quasi-variational inclusion (2.1) into an equivalent equation which is easier to handle. To do this, we multiply all the terms in (2.1) with some  $\rho > 0$  and add  $w$  and then we obtain

$$w - \rho N(x, y) \in w + \rho M(w, z).$$

Therefore we have the following:

LEMMA 3.1 *( $u, x, y, z, w$ ) is a solution of the problem (2.1) if and only if ( $u, x, y, z, w$ ) satisfies the relation*

$$w = J_{\rho}^{M(\cdot, z)}(w - \rho N(x, y)),$$

where  $\rho > 0$  is a constant,  $J_{\rho}^{M(\cdot, z)} = (I + \rho M(\cdot, z))^{-1}$  and  $I$  is the identity mapping on  $H$ .

Based on Lemma 3.1 and Nadler's result [32], we now suggest and analyze the following new general and unified algorithms for the problem (2.1).

Let  $N: H \times H \rightarrow H$  be a mapping and  $G, P, S, T: H \rightarrow CB(H)$  be set-valued mappings, where  $CB(H)$  is the family of all nonempty bounded closed subsets of  $H$ . For given  $u_0 \in H$ , we take  $x_0 \in Su_0$ ,  $y_0 \in Tu_0$ ,  $z_0 \in Gu_0$ ,  $w_0 \in Pu_0$ , and let

$$u_1 = u_0 - w_0 + J_{\rho}^{M(\cdot, z_0)}(w_0 - \rho N(x_0, y_0)).$$

Since  $x_0 \in Su_0 \in CB(H)$ ,  $y_0 \in Tu_0 \in CB(H)$ ,  $z_0 \in Gu_0 \in CB(H)$ , and  $w_0 \in Pu_0 \in CB(H)$ , by [32], Nadler's result, there exist  $x_1 \in Su_1$ ,  $y_1 \in Tu_1$ ,  $z_1 \in Gu_1$  and  $w_1 \in Pu_1$  such that

$$\begin{aligned} \|x_0 - x_1\| &\leq (1 + 1)H(Su_0, Su_1), \\ \|y_0 - y_1\| &\leq (1 + 1)H(Tu_0, Tu_1), \\ \|z_0 - z_1\| &\leq (1 + 1)H(Gu_0, Gu_1), \\ \|w_0 - w_1\| &\leq (1 + 1)H(Pu_0, Pu_1), \end{aligned}$$

where  $H(\cdot, \cdot)$  is the Hausdorff metric on  $CB(H)$ . By induction, we can obtain our algorithm for the problem (2.1) as follows:

ALGORITHM 3.1 *Suppose that  $N: H \times H \rightarrow H$  is a mapping and  $G, P, S, T: H \rightarrow CB(H)$  are set-valued mappings. For given  $u_0 \in H, x_0 \in Su_0, y_0 \in Tu_0, z_0 \in Gu_0,$  and  $w_0 \in Pu_0,$  compute  $\{u_n\}, \{x_n\}, \{y_n\}, \{z_n\},$  and  $\{w_n\}$  from the iterative schemes*

$$\left\{ \begin{array}{l} u_{n+1} = u_n - w_n + J_p^{M(\cdot, z_n)}(w_n - \rho N(x_n, y_n)) \\ \|x_n - x_{n+1}\| \leq (1 + (n + 1)^{-1})H(Su_n, Su_{n+1}), \quad x_n \in Su_n, \\ \|y_n - y_{n+1}\| \leq (1 + (n + 1)^{-1})H(Tu_n, Tu_{n+1}), \quad y_n \in Tu_n, \\ \|z_n - z_{n+1}\| \leq (1 + (n + 1)^{-1})H(Gu_n, Gu_{n+1}), \quad z_n \in Gu_n, \\ \|w_n - w_{n+1}\| \leq (1 + (n + 1)^{-1})H(Pu_n, Pu_{n+1}), \quad z_n \in Pu_n, \end{array} \right. \quad (3.1)$$

for  $n = 0, 1, 2, \dots,$  where  $\rho > 0$  is a constant.

From Algorithm 3.1, we can get an algorithm for the problem (2.2) as follows:

ALGORITHM 3.2 *Suppose that  $N: H \times H \rightarrow H$  is a mapping and  $G, P, S, T: H \rightarrow CB(H)$  are set-valued mappings. For given  $u_0 \in H, x_0 \in Su_0, y_0 \in Tu_0, z_0 \in Gu_0,$  and  $w_0 \in Pu_0,$  compute  $\{u_n\}, \{x_n\}, \{y_n\}, \{z_n\},$  and  $\{w_n\}$  from the iterative schemes*

$$\left\{ \begin{array}{l} u_{n+1} = u_n - p(u_n) + J_p^{\partial\varphi(\cdot, z_n)}(p(u_n) - \rho N(x_n, y_n)) \\ \|x_n - x_{n+1}\| \leq (1 + (n + 1)^{-1})H(Su_n, Su_{n+1}), \quad x_n \in Su_n, \\ \|y_n - y_{n+1}\| \leq (1 + (n + 1)^{-1})H(Tu_n, Tu_{n+1}), \quad y_n \in Tu_n, \\ \|z_n - z_{n+1}\| \leq (1 + (n + 1)^{-1})H(Gu_n, Gu_{n+1}), \quad z_n \in Gu_n, \\ \|w_n - w_{n+1}\| \leq (1 + (n + 1)^{-1})H(Pu_n, Pu_{n+1}), \quad z_n \in Pu_n, \end{array} \right. \quad (3.2)$$

for  $n = 0, 1, 2, \dots,$  where  $\rho > 0$  is a constant and  $J_p^{\partial\varphi(\cdot, z)} = (I + \rho\partial\varphi(\cdot, z))^{-1}.$

For a suitable choice of the mappings  $S, T, G, N, P, M$  and the space  $H,$  many known iterative algorithms for solving various classes of variational inequalities and complementarity problems in [1], [13], [14], [20–22], [24], [26], [34], [37], [38], [46], [47], [49] can be obtained as special cases of Algorithms 3.1 and 3.2.

#### 4 EXISTENCE AND CONVERGENCE THEOREMS

In this section, we prove the existence of a solution of the problem (2.1) and the convergence of iterative sequence generated by Algorithm 3.1.

DEFINITION 4.1 *A mapping  $g : H \rightarrow H$  is said to be*

- (1) *strongly monotone if there exists a number  $\delta > 0$  such that*

$$\langle g(u_1) - g(u_2), u_1 - u_2 \rangle \geq \delta \|u_1 - u_2\|^2$$

*for all  $u_i \in H$ ,  $i = 1, 2$ ,*

- (2) *Lipschitz continuous if there exists a number  $\sigma > 0$  such that*

$$\|g(u_1) - g(u_2)\| \leq \sigma \|u_1 - u_2\|$$

*for all  $u_i \in H$ ,  $i = 1, 2$ .*

DEFINITION 4.2 *A set-valued mapping  $S : H \rightarrow CB(H)$  is said to be*

- (1) *H-Lipschitz continuous if there exists a number  $\eta > 0$  such that*

$$H(S(u_1), S(u_2)) \leq \eta \|u_1 - u_2\|$$

*for all  $u_i \in H$ ,  $i = 1, 2$ ,*

- (2) *strongly monotone if there exists a number  $\gamma > 0$  such that*

$$\langle x_1 - x_2, u_1 - u_2 \rangle \geq \gamma \|u_1 - u_2\|^2$$

*for all  $x_i \in S(u_i)$ ,  $i = 1, 2$ ,*

- (3) *strongly monotone with respect to the first argument of  $N(\cdot, \cdot) : H \times H \rightarrow H$ , if there exists a number  $\delta > 0$  such that*

$$\langle N(x_1, \cdot) - N(x_2, \cdot), u_1 - u_2 \rangle \geq \alpha \|u_1 - u_2\|^2$$

*for all  $x_i \in S(u_i)$ ,  $i = 1, 2$ .*

DEFINITION 4.3 *The operator  $N : H \times H \rightarrow H$  is said to be Lipschitz continuous with respect to the first argument if there exists a constant  $\beta > 0$  such that*

$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \leq \beta \|u_1 - u_2\|$$

*for all  $u_i \in H$ ,  $i = 1, 2$ .*



In a similar way, we can define Lipschitz continuity of the operator  $N(\cdot, \cdot)$  with respect to the second argument.

**THEOREM 4.1** *Let  $N$  be Lipschitz continuous with respect to the first and second arguments with the constants  $\beta, \xi$ , respectively. Let  $S : H \rightarrow CB(H)$  be strongly monotone with respect to the first argument of  $N(\cdot, \cdot)$  with the constant  $\alpha$ . Let  $S, T, G : H \rightarrow CB(H)$  be  $H$ -Lipschitz with the constants  $\eta, \gamma$  and  $s$ , respectively,  $P : H \rightarrow CB(H)$  be strongly monotone and  $H$ -Lipschitz continuous with the constants  $\delta$  and  $\sigma$ , respectively. Suppose that there exist numbers  $\lambda > 0$  and  $\rho > 0$  such that, for each  $x, y, z \in H$ ,*

$$\|J_p^{M(\cdot, x)}(z) - J_p^{M(\cdot, y)}(z)\| \leq \lambda \|x - y\| \tag{4.1}$$

and

$$\left\{ \begin{array}{l} \left| \rho - \frac{\alpha + \xi\gamma(k-1)}{\eta^2\beta^2 - \xi^2\gamma^2} \right| < \frac{\sqrt{(\alpha + \xi\gamma(k-1))^2 - (\eta^2\beta^2 - \xi^2\gamma^2)k(2-k)}}{\eta^2\beta^2 - \xi^2\gamma^2}, \\ \alpha > (1-k)\xi\gamma + \sqrt{(\eta^2\beta^2 - \xi^2\gamma^2)k(2-k)}, \quad \eta\beta > \xi\gamma, \\ \rho\xi\gamma < 1-k, \quad k = \lambda s + 2\sqrt{1-2\delta + \sigma^2}, \quad k < 1. \end{array} \right. \tag{4.2}$$

Then there exist  $u \in H, x \in Su, y \in Tu, z \in Gu$ , and  $w \in Pu$  satisfying the problem (2.1). Moreover,

$$u_n \rightarrow u, x_n \rightarrow x, y_n \rightarrow y, z_n \rightarrow z, w_n \rightarrow w \quad \text{as } n \rightarrow \infty,$$

where  $\{u_n\}, \{x_n\}, \{y_n\}, \{z_n\}$ , and  $\{w_n\}$  are sequences defined in Algorithm 3.1.

*Proof* From Algorithm 3.1 and (4.1), we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|u_n - u_{n-1} - (w_n - w_{n-1}) + J_\rho^{M(\cdot, z_n)}(w_n - \rho N(x_n, y_n)) \\
&\quad - J_\rho^{M(\cdot, z_{n-1})}(w_{n-1} - \rho N(x_{n-1}, y_{n-1}))\| \\
&\leq \|u_n - u_{n-1} - (w_n - w_{n-1})\| + \|J_\rho^{M(\cdot, z_n)}(w_n - \rho N(x_n, y_n)) \\
&\quad - J_\rho^{M(\cdot, z_{n-1})}(w_{n-1} - \rho N(x_{n-1}, y_{n-1}))\| \\
&\leq \|u_n - u_{n-1} - (w_n - w_{n-1})\| \\
&\quad + \|J_\rho^{M(\cdot, z_n)}(w_{n-1} - \rho N(x_{n-1}, y_{n-1})) \\
&\quad - J_\rho^{M(\cdot, z_{n-1})}(w_{n-1} - \rho N(x_{n-1}, y_{n-1}))\| + \|J_P^{M(\cdot, z_n)} \\
&\quad \times (w_n - \rho N(x_n, y_n)) - J_P^{M(\cdot, z_{n-1})}(w_{n-1} - \rho N(x_{n-1}, y_{n-1}))\| \\
&\leq \|u_n - u_{n-1} - (w_n - w_{n-1})\| + \lambda \|z_n - z_{n-1}\| \\
&\quad + \|(w_n - \rho N(x_n, y_n)) - (w_{n-1} - \rho N(x_{n-1}, y_{n-1}))\| \\
&\leq 2\|u_n - u_{n-1} - (w_n - w_{n-1})\| + \lambda \|z_n - z_{n-1}\| \\
&\quad + \|u_n - u_{n-1} - \rho(N(x_n, y_n) - N(x_{n-1}, y_{n-1}))\| \\
&\leq 2\|u_n - u_{n-1} - (w_n - w_{n-1})\| + \lambda \|z_n - z_{n-1}\| \\
&\quad + \|u_n - u_{n-1} - \rho(N(x_n, y_n) - N(x_{n-1}, y_{n-1}))\| \\
&\quad + \rho \|N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1})\|.
\end{aligned} \tag{4.3}$$

By the  $H$ -Lipschitz continuity and strong monotonicity of  $P$  and Algorithm 3.1, we obtain

$$\begin{aligned}
&\|u_n - u_{n-1} - (w_n - w_{n-1})\|^2 \\
&= \|u_n - u_{n-1}\|^2 - 2\langle u_n - u_{n-1}, w_n - w_{n-1} \rangle + \|w_n - w_{n-1}\|^2 \\
&\leq \|u_n - u_{n-1}\|^2 - 2\delta \|u_n - u_{n-1}\|^2 + (1 + n^{-1})^2 [H(Pu_n, Pu_{n-1})]^2 \\
&\leq (1 - 2\delta + \sigma^2(1 + n^{-1})^2) \|u_n - u_{n-1}\|^2.
\end{aligned} \tag{4.4}$$

Since  $S$  is  $H$ -Lipschitz continuous and strongly monotone with respect to the first argument of  $N$  and  $N$  is Lipschitz continuous with respect to the first argument, we have

$$\begin{aligned} & \|u_n - u_{n-1} - \rho(N(x_n, y_n) - N(x_{n-1}, y_n))\|^2 \\ &= \|u_n - u_{n-1}\|^2 - 2\rho \langle u_n - u_{n-1}, N(x_n, y_n) - N(x_{n-1}, y_n) \rangle \\ &\quad + \rho^2 \|N(x_n, y_n) - N(x_{n-1}, y_n)\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\eta^2(1 + n^{-1})^2\beta^2)\|u_n - u_{n-1}\|^2. \end{aligned} \tag{4.5}$$

Further, since  $T, G$  are  $H$ -Lipschitz continuous and  $N$  is Lipschitz continuous with respect to the second argument, we get

$$\begin{aligned} \|N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1})\| &\leq \xi \|y_n - y_{n-1}\| \\ &\leq \xi\gamma(1 + n^{-1})\|u_n - u_{n-1}\| \end{aligned} \tag{4.6}$$

and

$$\|z_n - z_{n-1}\| \leq s(1 + n)^{-1}\|u_n - u_{n-1}\|. \tag{4.7}$$

From (4.3)~(4.7), it follows that

$$\|u_n - u_{n+1}\| \leq \theta_n \|u_n - u_{n-1}\|, \tag{4.8}$$

where

$$\begin{aligned} \theta_n &= \lambda s(1 + n^{-1}) + 2\sqrt{1 - 2\delta + \sigma^2(1 + n^{-1})^2} \\ &\quad + \sqrt{1 - 2\rho\alpha + \rho^2\eta^2\beta^2(1 + n^{-1})^2 + \rho\xi\gamma(1 + n^{-1})}. \end{aligned}$$

Letting

$$\theta = k + \sqrt{1 - 2\rho\alpha + \rho^2\eta^2\beta^2 + \rho\xi\gamma},$$

where  $k = \lambda s + 2\sqrt{1 - 2\delta + \sigma^2}$ , we know  $\theta_n \searrow \theta$ . It follows from (4.2) that  $\theta < 1$ . Hence  $\theta_n < 1$  for  $n$  sufficiently large. Therefore, (4.8) implies that  $\{u_n\}$  is a Cauchy sequence in  $H$  and we can suppose that  $u_n \rightarrow u \in H$ .

Now we prove that  $x_n \rightarrow x \in Su$ ,  $y_n \rightarrow y \in Tu$ ,  $z_n \rightarrow z \in Gu$  and  $w_n \rightarrow w \in Pu$ . In fact, it follows from Algorithm 3.1 that

$$\begin{aligned}\|x_n - x_{n-1}\| &\leq (1 + n^{-1})\eta\|u_n - u_{n-1}\|, \\ \|y_n - y_{n-1}\| &\leq (1 + n^{-1})\gamma\|u_n - u_{n-1}\|, \\ \|z_n - z_{n-1}\| &\leq (1 + n^{-1})s\|u_n - u_{n-1}\|, \\ \|w_n - w_{n-1}\| &\leq (1 + n^{-1})\sigma\|u_n - u_{n-1}\|,\end{aligned}$$

which imply that  $\{x_n\}$ ,  $\{y_n\}$ ,  $\{z_n\}$  and  $\{w_n\}$  are all Cauchy sequences in  $H$ . Let  $x_n \rightarrow x$ ,  $y_n \rightarrow y$ ,  $z_n \rightarrow z$  and  $w_n \rightarrow w$ . Furthermore,

$$\begin{aligned}d(x, Su) &= \inf\{\|x - v\| : v \in Su\} \\ &\leq \|x - x_n\| + d(x_n, Su) \\ &\leq \|x - x_n\| + H(Su_n, Su) \\ &\leq \|x - x_n\| + \eta\|u_n - u\| \rightarrow 0.\end{aligned}$$

Hence, we have  $x \in Su$ . Similarly, we have  $y \in Tu$ ,  $z \in Gu$ , and  $w \in Pu$ . This completes the proof.

From Theorem 4.1, we have the following result:

**THEOREM 4.2** *Let  $N, S, T, G, P$  be the same as in Theorem 4.1. Suppose that there exist numbers  $\lambda > 0$  and  $\rho > 0$  such that, for each  $x, y, z \in H$ ,*

$$\|J_\rho^{\partial\varphi(\cdot, x)}(z) - J_\rho^{\partial\varphi(\cdot, y)}(z)\| \leq \lambda\|x - y\|$$

*and the condition (4.2) in Theorem 4.1 holds. Then there exist  $u \in H$ ,  $x \in Su$ ,  $y \in Tu$  and  $z \in Gu$ , satisfying the problem (2.2). Moreover,*

$$u_n \rightarrow u, \quad x_n \rightarrow x, \quad y_n \rightarrow y, \quad z_n \rightarrow z, \quad w_n \rightarrow w \quad \text{as } n \rightarrow \infty,$$

*where  $\{u_n\}$ ,  $\{x_n\}$ ,  $\{y_n\}$ ,  $\{z_n\}$ , and  $\{w_n\}$  are sequences defined in Algorithm 3.2.*

For an appropriate and suitable choice of the mappings  $S, T, G, N, P, M$  and the space  $H$ , we can obtain several known results in [1], [14], [20–22], [24], [26], [34], [37], [38], [46], [47], [49] as special cases of Theorems 4.1 and 4.2.

## 5 PERTURBED ALGORITHMS AND STABILITY

In this section, we construct a new perturbed iterative algorithm with errors for solving the generalized nonlinear mixed implicit quasi-variational inclusion (2.8) and prove the convergence and stability of the iterative sequence generated by the perturbed iterative algorithm with errors.

**DEFINITION 5.1** *Let  $T$  be a self-mapping of  $H$ ,  $x_0 \in H$  and let  $x_{n+1} = f(T, x_n)$  define an iteration procedure which yields a sequence of points  $\{x_n\}_{n=0}^\infty$  in  $H$ . Suppose that  $\{x \in H : Tx = x\} \neq \emptyset$  and  $\{x_n\}_{n=0}^\infty$  converges to a fixed point  $x^*$  of  $T$ . Let  $\{y_n\} \subset H$  and let  $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$ . If  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  implies that  $\lim_{n \rightarrow \infty} y_n = x^*$ , then the iteration procedure defined by  $x_{n+1} = f(T, x_n)$  is said to be  $T$ -stable or stable with respect to  $T$ . If  $\sum_{n=0}^\infty \varepsilon_n < +\infty$  implies that  $\lim_{n \rightarrow \infty} y_n = x^*$ , then the iterative procedure  $\{x_n\}$  is said to be almost  $T$ -stable.*

We remark that an iterative procedure  $\{x_n\}$  which is  $T$ -stable is almost  $T$ -stable and an iterative procedure  $\{x_n\}$  which is almost  $T$ -stable need not be  $T$ -stable (see [40]).

Some stability results of iterative procedures have been established by several authors (see [2], [17], [18], [26], [27] and [39]). As pointed out by Harder and Hicks [18], the study on the stability of various iterative procedures is both of theoretical and numerical interest.

**DEFINITION 5.2** *Let  $\{M^n\}$  and  $M$  be maximal monotone mappings for  $n = 0, 1, 2, \dots$ . The sequence  $\{M^n\}$  is said to be graph-convergence to  $M$  (we write  $M^n \xrightarrow{G} M$ ) if the following property holds: for every  $(x, y) \in \text{Graph}(M)$ , there exists a sequence  $(x_n, y_n) \in \text{Graph}(M^n)$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$  as  $n \rightarrow \infty$ .*

For our results, we need the following lemmas:

**LEMMA 5.1** (See [27]) *Let  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  be three sequences of nonnegative numbers satisfying the following conditions: there exists  $n_0$  such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n$$

for all  $n \geq n_0$ , where

$$t_n \in [0, 1], \quad \sum_{n=0}^{\infty} t_n = +\infty, \quad \lim_{n \rightarrow \infty} b_n = 0, \quad \sum_{n=0}^{\infty} c_n < +\infty.$$

Then  $a_n \rightarrow 0$  as  $n \rightarrow +\infty$ .

LEMMA 5.2 (See [4]) *Let  $\{M^n\}$  and  $M$  be maximal monotone mappings from  $H$  into the power of  $H$  for  $n = 0, 1, 2, \dots$ . Then  $M^n \xrightarrow{G} M$  if and only if*

$$J_{\lambda}^{M^n}(x) \rightarrow J_{\lambda}^M(x)$$

for every  $x \in H$  and  $\lambda > 0$ .

ALGORITHM 5.1 *Let  $N: H \times H \rightarrow H$  and  $S, T, P: H \rightarrow H$  be single-valued mappings. Suppose that  $M^n: H \times H \rightarrow 2^H$  is a sequence of set-valued mapping such that, for each  $y \in H$ ,  $M^n(\cdot, y): H \rightarrow 2^H$  is a maximal monotone mapping for  $n = 0, 1, 2, \dots$ . For given  $u_0 \in H$ , the perturbed iterative sequence  $\{u_n\}$  are defined by*

$$\left\{ \begin{array}{l} u_{n+1} = (1 - \alpha_n)u_n + \alpha_n[v_n - Pv_n \\ \quad + J_{\rho}^{M^n(\cdot, v_n)}(Pv_n - \rho N(Sv_n, Tv_n))] + \alpha_n e_n, \\ v_n = (1 - \beta_n)u_n + \beta_n[w_n - Pw_n \\ \quad + J_{\rho}^{M^n(\cdot, w_n)}(Pw_n - \rho N(Sw_n, Tw_n))] + \beta_n f_n, \\ w_n = (1 - \gamma_n)u_n + \gamma_n[u_n - Pu_n \\ \quad + J_{\rho}^{M^n(\cdot, u_n)}(Pu_n - \rho N(Su_n, Tu_n))] + \gamma_n g_n \end{array} \right. \quad (5.1)$$

for  $n = 0, 1, 2, \dots$ , where  $\{e_n\}$ ,  $\{f_n\}$ , and  $\{g_n\}$  are three sequences of the elements of  $H$  introduced to take into account possible inexact computation and the sequences  $\{\alpha_n\}$ ,  $\{\beta_n\}$ , and  $\{\gamma_n\}$  satisfy the following conditions:

$$0 \leq \alpha_n, \beta_n, \gamma_n \leq 1 \quad (n \geq 0), \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

If  $\gamma_n = 0$  for  $n = 0, 1, 2, \dots$ , then Algorithm 5.1 reduces to the following algorithm:

**ALGORITHM 5.2** *Let  $N, S, T, P$  and  $M^n$  be the same as in Algorithm 5.1. For given  $u_0 \in H$ , the perturbed iterative sequence  $\{u_n\}$  are defined by*

$$\left\{ \begin{array}{l} u_{n+1} = (1 - \alpha_n)u_n + \alpha_n[v_n - Pv_n + J_\rho^{M^n(\cdot, v_n)}(Pv_n - \rho N(Sv_n, Tv_n))] \\ \quad + \alpha_n e_n, \\ v_n = (1 - \beta_n)u_n + \beta_n[u_n - Pu_n + J_\rho^{M^n(\cdot, u_n)}(Pu_n - \rho N(Su_n, Tu_n))] \\ \quad + \beta_n f_n \end{array} \right.$$

for  $n = 0, 1, 2, \dots$ , where  $\{e_n\}$  and  $\{f_n\}$  are two sequences of the elements of  $H$  introduced to take into account possible inexact computation and the sequences  $\{\alpha_n\}$  and  $\{\beta_n\}$  satisfy the following conditions:

$$0 \leq \alpha_n, \beta_n \leq 1 \quad (n \geq 0), \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

**THEOREM 5.1** *Let  $N$  be Lipschitz continuous with respect to the first and second arguments with the constants  $\beta, \xi$ , respectively. Let  $S: H \rightarrow H$  be strongly monotone with respect to the first argument of  $N$  with the constant  $\alpha$ . Let  $S, T: H \rightarrow H$  be Lipschitz continuous with the constants  $\eta$  and  $\gamma$ , respectively,  $P: H \rightarrow H$  be strongly monotone and Lipschitz continuous with the constants  $\delta$  and  $\sigma$ , respectively. Suppose that  $M^n: H \times H \rightarrow 2^H$  is a sequence of set-valued mapping such that, for each  $y \in H, M^n(\cdot, y): H \rightarrow 2^H$  is a maximal monotone mapping for  $n = 0, 1, 2, \dots, M^n(\cdot, y) \xrightarrow{G} M(\cdot, y)$ , and there exist numbers  $\lambda > 0$  and  $\rho > 0$  such that, for each  $x, y, z \in H$ ,*

$$\|J_\rho^{M^n(\cdot, x)}(z) - J_\rho^{M^n(\cdot, y)}(z)\| \leq \lambda \|x - y\|, \tag{5.2}$$

$$\|J_\rho^{M^n(\cdot, x)}(z) - J_\rho^{M^n(\cdot, y)}(z)\| \leq \lambda \|x - y\| \tag{5.3}$$

and

$$\left\{ \begin{array}{l} \left| \rho - \frac{\alpha + \xi\gamma(k-1)}{\eta^2\beta^2 - \xi^2\gamma^2} \right| < \frac{\sqrt{(\alpha + \xi\gamma(k-1))^2 - (\eta^2\beta^2 - \xi^2\gamma^2)k(2-k)}}{\eta^2\beta^2 - \xi^2\gamma^2}, \\ \alpha > (1-k)\xi\gamma + \sqrt{(\eta^2\beta^2 - \xi^2\gamma^2)k(2-k)}, \quad \eta\beta > \xi\gamma, \\ \rho\xi\gamma < 1-k, \quad k = \lambda + 2\sqrt{1-2\delta + \sigma^2}, \quad k < 1. \end{array} \right. \quad (5.4)$$

Let  $\{y_n\}$  be any sequence in  $H$  and define  $\{\varepsilon_n\}$  by

$$\left\{ \begin{array}{l} \varepsilon_n = \|y_{n+1} - \{(1-\alpha_n)y_n + \alpha_n[x_n - Px_n + J_\rho^{M(\cdot, x_n)} \\ \quad \times (Px_n - \rho N(Sx_n, Tx_n))] + \alpha_n e_n\}\|, \\ x_n = (1-\beta_n)y_n + \beta_n[z_n - Pz_n + J_\rho^{M(\cdot, z_n)} \\ \quad \times (Pz_n - \rho N(Sz_n, Tz_n))] + \beta_n f_n, \\ z_n = (1-\gamma_n)y_n + \gamma_n[y_n - Py_n + J_\rho^{M(\cdot, y_n)} \\ \quad \times (Py_n - \rho N(Sy_n, Ty_n))] + \gamma_n g_n \end{array} \right. \quad (5.5)$$

for  $n = 0, 1, 2, \dots$ . If  $\lim_{n \rightarrow \infty} \|e_n\| = 0$ ,  $\lim_{n \rightarrow \infty} \|f_n\| = 0$  and  $\lim_{n \rightarrow \infty} \|g_n\| = 0$ , then

- (I) The sequence  $\{u_n\}$  defined by Algorithm 5.1 converges strongly to the unique solution  $u^*$  of the problem (2.8).
- (II) If  $\sum_{n=0}^{\infty} \alpha_n < \infty$ , then  $\lim_{n \rightarrow \infty} y_n = u^*$ .
- (III) If  $\lim_{n \rightarrow \infty} y_n = u^*$ , then  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ .

*Proof* (I) It follows from (5.2), (5.4) and Theorem 4.1 that there exists  $u^* \in H$  which is a solution of the problem (2.8) and so

$$Pu^* = J_\rho^{M(\cdot, u^*)}(Pu^* - \rho N(Su^*, Tu^*)). \quad (5.6)$$



From (5.3), (5.5), (5.6) and Algorithm 5.1, it follows that

$$\begin{aligned}
 & \|u_{n+1} - u^*\| \\
 &= \|(1 - \alpha_n)u_n + \alpha_n[v_n - Pv_n + J_\rho^{M^n(\cdot, v_n)} \\
 &\quad \times (Pv_n - \rho N(Sv_n, Tv_n))] + \alpha_n e_n - (1 - \alpha_n)u^* \\
 &\quad - \alpha_n[u^* - Pu^* + J_\rho^{M(\cdot, u^*)}(Pu^* - \rho N(Su^*, Tu^*))]\| \\
 &\leq (1 - \alpha_n)\|u_n - u^*\| + \alpha_n\|v_n - Pv_n - (u^* - Pu^*)\| + \alpha_n\|e_n\| \\
 &\quad + \alpha_n\|J_\rho^{M^n(\cdot, v_n)}(Pv_n - \rho N(Sv_n, Tv_n)) \\
 &\quad - J_\rho^{M(\cdot, u^*)}(Pu^* - \rho N(Su^*, Tu^*))\| \\
 &\leq (1 - \alpha_n)\|u_n - u^*\| + \alpha_n\|v_n - u^* - (Pv_n - Pu^*)\| + \alpha_n\|e_n\| \\
 &\quad + \alpha_n\|J_\rho^{M^n(\cdot, v_n)}(Pv_n - \rho N(Sv_n, Tv_n)) \\
 &\quad - J_\rho^{M^n(\cdot, v_n)}(Pu^* - \rho N(Su^*, Tu^*))\| \\
 &\quad + \alpha_n\|J_\rho^{M^n(\cdot, v_n)}(Pu^* - \rho N(Su^*, Tu^*)) \\
 &\quad - J_\rho^{M^n(\cdot, v_n)}(Pu^* - \rho N(Su^*, Tu^*))\| \\
 &\quad + \alpha_n\|J_\rho^{M^n(\cdot, v_n)}(Pu^* - \rho N(Su^*, Tu^*)) \\
 &\quad - J_\rho^{M^n(\cdot, v_n)}(Pu^* - \rho N(Su^*, Tu^*))\| \\
 &\quad + \alpha_n\|J_\rho^{M^n(\cdot, v_n)}(Pu^* - \rho N(Su^*, Tu^*)) \\
 &\quad - J_\rho^{M^n(\cdot, v_n)}(Pu^* - \rho N(Su^*, Tu^*))\| \\
 &\leq (1 - \alpha_n)\|u_n - u^*\| + 2\alpha_n\|v_n - u^* - (Pv_n - Pu^*)\| \\
 &\quad + \alpha_n(\|e_n\| + h_n) + \alpha_n\|v_n - u^* - \rho(N(Sv_n, Tu^*) \\
 &\quad - N(Su^*, Tu^*))\| + \lambda\alpha_n\|v_n - u^*\| \\
 &\leq (1 - \alpha_n)\|u_n - u^*\| + 2\alpha_n\|v_n - u^* - (Pv_n - Pu^*)\| + \alpha_n(\|e_n\| \\
 &\quad + h_n) + \alpha_n\|v_n - u^* - \rho(N(Sv_n, Tv_n) - N(Su^*, Tv_n))\| \\
 &\quad + \alpha_n\rho\|N(Su^*, Tv_n) - N(Su^*, Tu^*)\| + \lambda\alpha_n\|v_n - u^*\|,
 \end{aligned}
 \tag{5.7}$$

where

$$\begin{aligned}
 h_n &= \|J_\rho^{M^n(\cdot, u^*)}(Pu^* - \rho N(Su^*, Tu^*)) \\
 &\quad - J_\rho^{M(\cdot, u^*)}(Pu^* - \rho N(Su^*, Tu^*))\|.
 \end{aligned}$$

From Lemma 5.2, we know that  $h_n \rightarrow 0$  as  $n \rightarrow \infty$ . By the Lipschitz continuity of  $N$ ,  $S$ ,  $T$ ,  $P$  and the strong monotonicity of  $S$  and  $P$ , we obtain

$$\|v_n - u^* - (Pv_n - Pu^*)\|^2 \leq (1 - 2\delta + \sigma^2)\|v_n - u^*\|^2, \quad (5.8)$$

$$\begin{aligned} \|v_n - u^* - \rho(N(Sv_n, Tv_n) - N(Su^*, Tu^*))\|^2 \\ \leq (1 - 2\rho\alpha + \rho^2\eta^2\beta^2)\|v_n - u^*\|^2 \end{aligned} \quad (5.9)$$

and

$$\|N(Su^*, Tv_n) - N(Su^*, Tu^*)\| \leq \xi\gamma\|v_n - u^*\|. \quad (5.10)$$

It follows from (5.7)~(5.10) that

$$\|u_{n+1} - u^*\| \leq (1 - \alpha_n)\|u_n - u^*\| + \theta\alpha_n\|v_n - u^*\| + \alpha_n(\|e_n\| + h_n), \quad (5.11)$$

where

$$\theta = \lambda + 2\sqrt{1 - 2\delta + \sigma^2} + \sqrt{1 - 2\rho\alpha + \rho^2\eta^2\beta^2} + \rho\xi\gamma.$$

Similarly, we have

$$\|v_n - u^*\| \leq (1 - \beta_n)\|u_n - u^*\| + \theta\beta_n\|w_n - u^*\| + \beta_n\|f_n\|$$

and

$$\|w_n - u^*\| \leq (1 - \gamma_n)\|u_n - u^*\| + \theta\gamma_n\|u_n - u^*\| + \gamma_n\|g_n\|.$$

The condition (5.4) implies that  $0 < \theta < 1$ . It follows that

$$\|w_n - u^*\| \leq \|u_n - u^*\| + \gamma_n\|g_n\|$$

and so

$$\|v_n - u^*\| \leq \|u_n - u^*\| + \beta_n(\gamma_n\|g_n\| + \|f_n\|). \quad (5.12)$$

From (5.11) and (5.12), we have

$$\begin{aligned} \|u_{n+1} - u^*\| &\leq [1 - \alpha_n(1 - \theta)]\|u_n - u^*\| + \alpha_n\beta_n\theta(\gamma_n\|g_n\| + \|f_n\|) \\ &\quad + \alpha_n(\|e_n\| + h_n) \\ &\leq [1 - \alpha_n(1 - \theta)]\|u_n - u^*\| + \alpha_n(1 - \theta)d_n, \end{aligned} \tag{5.13}$$

where

$$d_n = \frac{\beta_n(\gamma_n\|g_n\| + \|f_n\|) + \|e_n\| + h_n}{1 - \theta} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It follows from (5.13) and Lemma 5.1 that  $u_n \rightarrow u^*$  as  $n \rightarrow \infty$ .

Now we prove that  $u^*$  is a unique solution of the problem (2.8). In fact, if  $u$  is also a solution of the problem (2.8), then

$$Pu = J_\rho^{M(\cdot, u)}(Pu - \rho N(Su, Tu))$$

and, as in the proof of (5.11), we have

$$\|u^* - u\| \leq \theta\|u^* - u\|,$$

where  $0 < \theta < 1$ . Therefore,  $u^* = u$ . This completes the proof of the conclusion (I).

Next we prove the conclusion (II). Using (5.2) we obtain

$$\begin{aligned} \|y_{n+1} - u^*\| &\leq \|y_{n+1} - \{(1 - \alpha_n)y_n + \alpha_n[x_n - Px_n + J_\rho^{M^n(\cdot, x_n)} \\ &\quad \times (Px_n - \rho N(Sx_n, Tx_n))] + \alpha_n e_n\}\| \\ &\quad + \|(1 - \alpha_n)y_n + \alpha_n[x_n - Px_n + J_\rho^{M^n(\cdot, x_n)} \\ &\quad \times (Px_n - \rho N(Sx_n, Tx_n))] + \alpha_n e_n - u^*\| \\ &= \|(1 - \alpha_n)y_n + \alpha_n[x_n - Px_n + J_\rho^{M^n(\cdot, x_n)} \\ &\quad \times (Px_n - \rho N(Sx_n, Tx_n))] + \alpha_n e_n - u^*\| + \varepsilon_n. \end{aligned} \tag{5.14}$$

As in the proof of the inequality (5.13), we have

$$\begin{aligned} \|(1 - \alpha_n)y_n + \alpha_n[x_n - Px_n + J_\rho^{M^n(\cdot, x_n)}(Px_n - \rho N(Sx_n, Tx_n))] + \alpha_n e_n - u^*\| \\ \leq (1 - \alpha_n(1 - \theta))\|y_n - u^*\| + \alpha_n(1 - \theta)d_n. \end{aligned} \tag{5.15}$$

It follows from (5.14) and (5.15) that

$$\|y_{n+1} - u^*\| \leq (1 - \alpha_n(1 - \theta))\|y_n - u^*\| + \alpha_n(1 - \theta)d_n + \varepsilon_n. \quad (5.16)$$

Since  $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ ,  $d_n \rightarrow 0$ , and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ , it follows from (5.16) and Lemma 5.1 that  $\lim_{n \rightarrow \infty} y_n = u^*$ .

Now we prove the conclusion (III). Suppose that  $\lim_{n \rightarrow \infty} y_n = u^*$ . Then we have

$$\begin{aligned} \varepsilon_n &= \|y_{n+1} - \{(1 - \alpha_n)y_n + \alpha_n[x_n - Px_n + J_\rho^{M^n(\cdot, x_n)} \\ &\quad \times (Px_n - \rho N(Sx_n, Tx_n))] + \alpha_n e_n]\|\} \\ &\leq \|y_{n+1} - u^*\| + \|(1 - \alpha_n)y_n + \alpha_n[x_n - Px_n + J_\rho^{M^n(\cdot, x_n)} \\ &\quad \times (Px_n - \rho N(Sx_n, Tx_n))] + \alpha_n e_n - u^*\| \\ &\leq \|y_{n+1} - u^*\| + (1 - \alpha_n(1 - \theta))\|y_n - u^*\| + \alpha_n(1 - \theta)d_n \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . This completes the proof.

From Theorem 5.1, we have the following result:

**THEOREM 5.2** *Suppose that  $N, S, T, P$ , and  $M^n$  are the same as in Theorem 5.1. Let  $\{y_n\}$  be any sequence in  $H$  and define  $\{\varepsilon_n\}$  by*

$$\left\{ \begin{array}{l} \varepsilon_n = \|y_{n+1} - \{(1 - \alpha_n)y_n + \alpha_n[x_n - Px_n + J_\rho^{M^n(\cdot, x_n)} \\ \quad \times (Px_n - \rho N(Sx_n, Tx_n))] + \alpha_n e_n]\|\}, \\ x_n = (1 - \beta_n)y_n + \beta_n[y_n - Py_n + J_\rho^{M^n(\cdot, y_n)} \\ \quad \times (Py_n - \rho N(Sy_n, Ty_n))] + \beta_n f_n \end{array} \right.$$

for  $n = 0, 1, 2, \dots$ . If the conditions (5.3)  $\sim$  (5.5) hold, then

- (I) *The sequence  $\{u_n\}$  defined by Algorithm 5.2 converges strongly to the unique solution  $u^*$  of the problem (2.8).*
- (II) *If  $\sum_{n=0}^{\infty} \alpha_n < \infty$ , then  $\lim_{n \rightarrow \infty} y_n = u^*$ .*
- (III) *If  $\lim_{n \rightarrow \infty} y_n = u^*$ , then  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ .*

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