J. of Inequal. & Appl., 2001, Vol. 6, pp. 17–27
Reprints available directly from the publisher
Photocopying permitted by license only

On the Inequality of Weierstrass for Nonlocal Functionals

G.A. KAMENSKII* and JU.P. ZABRODINA

Department of Differential Equations, State Moscow Institute of Aviation, Volokolamskoe Shosse 4, Moscow 125871, Russia

(Received 28 February 1999; Revised 20 September 1999)

The aim of this paper is to consider the problem of extremum of the nonlocal functional, which depends on a function $u(t,s)$ and its derivative with respect to t at several values of s.

For this problem the generalized Weierstrass inequality, the principle of minimum and the generalized conditions of Weierstrass–Erdmann are derived.

Keywords: Calculus of variations; Nonlocal functionals; Minimum principle; Necessary Weierstrass condition

1991 Mathematics Subject Classification: 34K10, 49K25

The problem of extremum of the nonlocal functional

$$\begin{aligned} J(u) = \int_{t_0}^{t_1} \mathrm{d}t \int_{s_0}^{s_1} & F[t, s, u(t, s-r_1), \ldots, u(t,s) \ldots, \\ & u(t, s+r_2), u'(t, s-r_1), \ldots, \\ & u'(t,s), \ldots, u'(t, s+r_2)] \,\mathrm{d}s, \end{aligned} \tag{1}$$

where $s_1 - s_0 > r_1 + r_2, r_1, r_2$ are some integers is considered. Here and later $u'(t,s)$ denotes the partial derivative of $u(t,s)$ with respect to t.

The analog of the necessary Euler condition for this problem was derived in [1]. The boundary value problems arising from this variational

* Corresponding author.

problem were also studied. The necessity of Weierstrass inequality for other type of nonlocal functionals was proved in [2]. The review of the theory of mixed functional-differential equations, connected with the theory considered here is published in [3].

On the set $E_0=\{(t,s)\mid t\in[t_0,t_1],\ s\in[s_0-r_1,s_0+r_2)\}$ the boundary value function $\varphi(t,s)$ is given; on the set $E_1=\{(t,s)\mid t\in[t_0,t_1], s\in(s_1-r_1, s_1+r_2]\}$ the boundary value function $\psi(t,s)$ is given. The functions $\varphi(t,s)$ and $\psi(t,s)$ are supposed to be continuous with respect to s and continuously differentiable with respect to t. On the intervals $G_0=\{(t,s)\mid t=t_0,\ s\in(s_0+r_2,s_1-r_1)\}$ and $G_1=\{(t,s)\mid t=t_1,\ s\in(s_0+r_2, s_1-r_1)\}$ the piecewise continuous functions $\mu(s)$ and $\nu(s)$ are given. Denote by ρ_μ and ρ_ν the sets of their discontinuity points and let $R_\mu=[t_0,t_1]\times\rho_\mu$, $R_\nu=[t_0,t_1]\times\rho_\nu$. Let $E(z)$ mean the integral part of z. Define the sets

$$Q=[t_0,t_1]\times(s_0+r_2,s_1-r_1),$$

$$R_0=\{(t,s)\mid t\in[t_0,t_1],\ s=s_0+i,\ i=r_2,r_2+1,\dots,E(s_1-s_0)-r_1\},$$

$$R_1=\{(t,s)\mid t\in[t_0,t_1],\ s=s_1-i,\ i=r_1,r_1+1,\dots,E(s_1-s_0)-r_2\},$$

$$R=R_0\cup R_1\cup R_\mu\cup R_\nu.$$

Denote $\tilde{R}=R\cup\{(t,s)\in Q$, where $u'(t,s)$ does not exist$\}$.

The problem of extremum of the functional (1) is considered with boundary value conditions

$$u(t,s)=\varphi(t,s),\quad (t,s)\in E_0,\qquad u(t,s)=\psi(t,s),\quad (t,s)\in E_1, \tag{2}$$

$$u(t,s)=\mu(s),\quad (t,s)\in G_0,\qquad u(t,s)=\nu(s),\quad (t,s)\in G_1. \tag{3}$$

Define the space $\mathbb{H}^0(Q,R)$ of functions $u(t,s)$ on Q that are piecewise continuous with respect to s with the set of discontinuity points contained in R and continuous with respect to t for any $(t,s)\notin R$. On $\mathbb{H}^0(Q,R)$ we define the norm

$$\|u\|_{\mathbb{H}^0}=\max_{(t,s)\in Q\setminus R}|u(t,s)|.$$

Let $\mathbb{H}^1(Q, \tilde{R})$ be the space of function that belong to $\mathbb{H}^0(Q, R)$ and are continuously differentiable with respect to t with the norm

$$\|u\|_{\mathbb{H}^1} = \max_{(t,s)\in Q\setminus\tilde{R}} \{|u(t,s)|, |u'(t,s)|\}.$$

We shall suppose that the function F is continuous and has continuous first and second derivatives with respect to all of its arguments.

If the extremum of functional (1) is considered in the space $\mathbb{H}^1(Q, \tilde{R})$, then it is a weak extremum. If it is considered in the space $\mathbb{H}^0(Q, R)$, then it is a strong extremum.

Denote

$$\begin{aligned} &F(t, s, [u(t, s-j)], [u'(t, s-j)], j = -r_2, \dots, r_1) \\ &\quad := F[t, s, u(t, s-r_1), \dots, u(t,s) \dots, u(t, s+r_2), \\ &\qquad u'(t, s-r_1), \dots, u'(t,s), \dots, u'(t, s+r_2)]. \end{aligned}$$

It was proved in [1] that if the function $u(t, s)$ furnishes the functional (1) with an extremum, then there exists a function $C(s)$ such that $u(t, s)$ satisfies on $Q\tilde{R}$ the equation

$$\Phi_{u'(t,s)} = \int_{t_0}^{t} \Phi_{u(t,s)}\, \mathrm{d}t + C(s), \tag{4}$$

where

$$\begin{aligned} &\Phi(t, s, u(t, s-r_1-r_2), \dots, u(t,s), \dots, u(t, s+r_1+r_2), \\ &\quad u'(t, s-r_1-r_2), \dots, u'(t,s), \dots, u'(t, s+r_1+r_2)) \\ &\qquad := \sum_{i=-r_2}^{r_1} F(t, s+i, [u(t, s+i-j)], [u'(t, s+i-j)], \\ &\qquad\qquad j = -r_2, \dots, r_1). \end{aligned} \tag{5}$$

Equation (4) can be written in the differential form

$$\Phi_{u(t,s)} - \frac{\mathrm{d}}{\mathrm{d}t}\Phi_{u'(t,s)} = 0, \tag{6}$$

which is also satisfied by $u(t,s)$ on $Q\tilde{R}$, and is an analog of the Euler equation for the considered problem. Let

$$\begin{aligned}\Phi[t,s,\alpha,\beta] := \Phi(&t,s,u(t,s-r_1-r_2),\dots,u(t,s-1),\\ &\alpha,u(t,s+1),\dots,u(t,s+r_1+r_2),u'(t,s-r_1-r_2),\dots,\\ &u'(t,s-1),\beta,u'(t,s+1),\dots,u'(t,s+r_1+r_2)).\end{aligned}$$

The generalized function of Weierstrass will be called the function

$$\begin{aligned}E(t,s,[u(t,s)],[u'(t,s)],\eta) := {}&\Phi[t,s,u(t,s),\eta] - \Phi[t,s,u(t,s),u'(t,s)]\\ &- (\eta - u'(t,s))\Phi_{u'(t,s)}[t,s,u(t,s),u'(t,s)].\end{aligned} \tag{7}$$

For any admissible $u(t,s)$ the function $E(t,s,[u(t,s)],[u'(t,s)],\eta)$ is defined $Q\tilde{R}$.

THEOREM 1 *If the functional* (1) *attains on* $u(t,s)$ *a strong minimum, the* $u(t,s)$ *satisfies on* $Q\tilde{R}$ *the generalized Weierstrass condition*

$$E(t,s,[u(t,s)],[u'(t,s)],\eta) \geq 0 \tag{8}$$

for any $\eta \in R^1$.

Proof Let $(\bar{t},\bar{s}) \in Q\backslash\tilde{R}$. Take $a \in (t_0,\bar{t})$ and $h < \bar{t} - a$ such that the segments $[\bar{s}-h+j,\bar{s}+h+j], (j=-r_2,\dots,r_1)$ have empty intersections. These segments are also disjoint from $\tilde{R}$.

Denote

$$\Theta_h(t,s) = \begin{cases} -\dfrac{wh(t-a)}{\bar{t}-h-a} - w(s-\bar{s}), & (t,s)\in(I),\\ -\dfrac{wh(t-a)}{\bar{t}-h-a} + w(s-\bar{s}), & (t,s)\in(II),\\ w(t-\bar{t}) - w(s-\bar{s}), & (t,s)\in(III),\\ w(t-\bar{t}) + w(s-\bar{s}), & (t,s)\in(IV),\\ 0, & (t,s)\in Q\backslash D_h,\end{cases}$$

where

$$
\begin{aligned}
D_h &= (I) \cup (II) \cup (III) \cup (IV), \\
(I) &= \{(t,s) \mid s \in [\bar{s}, \bar{s}+h], \quad t \in [\Gamma_1(s,h), \bar{t}-h]\}, \\
(II) &= \{(t,s) \mid s \in [\bar{s}-h, \bar{s}], \quad t \in [\Gamma_2(s,h), \bar{t}-h]\}, \\
(III) &= \{(t,s) \mid s \in [\bar{s}-h, \bar{s}], \quad t \in [\bar{t}-h, \Gamma_3(s),]\}, \\
(IV) &= \{(t,s) \mid s \in [\bar{s}, \bar{s}+h], \quad t \in [\bar{t}-h, \Gamma_4(s),]\}.
\end{aligned}
$$

Here Γ_i $(i=1,\ldots,4)$ is a boundary of D_h:

$$
\begin{aligned}
\Gamma_1(t,s) &= (s-\bar{s})\frac{\bar{t}-h-a}{h} + a, \\
\Gamma_2(t,s) &= -(s-\bar{s})\frac{\bar{t}-h-a}{h} + a, \\
\Gamma_3(t,s) &= (s-\bar{s}) + \bar{t}, \\
\Gamma_4(t,s) &= -(s-\bar{s}) + \bar{t}.
\end{aligned}
$$

Thus the graph $\Theta_h(t,s)$ is a "pyramid" with base D_h and height wh, where $w \in R^1$ is an arbitrary number.

If $u(t,s)$ is an admissible function, then $u(t,s)+\Theta_h(t,s)$ is also an admissible function. Denote $\Psi(h) = J(u+\Theta_h) - J(u)$. Let A be a domain. Denote

$$(A+i) = \{(t,s) \mid (t, s-i) \in A\}.$$

Then

$$
\begin{aligned}
\Psi(h) = \sum_{i=-r_2}^{r_1} \Bigg\{ &\iint_{(D_h+i)} F(t,s,[u(t,s-j) + \Theta_h(t,s-i)\delta_i^j], \\
&[u'(t,s-j) + \Theta_h'(t,s-i)\delta_i^j], j=-r_2,\ldots,r_1)\,\mathrm{d}t\,\mathrm{d}s \\
&- \iint_{(D_h+i)} F(t,s,[u(t,s-j)],[u'(t,s-j)], j=-r_2,\ldots,r_1)\,\mathrm{d}t\,\mathrm{d}s \Bigg\}.
\end{aligned}
$$

Here δ_i^j is a Kronecker symbol, $\delta_i^j = 1$ if $i=j$, and $\delta_i^j = 0$ if $i \neq j$.

Changing the variable of integration $s = z + i$ and denoting again the variable of integration by s, we receive

$$\iint_{(D_h+i)} F(t, s, [u(t, s-j) + \Theta_h(t, s-i)\delta_i^j],$$

$$[u'(t, s-j) + \Theta_h'(t, s-i)\delta_i^j], j = -r_2, \dots, r_1)\,\mathrm{d}t\,\mathrm{d}s$$

$$= \iint_{D_h} F(t, s+i, [u(t, s+i-j) + \Theta_h(t, s)\delta_i^j],$$

$$[u'(t, s+i-j) + \Theta_h'(t, s)\delta_i^j], j = -r_2, \dots, r_1)\,\mathrm{d}t\,\mathrm{d}s,$$

$$\iint_{(D_h+i)} F(t, s, [u(t, s-j)], [u'(t, s-j)], j = -r_2, \dots, r_1)\,\mathrm{d}t\,\mathrm{d}s$$

$$= \iint_{D_h} F(t, s+i, [u(t, s+i-j)], [u'(t, s+i-j)],$$

$$j = -r_2, \dots, r_1)\,\mathrm{d}t\,\mathrm{d}s \quad (i = -r_2, \dots, r_1).$$

Thus

$$\Psi(h) = \iint_{D_h} \Phi[t, s, u(t, s) + \Theta_h(t, s), u'(t, s) + \Theta_h'(t, s)]\,\mathrm{d}t\,\mathrm{d}s$$

$$- \iint_{D_h} \Phi[t, s, u(t, s), u'(t, s)]\,\mathrm{d}t\,\mathrm{d}s.$$

It is easy to see that

$$\iint_{D_h} \Phi[t, s, u(t, s) + \Theta_h(t, s), u'(t, s) + \Theta_h'(t, s)]\,\mathrm{d}t\,\mathrm{d}s$$

$$= J_{(I)}(h) + J_{(II)}(h) + J_{(III)}(h) + J_{(IV)}(h)$$

and

$$\iint_{D_h} \Phi[t, s, u(t, s), u'(t, s)]\,\mathrm{d}t\,\mathrm{d}s$$

$$= \tilde{J}_{(I)}(h) + \tilde{J}_{(II)}(h) + \tilde{J}_{(III)}(h) + \tilde{J}_{(IV)}(h),$$

where

$$J_{(I)}(h) = \int_{\bar{s}}^{\bar{s}+h} \mathrm{d}s \int_{\Gamma_1(s,h)}^{\bar{t}-h} \Phi\left[t, s, u(t,s) - \frac{wh(t-a)}{\bar{t}-h-a} + w(s-\bar{s}), u'(t,s) - \frac{wh}{\bar{t}-h-a}\right] \mathrm{d}t,$$

$$J_{(II)}(h) = \int_{\bar{s}-h}^{\bar{s}} \mathrm{d}s \int_{\Gamma_2(s,h)}^{\bar{t}-h} \Phi\left[t, s, u(t,s) - \frac{wh(t-a)}{\bar{t}-h-a} - w(s-\bar{s}), u'(t,s) - \frac{wh}{\bar{t}-h-a}\right] \mathrm{d}t,$$

$$J_{(III)}(h) = \int_{\bar{s}-h}^{\bar{s}} \mathrm{d}s \int_{\bar{t}-h}^{\Gamma_3(s)} \Phi[t, s, u(t,s) + w(t-\bar{t}) - w(s-\bar{s}), u'(t,s) + w] \,\mathrm{d}t,$$

$$J_{(IV)}(h) = \int_{\bar{s}}^{\bar{s}+h} \mathrm{d}s \int_{\bar{t}-h}^{\Gamma_4(s)} \Phi[t, s, u(t,s) + w(t-\bar{t}) + w(s-\bar{s}), u'(t,s) + w] \,\mathrm{d}t.$$

The integrals $\tilde{J}_{(I)}(h), \ldots, \tilde{J}_{(IV)}(h)$ are defined in an analogous way. Then we have

$$\Psi'(h) = J'_{(I)}(h) + J'_{(II)}(h) + J'_{(III)}(h) + J'_{(IV)}(h) - \tilde{J}'_{(I)}(h) - \tilde{J}'_{(II)}(h) - \tilde{J}'_{(III)}(h) - \tilde{J}'_{(IV)}(h).$$

and

$$\begin{aligned}\Psi'(h) = 2h\Bigg\{ &\Phi[\bar{t}, \bar{s}, u(\bar{t}-h, \bar{s}) - wh, u'(\bar{t}-h, \bar{s}) + w] \\ &- \Phi\left[\bar{t}, \bar{s}, u(\bar{t}-h, \bar{s}) - wh, u'(\bar{t}-h, \bar{s}) - \frac{wh}{\bar{t}-h-a}\right] \\ &- \int_a^{\bar{t}-h} \Bigg[\Phi_{u(t,s)}\left[t, \bar{s}, u(t,\bar{s}) - \frac{wh(t-a)}{\bar{t}-h-a}, u'(t,\bar{s}) - \frac{wh}{\bar{t}-h-a}\right] \\ &\times \frac{w(\bar{t}-a)(t-a)}{(\bar{t}-h-a)^2} + \Phi_{u'(t,s)}\left[t, \bar{s}, u(t,\bar{s}) - \frac{wh(t-a)}{\bar{t}-h-a}, u'(t,\bar{s}) \right. \\ &\left. - \frac{wh}{\bar{t}-h-a}\right] \frac{w(\bar{t}-a)}{(\bar{t}-h-a)^2}\Bigg] \mathrm{d}t \Bigg\}.\end{aligned}$$

If the functional $J(u)$ attains on $u(t,s)$ a minimum in the space $\mathbb{H}^0(Q,R)$, then $\Psi(h)\geq 0$, and from $\Psi(0)=0$ it follows that $\Psi'(0+)\geq 0$. Thus $\Omega(h)=\Psi'(h)/2h\geq 0$ and therefore $\Omega(0+)\geq 0$.

$$\begin{aligned}\Omega(0+) &= \Phi[\bar{t},\bar{s},u(\bar{t},\bar{s}),u'(\bar{t},\bar{s})+w] - \Phi[\bar{t},\bar{s},u(\bar{t},\bar{s}),u'(\bar{t},\bar{s})] \\ &\quad - \int_a^{\bar{t}} \left\{ \Phi_{u(t,s)}[t,\bar{s},u(t,\bar{s}),u'(t,\bar{s})] \frac{w(t-a)}{\bar{t}-a} \right. \\ &\quad \left. + \Phi_{u'(t,s)}[t,\bar{s},u(t,\bar{s}),u'(t,\bar{s})]\frac{w}{\bar{t}-a} \right\} \mathrm{d}t.\end{aligned}$$

If the functional (1) attains on $u(t,s)$ a strong minimum, then it attains also a weak minimum and therefore $u(t,s)$ satisfies the generalized Euler equation (6). Then we have

$$\begin{aligned}&\int_a^{\bar{t}} \left\{ \Phi_{u(t,s)}[t,\bar{s},u(t,\bar{s}),u'(t,\bar{s})] \frac{w(t-a)}{\bar{t}-a} \right. \\ &\quad \left. + \Phi_{u'(t,s)}[t,\bar{s},u(t,\bar{s}),u'(t,\bar{s})]\frac{w}{\bar{t}-a} \right\} \mathrm{d}t \\ &= \frac{w}{\bar{t}-a}\int_a^{\bar{t}} \left\{ (t-a)\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{u'(t,s)}[t,\bar{s},u(t,\bar{s}),u'(t,\bar{s})] \right. \\ &\quad \left. + \Phi_{u'(t,s)}[t,\bar{s},u(t,\bar{s}),u'(t,\bar{s})] \right\} \mathrm{d}t \\ &= \frac{w}{\bar{t}-a}\int_a^{\bar{t}} \frac{\mathrm{d}}{\mathrm{d}t}\{(t-a)\Phi_{u'(t,s)}[t,\bar{s},u(t,\bar{s}),u'(t,\bar{s})]\}\,\mathrm{d}t \\ &= \frac{w}{\bar{t}-a}\{(t-a)\Phi_{u'(t,s)}[t,\bar{s},u(t,\bar{s}),u'(t,\bar{s})]\}\big|_a^{\bar{t}} \\ &= w\Phi_{u'(t,s)}[\bar{t},\bar{s},u(\bar{t},\bar{s},u(\bar{t},\bar{s}),u'(\bar{t},\bar{s})].\end{aligned}$$

Denoting $\eta = w + u'(\bar{t},\bar{s})$, we receive

$$\begin{aligned}\Omega(0+) &= \Phi[\bar{t},\bar{s},u(\bar{t},\bar{s}),\eta] - \Phi[\bar{t},\bar{s},u(\bar{t},\bar{s}),u'(\bar{t},\bar{s})] \\ &\quad - (\eta - u'(\bar{t},\bar{s}))\Phi_{u'(t,s)}[\bar{t},\bar{s},u(\bar{t},\bar{s}),u'(\bar{t},\bar{s})].\end{aligned}$$

Therefore while $\Omega(0+)\geq 0$ and $(\bar{t},\bar{s})$ is arbitrary point of $Q\backslash\tilde{R}$, we conclude that (8) holds everywhere on $Q\backslash\tilde{R}$.

DEFINITION *Define the function* $H(t, s, \eta) : R^3 \to R^1$ *for a given admissible function* $u(t, s)$ *by the equality*

$$H(t, s, \eta) := \Phi[t, s, u(t, s), \eta] - \eta\left\{ \int_{t_0}^{t} \Phi_{u(t,s)}[x, s, u(x, s), u'(x, s)]\,\mathrm{d}x + \Phi_{u(t,s)}[t_0, s, u(t_0, s), u'(t_0, s)] \right\} \tag{9}$$

for $(t, s) \in Q \backslash \tilde{R}$.

The function u satisfies the minimum principle if

$$\min_{\eta \in R^1} H(t, s, \eta) = H(t, s, u'(t, s)), \quad (t, s) \in Q \backslash \tilde{R}. \tag{10}$$

THEOREM 2 *The function u satisfies the minimum principle iff it satisfies the generalized Euler equation* (6) *and the Weierstrass inequality* (8).

Proof Let $(\bar{t}, \bar{s}) \in Q \backslash \tilde{R}$. The function $H(\bar{t}, \bar{s}, \eta)$ attains a minimum at $\eta = u'(\bar{t}, \bar{s})$ and therefore $\partial H/\partial \eta = 0$. By differentiating (9) and putting $u'(\bar{t}, \bar{s}) = \eta$, we show that u satisfies (4) and consequently (6) on $Q \backslash \tilde{R}$. By using (6) we may write $H(t, s, \eta)$ in the form

$$H(t, s, \eta) = \Phi[t, s, u(t, s), \eta] - \eta \Phi_{u'(t,s)}[t, s, u(t, s), u'(t, s)],$$

and accordingly

$$E(t, s, [u(t, s)], [u'(t, s)], \eta) = H(t, s, \eta) - H(t, s, u'(t, s)). \tag{11}$$

From (11) and the minimum principle (8) follows.

On the other hand, if u satisfies (6), then from (11) it follows that the minimum principle is valid on u.

THEOREM 3 *If the functional* (1) *attains on u a strong minimum, then u satisfies the minimum principle.*

Proof Since the functional (1) also attains on $u(t, s)$ a weak minimum, the function u satisfies (6), and from Theorems 1 and 2 the assertion of Theorem 3 follows.

Now we shall prove the generalized conditions of Weierstrass–Erdman at the corner points of the solutions of the problems, (1)–(3). It is natural to suppose that these corner points are isolated. Therefore in Theorems 4 and 5 the space $\tilde{D}$ of admissible functions will be the space of functions that are continuous with respect to t on $Q\backslash\tilde{R}$ and having two piecewise continuous derivatives with respect to t with the finite fixed set G of possible corner points.

THEOREM 4 *Let $u \in \tilde{D}$ satisfy the minimum principle. Then there exists a function $C_1(s)$, such that $u(t,s)$ satisfies the equation*

$$\Phi - u'\Phi_{u'(t,s)} = \int_{t_0}^{t} H_t(t,s,u'(t,s))\,\mathrm{d}t + C_1(s) \tag{12}$$

on $Q\backslash(\tilde{R} \cup G)$.

Proof Let $(\bar{t},\bar{s}) \in Q\backslash(\tilde{R} \cup G)$. Then for almost all sufficiently small $h > 0$ we can define the function

$$D(h) = H(\bar{t}+h,\bar{s},u'(\bar{t}+h,\bar{s})) - H(\bar{t},\bar{s},u'(\bar{t},\bar{s})).$$

The inequalities

$$\begin{aligned} H(\bar{t}+h,\bar{s},u'(\bar{t}+h,\bar{s})) - H(\bar{t},\bar{s},u'(\bar{t}+h,\bar{s})) &\leq D(h), \\ D(h) &\leq H(\bar{t}+h,\bar{s},u'(\bar{t},\bar{s})) - H(\bar{t},\bar{s},u'(\bar{t},\bar{s})) \end{aligned} \tag{13}$$

follow from the minimum principle. From the suppositions of the theorem it follows that in a sufficiently small vicinity of the point $(\bar{t},\bar{s})$, the function H has a partial derivative in respect to t. From (13) and the mean value theorem, we obtain

$$\begin{aligned} &H_t(\bar{t}+\Theta_1(h)h,\bar{s},u'(\bar{t}+h,\bar{s})) \\ &\quad\leq \frac{D(h)}{h} \leq H_t(\bar{t}+\Theta_2(h)h,\bar{s},u'(\bar{t},\bar{s})), \end{aligned}$$

where $0 < \Theta_i(h) < 1$, $(i = 1,2)$. Taking the limit as $h \to 0$ we receive

$$\frac{\mathrm{d}H(\bar{t},\bar{s},u'(\bar{t},\bar{s}))}{\mathrm{d}t} = H_t(\bar{t},\bar{s},u'(\bar{t},\bar{s})) \tag{14}$$

for $(\bar{t}, \bar{s}) \in Q\backslash(\tilde{R} \cup G)$. By integrating (14), we obtain

$$H(t, s, u'(t, s)) = \int_{t_0}^{t} H_t(t, s, u'(t, s))\,\mathrm{d}t + C_1(s) \tag{15}$$

for $(t, s) \in Q\backslash(\tilde{R} \cup G)$.

If the functional (1) attains an extremum on $u(t, s)$, then from (6) it follows that

$$\begin{aligned} H(t, s, u'(t, s)) = \Phi[t, s, u(t, s), u'(t, s)] \\ - u'(t, s)\Phi_{u'(t,s)}[t, s, u(t, s), u'(t, s)], \end{aligned} \tag{16}$$

for $(t, s) \in Q\backslash(\tilde{R} \cup G)$. From (15) and (16) we receive the assertion of the theorem.

THEOREM 5 *Let the functional* (1) *attain on u an extremum, and* $(\bar{t}, \bar{s})$ *be a corner point of* $u(t, s)$. *Then at this point the generalized Weierstrass–Erdman conditions*

$$\Phi_{u'(t,s)}|_{(\bar{t}-0,\bar{s})} = \Phi_{u'(t,s)}|_{(\bar{t}+0,\bar{s})}, \tag{17}$$

$$\Phi - u'\Phi_{u'(t,s)}|_{(\bar{t}-0,\bar{s})} = \Phi - u'\Phi_{u'(t,s)}|_{(\bar{t}+0,\bar{s})} \tag{18}$$

are fulfilled.

Proof It follows from (4) and (12) that functions $\Phi_{u'(t,s)}$ and $\Phi - u'\Phi_{u'(t,s)}$ are equal to indefinite integrals on $Q\backslash\tilde{R}$. Therefore, if we define additionally $\Phi_{u'(t,s)}$ and $\Phi - u'\Phi_{u'(t,s)}$ on the set of null measure, we can make them continuous with respect to t on the whole set $Q\backslash\tilde{R}$. Hence the conditions (17) and (18) are fulfilled.

References

[1] G.A. Kamenskii, Boundary value problems for differential-difference equations arising from variational problems. *Nonlinear Analysis, TMA*, **18**(8) (1992), 801–813.

[2] G.A. Kamenskii, On some necessary conditions of extrema of functionals with deviating argument. *Nonlinear Analysis, TMA*, **17**(5) (1991), 457–464.

[3] G.A. Kamenskii, A review of the theory of mixed functional-difference equations. *Problems of Nonlinear Analysis in Engineering Systems*, **2**(8) (1998), 1–16.