

Hyperbolic Sets with the Strong Limit Shadowing Property*

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Let ϕ be a C^1 dynamical system on a compact smooth manifold M . In this paper we introduce the notions of weak limit shadowing property and strong limit shadowing property of subsets of M which are not equivalent with that of shadowing property, and show that for any hyperbolic submanifold Λ of M the restriction $\phi|_{\Lambda}$ is Anosov if and only if Λ has the strong limit shadowing property. Moreover we find hyperbolic sets which have the strong limit shadowing property.

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Let M be a compact smooth manifold with a metric r and $\phi: M \rightarrow M$ a C^1 diffeomorphism (or dynamical system). It is said that a sequence $\xi = \{x_k: k \in \mathbb{Z}\}$ in M is ε -shadowed by a point $x^* \in M$ if

$$r(\phi^k(x^*), x_k) < \varepsilon$$

for all $k \in \mathbb{Z}$. We say that a subset Y of M has the *shadowing property* or *pseudo orbit tracing property* (POTP) for ϕ if for given $\varepsilon > 0$ there exists $\delta > 0$ such that any δ -pseudotrajectory in Y , i.e., a sequence $\xi = \{x_k \in Y: k \in \mathbb{Z}\}$ with $r(\phi(x_k), x_{k+1}) < \delta$, is ε -shadowed by a point

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$x^* \in M$. In this case, the point x^* is called a *shadowing point* of ξ . If this property holds with $Y = M$, we say that ϕ has the POTP.

Often pseudotrajectories are obtained as results of numerical studies of dynamical systems. In the context POTP means that numerically found trajectories with uniformly small errors are close to real trajectories. In dynamical system theory, there are various types of shadowing property.

In 1997, Eirola, Nevanlinna and Pilyugin introduced the concept of the limit shadowing property and studied their properties (see [1]).

DEFINITION 1 A subset Y of M has the *limit shadowing property* (LmSP) for ϕ if for any sequence $\xi = \{x_k : k \in \mathbb{Z}\}$ in Y with

$$r(\phi(x_k), x_{k+1}) \rightarrow 0 \quad \text{as } |k| \rightarrow \infty$$

there is a point $x^* \in M$ such that

$$r(\phi^k(x^*), x_k) \rightarrow 0 \quad \text{as } |k| \rightarrow \infty.$$

If this property holds with $Y = M$, we say that ϕ has the LmSP.

From the numerical point of view, this property of a dynamical system ϕ means the following: if we apply a numerical method that approximates ϕ with “improving accuracy”, so that one-step errors tend to zero as time go to positive and negative infinity then the numerically obtained trajectories tend to real ones.

A closed invariant set $\Lambda \subset M$ is said to be *hyperbolic* for ϕ if $T_\Lambda M$ has a continuous splitting $T_\Lambda M = E_\Lambda^s \oplus E_\Lambda^u$ satisfying:

- (1) E^s and E^u are invariant under the derivative map $T\phi$;
- (2) there exist constants $c > 0$ and $0 < \lambda < 1$ such that for any $n \in \mathbb{Z}^+$,

$$\max \{ \|T\phi^n|_{E^s}\|, \|T\phi^{-n}|_{E^u}\| \} < c\lambda^n.$$

We say that $\Lambda \subset M$ is a *hyperbolic manifold* for ϕ if Λ is a C^1 compact invariant submanifold of M with a hyperbolic structure as a subset of M . If M is hyperbolic for ϕ then ϕ is called *Anosov*.

Hirsch asks in [3], if $\Lambda \subset M$ is a hyperbolic manifold for ϕ , does it follow that ϕ restricted to Λ is Anosov (has a hyperbolic structure)? The answer given by Franks and Robinson in [2] was negative.

Recently, Lee and Kim [6] showed that if Λ has the strong shadowing property then $\phi|_{\Lambda}: \Lambda \rightarrow \Lambda$ is Anosov, and found hyperbolic sets which have the strong shadowing property. It is a well-known fact in dynamical system theory that if Λ is hyperbolic for ϕ then it has the shadowing property, *i.e.*, for given $\varepsilon > 0$ there exists $\delta > 0$ such that any δ – pseudotrajectory in Λ is ε – shadowed by a point $x^* \in M$. In general, the shadowing point x^* need not belong to Λ . We say that Λ has the strong shadowing property if the shadowing point x^* belongs to Λ (for more details, see [6]).

In this paper we introduce the concept of the weak limit shadowing property which is different from that of POTP. Moreover we will discuss that $\phi|_{\Lambda}$ is Anosov if and only if Λ has the strong limit shadowing property, and find hyperbolic sets which have the strong limit shadowing property.

DEFINITION 2 A subset Y of M has the weak limit shadowing property (weak LmSP) for ϕ if there exists a constant $d_0 > 0$ such that for any $0 < d < d_0$ and d -pseudotrajectory $\xi = \{x_k : k \in \mathbb{Z}\}$ in Y with

$$r(\phi(x_k), x_{k+1}) \rightarrow 0 \text{ as } |k| \rightarrow \infty$$

there is a point $x^* \in M$ such that

$$r(\phi^k(x^*), x_k) \rightarrow 0 \text{ as } |k| \rightarrow \infty.$$

In this case, the point x^* is called a *limit shadowing point* of ξ . If this property holds with $Y = M$, we say that ϕ has the weak LmSP.

It is easy to show that there exist systems which do not have the weak LmSP.

Example 3 Consider the circle S^1 with coordinate $x \in [0, 1)$ and a diffeomorphism ϕ of S^1 given by $\phi(x) = x$. For any $d_0 > 0$ and $0 < d < d_0$, choose $n \in \mathbb{N}$ satisfying $(1/n) < d$. Let $\xi = \{x_k : k \in \mathbb{Z}\}$ be a sequence in S^1 defined by

$$x_k = \begin{cases} 0 & \text{if } k = 0 \\ x_{k-1} + (1/n + k) \pmod{1} & \text{if } k \geq 1 \\ x_{k+1} - (1/n - k) \pmod{1} & \text{if } k \leq -1 \end{cases}$$

Then ξ is a d -pseudotrajectory in S^1 with

$$r(\phi(x_k), x_{k+1}) = \frac{1}{n+k} \rightarrow 0 \quad \text{as } |k| \rightarrow \infty$$

But we can see that for any point $x^* \in S^1$,

$$r(\phi^k(x^*), x_k) \not\rightarrow 0 \quad \text{as } |k| \rightarrow \infty.$$

Clearly we know that the LmSP implies the weak LmSP. However the following example shows that the weak LmSP is not equivalent to the LmSP. Moreover we can see that a dynamical system which has the weak LmSP need not have the POTP.

Example 4 Consider the circle S^1 with coordinate $x \in [0, 1)$ and a diffeomorphism on S^1 with the following properties;

$$\begin{aligned} \phi(x) &= x \quad \text{if } x \in \left\{0, \frac{1}{4}, \frac{1}{2}\right\}, \\ \phi(x) &> x \quad \text{if } x \in \left(0, \frac{1}{2}\right) - \left\{\frac{1}{4}\right\}; \quad \text{and} \quad \phi(x) < x \quad \text{if } x \in \left(\frac{1}{2}, 1\right). \end{aligned}$$

Then we can see that ϕ does not have the LmSP. In fact, let $\xi = \{x_k : k \in \mathbb{Z}\}$ be a sequence in S^1 given by

$$x_k = \begin{cases} 0 & \text{if } k \geq 0, \\ \frac{1}{4} & \text{if } k < 0. \end{cases}$$

Then ξ is a pseudotrajectory with

$$r(\phi(x_k), x_{k+1}) \rightarrow 0 \quad \text{as } |k| \rightarrow \infty.$$

But we can see that for any point $x^* \in S^1$, we have

$$r(\phi^k(x^*), x_k) \not\rightarrow 0 \quad \text{as } |k| \rightarrow \infty.$$

Moreover it is easy to show that ϕ does not have the POTP.

To show that ϕ has the weak LmSP, we let $a=0$, $b=(1/4)$, $c=(1/2)$. For any $n \in \mathbb{N}$, we let V_s^n denote the $(1/n)$ -neighborhood of the points

$s \in \{a, b, c\}$. Set

$$\begin{aligned} W_1^n &= \left(0, \frac{1}{4}\right) - \left(V_a^n \cup V_b^n\right), \\ W_2^n &= \left(\frac{1}{4}, \frac{1}{2}\right) - \left(V_b^n \cup V_c^n\right) \quad \text{and} \\ W_3^n &= \left(\frac{1}{2}, 1\right) - \left(V_c^n \cup V_a^n\right) \end{aligned}$$

For each $n \in \mathbb{N}$, Choose $\alpha_n > 0$ such that

$$\alpha_n < \frac{1}{2} \inf \left\{ r(\phi(x), x), r(\phi^{-1}(x), x) : x \in \bigcup_{i=1}^3 W_i^n \right\}.$$

For any $0 < d < \alpha_{10}$, let $\xi = \{x_k : k \in \mathbb{Z}\}$ be a d -pseudotrajectory in S^1 with $r(\phi(x_k), x_{k+1}) \rightarrow 0$ as $|k| \rightarrow \infty$. For each integer $n \geq 10$, we can find $k_n > 0$ such that

$$|k| \geq k_n \quad \text{implies} \quad r(\phi(x_k), x_{k+1}) < \alpha_n$$

Then we can consider four possible cases.

Case 1 Suppose $\{x_k : |k| \geq k_n\} \subset (V_a^n \cup V_b^n \cup V_c^n)$. Let $x_k \in V_s^n$ for some $s \in \{a, b, c\}$. Then, by the choice of d , both x_{k-1} and x_{k+1} cannot belong to V_u^n for $u \neq s$. And so we have $\{x_k : |k| \geq k_n\} \subset V_s^n$ for some fixed $s \in \{a, b, c\}$. This means that $x_k \rightarrow s$ as $|k| \rightarrow \infty$

Case 2 Suppose $x_k \in W_3^n$ for some $|k| \geq k_n$. Since $r(\phi(x_k), x_k) > 2\alpha_n$ and $r(\phi(x_k), x_{k+1}) < \alpha_n$, we have

$$r(x_k, x_{k+1}) > \alpha_n \quad \text{and} \quad r(x_k, x_{k-1}) > \alpha_n$$

This means that

$$x_i \in V_c^n \quad \text{and} \quad x_{-i} \in V_a^n,$$

for some $i > k_n$. By the choice of α_n , we can find $h_n > 0$, $n \geq 10$, such that if $k > h_n$ then

$$x_k \in V_c^n \quad \text{and} \quad x_{-k} \in V_a^n.$$

This means that $x_k \rightarrow c$ and $x_{-k} \rightarrow a$, as $k \rightarrow \infty$.

Case 3 Suppose $x_k \in W_2^n$ for some $|k| \geq k_n$. As in the Case 2, we can show that either $x_k \rightarrow c$ and $x_{-k} \rightarrow b$ hold, or $x_k \rightarrow c$ and $x_{-k} \rightarrow a$ hold, as $k \rightarrow \infty$.

Case 4 Suppose $x_k \in W_1^n$ for some $|k| \geq k_n$. Then we can see that either $x_k \rightarrow b$ and $x_{-k} \rightarrow a$ hold, or $x_k \rightarrow c$ and $x_{-k} \rightarrow a$ hold, as $k \rightarrow \infty$.

At any case, we can easily find $x^* \in S^1$ such that

$$r(\phi^k(x^*), x_k) \rightarrow 0 \quad \text{as } |k| \rightarrow \infty$$

This means that ϕ has the weak LmSP.

One of the main results about shadowing near a hyperbolic set of a dynamical system is the so-called the Shadowing Lemma; which means that if Λ is a hyperbolic set for ϕ then it has a neighborhood U which has the POTP (Shadowing property).

In [1], Eirola, Nevanlinna and Pilyugin obtained the similar result with the Shadowing Lemma as follows.

THEOREM 5 ([1], Theorem 2.1) *If Λ is hyperbolic for ϕ then there exists a neighborhood U of Λ such that if a sequence $\{x_k : k \in \mathbb{Z}\}$ belongs to U and if $r(\phi(x_k), x_{k+1}) \rightarrow 0$ as $k \rightarrow \infty$ then there exists a point $x^* \in M$ such that $r(\phi^k(x^*), x_k) \rightarrow 0$ as $k \rightarrow \infty$.*

Similarly, we can consider the above theorem for two-sided sequence. The proof of the following theorem is similar to that of Theorem 5, and we omit it here.

THEOREM 6 *If Λ is hyperbolic for ϕ then there exists a neighborhood U of Λ which has the weak LmSP.*

Remarks 7 If $\Lambda \subset M$ is hyperbolic for ϕ then it has a neighborhood U which has the weak LmSP. However the limit shadowing point need not belong to Λ .

DEFINITION 8 A hyperbolic set $\Lambda \subset M$ has the *strong limit shadowing property* (strong LmSP) for ϕ if there exist a neighborhood U of Λ and a constant $d_0 > 0$ such that for any $0 < d < d_0$ and any d -pseudo-trajectory $\xi = \{x_k : k \in \mathbb{Z}\}$ in U with $r(\phi(x_k), x_{k+1}) \rightarrow 0$ as $|k| \rightarrow \infty$, there is $x^* \in \Lambda$ satisfying $r(\phi^k(x^*), x_k) \rightarrow 0$ as $|k| \rightarrow \infty$.

THEOREM 9 *Let Λ be a hyperbolic manifold for ϕ . Then $\phi|_{\Lambda}: \Lambda \rightarrow \Lambda$ is Anosov if and only if Λ has the strong LmSP.*

Proof Suppose Λ has the strong LmSP. Since Λ is hyperbolic, ϕ is expansive on Λ , i.e., there exists an (expansive) constant $c > 0$ such that if $r(\phi^k(x), \phi^k(y)) < c$ for $x \in \Lambda$, $y \in M$ and all $k \in \mathbb{Z}$ then $x = y$.

We can see that for each $n \in \mathbb{Z} - \{0\}$, $\phi^n = \overbrace{\phi \circ \dots \circ \phi}^{n \text{ times}}$ is also expansive on Λ . Let

$$e(\phi^n) = \sup\{e > 0 : e \text{ is an expansive constant of } \phi^n \text{ w.r.t. } \Lambda\},$$

and let

$$c = \frac{1}{2} \inf\{e(\phi^n) : n \in \mathbb{Z} - \{0\}\}.$$

Then we have $c > 0$. Since Λ has the strong limit shadowing property, we can find a neighborhood U_0 of Λ and a constant $d_0 > 0$ such that for any $0 < d < d_0$ and any d -pseudotrajectory $\xi = \{x_k : k \in \mathbb{Z}\}$ in U_0 with $r(\phi(x_k), x_{k+1}) \rightarrow 0$ as $|k| \rightarrow \infty$, there is a point $x^* \in \Lambda$ satisfying

$$r(\phi^k(x^*), x_k) \rightarrow 0 \quad \text{as } |k| \rightarrow \infty.$$

Choose $d > 0$ such that

$$d < \min(c, d_0) \quad \text{and} \quad B(\Lambda, d) \subset U_0.$$

Put $U = B(\Lambda, d)$.

First we show that $\bigcap_{k \in \mathbb{Z}} \phi^k(U) = \Lambda$. It is clear that $\Lambda \subset \bigcap_{k \in \mathbb{Z}} \phi^k(U)$ since Λ is invariant. To show that $\bigcap_{k \in \mathbb{Z}} \phi^k(U) \subset \Lambda$, we let $y \in \bigcap_{k \in \mathbb{Z}} \phi^k(U)$. Then we have $\phi^k(y) \in U$ for all $k \in \mathbb{Z}$. Let $x_k = \phi^k(y)$ for $k \in \mathbb{Z}$. The $\xi = \{x_k : k \in \mathbb{Z}\}$ is a d -pseudotrajectory in U . Hence there exists a point $x^* \in \Lambda$ such that

$$r(\phi^k(x^*), x_k) \rightarrow 0 \quad \text{as } |k| \rightarrow \infty.$$

Choose $n > 0$ such that if $|k| \geq n$ then

$$r(\phi^k(x^*), x_k) = r(\phi^k(x^*), \phi^k(y)) < c.$$

Put $\phi^{2n}(x^*) = a \in \Lambda$ and $\phi^{2n}(y) = b$. Then we have

$$r(\phi^{2kn}(a), \phi^{2kn}(b)) < c \quad \text{for all } k \in \mathbb{Z}.$$

This means that $a = b$ and so $y = x^* \in \Lambda$.

Next we show that $\phi|_\Lambda$ is structurally stable. Let $\psi \in \text{Diff}^1(\Lambda)$ be C^1 near to $\phi|_\Lambda$. Then we can find $\tilde{\psi} \in \text{Diff}^1(M)$ such that

$$\tilde{\psi} \text{ is } C^1 \text{ near to } \phi \text{ and } \tilde{\psi}|_\Lambda = \psi.$$

If we apply [4, Theorem 7.3] which says that the maximal hyperbolic sets enjoy a type of structural stability, then we can find a homeomorphism $h : \bigcap_{k \in \mathbb{Z}} \phi^k(U) \rightarrow \bigcap_{k \in \mathbb{Z}} \tilde{\psi}^k(U)$ such that

- (1) $\tilde{\psi} \circ h = h \circ \phi$ on $\bigcap_{k \in \mathbb{Z}} \phi^k(U) = \Lambda$, and
- (2) h is C^0 near to the identity map on Λ .

Since $\tilde{\psi}(\Lambda) = \Lambda$, we have $\Lambda \subset \bigcap_{k \in \mathbb{Z}} \tilde{\psi}^k(U)$. Put $g = h^{-1}|_\Lambda$. Since Λ is a compact manifold and g is C^0 near to the identity map on Λ , g is surjective. Hence we get $h(\Lambda) = h(g(\Lambda)) = \Lambda$. This means that $\phi|_\Lambda$ is structurally stable.

If we apply [8, Theorem 5], we can see that $\phi|_\Lambda : \Lambda \rightarrow \Lambda$ is Anosov. The converse is obvious by Theorem 6, and so completes the proof of the theorem.

Hyperbolic manifold which do not have the strong limit shadowing property can be found in the example given by Franks and Robinson [2].

Now we wish to find hyperbolic sets which have the strong limit shadowing property. Put

$$C(\phi) = \overline{\{x \in M : x \in \omega(x) \cap \alpha(x)\}},$$

where $\omega(x)$ and $\alpha(x)$ denote the positive and negative limit set of x . Then $C(\phi)$ is a nonempty closed invariant subset of M . We say that a point $x \in M$ is called *nonwandering* if for any neighborhood U of x and an integer $n_0 > 0$ there exists an integer $n > n_0$ with $\phi^n(U) \cap U \neq \emptyset$. A point $x \in M$ is said to be *chain recurrent* if for any $\varepsilon > 0$ there exists an ε -pseudotrajectory for ϕ from x to x . The set of nonwandering points and the set of chain recurrent points of ϕ will be denoted by $\Omega(\phi)$ and $CR(\phi)$, respectively. Then we have the following inclusions

$$\overline{\text{Per}(\phi)} \subset C(\phi) \subset \Omega(\phi) \subset CR(\phi).$$

THEOREM 10 *If the set $C(\phi)$ is hyperbolic for ϕ then it has the strong limit shadowing property.*

Proof If the set $C(\phi)$ is hyperbolic then ϕ is expansive on $C(\phi)$. Then ϕ^n is also expansive on $C(\phi)$ for each nonzero integer n . Put

$$e(\phi^n) = \sup\{e > 0 : e \text{ is an expansive constant of } \phi^n \text{ w.r.t. } C(\phi)\},$$

and

$$e = \frac{1}{2} \inf\{e(\phi^n) : n \in \mathbb{Z} - \{0\}\}.$$

Then we have $e > 0$. Since $C(\phi)$ has the strong shadowing property for ϕ [6, Theorem 7], we can find $0 < \delta < (e/2)$ such that any δ -pseudotrajectory in $C(\phi)$ is $(e/2)$ -shadowed by a point in $C(\phi)$. By Theorem 6, there exist a neighborhood U of $C(\phi)$ and a constant $d' > 0$ such that for any $0 < d < d'$ and any d -pseudotrajectory $\{x_k : k \in \mathbb{Z}\}$ in U with $r(\phi(x_k), x_{k+1}) \rightarrow 0$ and $|k| \rightarrow \infty$, there exists a point $x^* \in M$ such that

$$r(\phi^k(x^*), x_k) \rightarrow 0 \quad \text{as } |k| \rightarrow \infty.$$

Choose $d_0 > 0$ such that

$$d_0 < \min \left\{ d', \frac{\alpha}{2} \right\} \quad \text{and} \quad B\left(\Lambda, \frac{1}{2}d_0\right) \subset U.$$

If we let $U_0 = B(\Lambda, (1/2)d_0)$ then U_0 and $(1/3)d_0$ are required. For any $0 < d < (1/3)d_0$, let $\xi = \{x_k : k \in \mathbb{Z}\}$ be a d -pseudotrajectory in U_0 with $r(\phi(x_k), x_{k+1}) \rightarrow 0$ as $|k| \rightarrow \infty$. Then we can find a point $x^* \in M$ satisfying

$$r(\phi^k(x^*), x_k) \rightarrow 0 \quad \text{as } |k| \rightarrow \infty \tag{1}$$

For each $k \in \mathbb{Z}$, choose $y_k \in \Lambda$ with $r(x_k, y_k) < (1/2)d_0$. Then $\xi' = \{y_k : k \in \mathbb{Z}\}$ is a α -pseudotrajectory in Λ . In fact, we have

$$\begin{aligned} r(\phi(y_k), y_{k+1}) &\leq r(\phi(y_k), \phi(x_k)) + r(\phi(x_k), x_{k+1}) + r(x_{k+1}, y_{k+1}) \\ &< \frac{\alpha}{2} + d + \frac{1}{2}d_0 < d \end{aligned}$$

Since $C(\phi)$ has the strong shadowing property, there exists a point $y^* \in C(\phi)$ such that

$$r(\phi^k(y^*), y_k) < \frac{1}{2}e \quad (2)$$

By the fact (1), we can choose $N > 0$ such that if $|k| \geq N$ then $r(\phi^k(x^*), x_k) < (1/2)d_0$. Then we have

$$\begin{aligned} r(\phi^k(x^*), \phi^k(y^*)) &\leq r(\phi(x^*), x_k) + r(x_k, y_k) + r(y_k, \phi^k(y^*)) \\ &< \frac{1}{2}d_0 + \frac{1}{2}d_0 + \frac{1}{2}e < e \end{aligned}$$

for each $|k| \geq N$. Put $\phi^{2N}(x^*) = a$ and $\phi^{2N}(y^*) = b \in C(\phi)$. Then we have

$$r(\phi^{2N}(a), \phi^{2N}(b)) < e$$

for all $k \in \mathbb{Z}$. Since ϕ^{2N} is expansive on $C(\phi)$ with an expansive constant e , we have $a = b$ and so $x^* = y^* \in C(\phi)$. This means that $C(\phi)$ has the strong limit shadowing property.

If the chain recurrent set $CR(\phi)$ is hyperbolic then we have $CR(\phi) = C(\phi)$. Hence we get the following corollary.

COROLLARY 11 *If the set $CR(\phi)$ is hyperbolic then it has the strong limit shadowing property.*

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