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Saddle Point Theorems on Generalized Convex Spaces

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A sharpened extension of Komiya's saddle point theorem is obtained for generalized convex spaces. Moreover, we show that convexity of the involved sets in his theorem can be replaced by acyclicity, and continuity of the involved functions by lower and upper semicontinuity.

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1. INTRODUCTION

The numerous applications and generalizations of John von Neumann's classical minimax theorem [4] constitute an important branch of modern convex analysis. One of the main purposes of these generalizations was to eliminate the underlying convexity structure from the original hypothesis.

On the other hand, the convexity of subsets of topological vector spaces was extended to convex spaces by Lassonde, to *C*-spaces (or *H*-spaces) by Horvath, and to *G*-convex spaces (or generalized convex spaces) by

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the second author; for the literature, see [6–9]. It is known that the KKM theory, fixed point theory, and other equilibrium results are now well-developed in these abstract convexities.

In the present paper, from a coincidence theorem due to Park and H. Kim [7, Theorem 1], we deduce saddle point theorems on G-convex spaces. The coincidence theorem is a far-reaching generalized form of the Fan-Browder fixed point theorem [1, Theorem 1] and Browder's coincidence theorem [1, Theorem 7]. This was used by Komiya [3] to obtain a saddle point theorem. We show that Komiya's theorem can be sharpened in several aspects; namely, under less restrictive hypothesis we can obtain the same conclusion for a generalized convex space.

2. PRELIMINARIES

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\Phi_A : \Delta_n \to X$ such that $\Phi_A(\Delta_J) \subset \Gamma(J)$ for every $J \in \langle A \rangle$. Note that $\Phi_A|_{\Delta_J}$ can be regarded as Φ_J.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n the standard n-simplex, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$. We write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X; \Gamma) = (X, X; \Gamma)$. A subset K of X is said to be Γ-*convex* if for each $A \in \langle D \rangle$, $A \subset K$ implies $\Gamma_A \subset K$. For details on G-convex spaces, see [5–9], where basic theory was extensively developed.

Major examples of other G-convex spaces than convex spaces or H-spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joo's pseudoconvex spaces, topological semilattices with path connected intervals, and so on. For the literature, see [6–8].

Recently, the second author [5] gave new examples of G-convex spaces and, simultaneously, showed that some abstract convexities of other authors are simple particular examples of our G-convexity. Such examples are L-spaces of Ben-El-Mechaiekh *et al.*, continuous images of G-convex spaces, Verma's generalized H-spaces, Kulpa's simplicial structures, $P_{1,1}$-spaces of Forgo and Joó, generalized H-spaces of

Stachó, and Llinares's *mc*-spaces. Further examples of L-spaces are spaces with B'-simplicial convexity, hyperconvex metric spaces due to Aronszajn and Panitchpakdi, and Takahashi's convexity in metric spaces. For the literature, see [5].

A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. For topological spaces X and Y a multimap $T: X \multimap Y$ is called an *acyclic map* if it is upper semicontinuous with compact acyclic values.

The following coincidence theorem is a particular form of Park and H. Kim [7, Theorem 1]:

THEOREM 1 *Let X be a Hausdorff compact topological space, $(Y, D; \Gamma)$ a G-convex space, and $F: X \multimap Y$, $G: X \multimap D$ multimaps and $T: Y \multimap X$ an acyclic map such that*

(1) *for each $x \in X$, $A \in \langle Gx \rangle$ implies $\Gamma_A \subset Fx$;*
(2) *$X = \bigcup\{Int_X G^- y: y \in D\}$, where $Int_X A$ denotes the interior of A in X.*

Then there exist points $x_0 \in X$ and $y_0 \in Y$ such that $y_0 \in Fx_0$ and $x_0 \in Ty_0$.

Note that if $Y = D$ is a convex space, $\Gamma = \mathrm{co}$, the convex hull, and T has convex values, then Theorem 1 reduces to Browder [1, Theorem 7]; and further if $X = Y$ and $T = id_X$, the identity map of X, then Theorem 1 reduces to a generalized form of the Fan–Browder fixed point theorem [1, Theorem 1].

3. THE SADDLE POINT THEOREM

We begin with the following lemma:

LEMMA *Let X and Y be topological spaces, $f: X \times Y \to \mathbb{R}$ a real function on the product space $X \times Y$, and*

$$h(x) := \inf_{y \in Y} f(x, y) \quad and \quad Fx := \{y \in Y: \ f(x, y) = h(x)\} \quad for \ x \in X.$$

(1) *If $f(\cdot, y)$ is upper semicontinuous on X for each $y \in Y$ and $\inf_{y \in Y} f(x, y)$ exists for each $x \in X$, then $h: X \to \mathbb{R}$ is upper semicontinuous.*
(2) *If $f(\cdot, y)$ is upper semicontinuous on X for each $y \in Y$ and f is lower semicontinuous on $X \times Y$ and if Y is a Hausdorff compact space, then $F: X \multimap Y$ is upper semicontinuous.*

(3) *If* $f(x,\cdot)$ *is lower semicontinuous on* Y *for each* $x \in X$ *and* $g(y) := \sup_{x\in X} f(x,y)$ *exists for each* $y \in Y$, *then* $g: Y \to \mathbb{R}$ *is lower semicontinuous.*

Proof (1) Let $x_0 \in X$ and $r \in \mathbb{R}$ such that $h(x_0) < r$. Then there is a point $y_0 \in Y$ such that $f(x_0, y_0) < r$. Since $f(\cdot, y_0)$ is upper semicontinuous on X, there exists a neighborhood U of x_0 in X such that $f(x, y_0) < r$ for all $x \in U$ and hence $h(x) \leq f(x, y_0) < r$ for all $x \in U$. Thus h is upper semicontinuous.

(2) Since $f(x,\cdot)$ is lower semicontinuous on the compact set Y, $\inf_{y\in Y} f(x,y)$ exists for each $x \in X$. We first claim that F has a closed graph. Let (x_α, y_α) be a net in the graph $\mathrm{Gr}(F)$ of F such that $(x_\alpha, y_\alpha) \to (x_0, y_0)$. Since f is lower semicontinuous on $X \times Y$, $(x_\alpha, y_\alpha) \in \mathrm{Gr}(F)$, and h is upper semicontinuous, we have

$$\begin{aligned} f(x_0,y_0) &\leq \liminf_\alpha f(x_\alpha, y_\alpha) \leq \limsup_\alpha h(x_\alpha) \\ &\leq h(x_0) \leq f(x_0, y_0) \end{aligned}$$

and hence $f(x_0,y_0) = h(x_0)$; that is $(x_0,y_0) \in \mathrm{Gr}(F)$. Thus F has closed graph. Since Y is compact, it is well known that F is upper semicontinuous.

(3) Let $y_0 \in Y$ and $r \in \mathbb{R}$ such that $g(y_0) > r$. Then there is a point $x_0 \in X$ such that $f(x_0, y_0) > r$. Since $f(x_0, \cdot)$ is lower semicontinuous on Y, there exists a neighborhood V of y_0 in Y such that $f(x_0, y) > r$ for all $y \in V$ and hence $g(y) \geq f(x_0, y) > r$ for all $y \in V$. Hence $g: Y \to \mathbb{R}$ is lower semicontinuous. This completes the proof.

Motivated by [3], we obtain the following:

THEOREM 2 *Let* $(X;\Gamma)$ *be a G-convex space and* Y *a Hausdorff compact space. Let* $f: X \times Y \to \mathbb{R}$ *be a lower semicontinuous real function such that*

(1) *for each* $y \in Y$, $\sup_{x\in X} f(x,y)$ *exists*;
(2) *for each* $y \in Y$, $f(\cdot, y)$ *is upper semicontinuous on* X;
(3) *for each* $y \in Y$ *and* $t \in \mathbb{R}$, *the set* $\{x \in X: f(x,y) > t\}$ *is* Γ*-convex*;
(4) *for each* $x \in X$, $f(x,\cdot) - \sup_{x\in X} f(x,\cdot)$ *is lower semicontinuous on* Y;
(5) *for each* $x \in X$, *the set* $\{y \in Y: f(x,y) = \min_{y\in Y} f(x,y)\}$ *is acyclic*; *and*

(6) *for each sequence* $\{x_k\}_{k\in\mathbb{N}}$ *in* X, *there exist a subsequence* $\{x_{k_n}\}_{n\in\mathbb{N}}$ *of* $\{x_k\}_{k\in\mathbb{N}}$ *and a point* $\bar{x}\in X$ *such that*

$$f(\bar{x},y)\geq \limsup_{n\to\infty} f(x_{k_n},y) \quad \textit{for all } y\in Y.$$

Then f has a saddle point $(x_0,y_0)\in X\times Y$; *that is,*

$$\max_{x\in X} f(x,y_0)=f(x_0,y_0)=\min_{y\in Y} f(x_0,y).$$

Proof I. Define a multimap $A: X \multimap Y$ by

$$Ax:=\{y\in Y:\ f(x,y)=\min_{y\in Y} f(x,y)\}.$$

Since Y is compact, A is upper semicontinuous by (2) and Lemma, and A has closed acyclic values by (5).

Define a function $g: Y\to\mathbb{R}$ by

$$g(y):=\sup_{x\in X} f(x,y).$$

Then g is lower semicontinuous by (1) and Lemma because $f(x,\cdot)$ is lower semicontinuous on Y for each $x\in X$.

For any $k\in\mathbb{N}$, let a multimap $B_k: Y\multimap X$ be defined by

$$B_k y:=\{x\in X: f(x,y)>g(y)-1/k\}.$$

Then B_k has Γ-convex values by (3) and open fibers since $f(x,\cdot)-g(\cdot)$ is lower semicontinuous by (4). By Theorem 1, there exist points $x_k\in X$ and $y_k\in Y$ such that $y_k\in Ax_k$ and $x_k\in B_k y_k$. Hence we have

$$f(x_k,y)\geq f(x_k,y_k)>g(y_k)-1/k \quad \text{for all } k\in\mathbb{N} \text{ and all } y\in Y.$$

II. By (6), there exist a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$ and a point $x_0\in X$ such that

$$f(x_0,y)\geq \limsup_{n\to\infty} f(x_{k_n},y) \quad \text{for all } y\in Y.$$

Since Y is compact there are a subnet $\{y_\alpha\}$ of $\{y_{k_n}\}_{n\in\mathbb{N}}$ and a point $y_0\in Y$ such that $y_\alpha\to y_0$. Since $\min_{y\in Y} f(x_0,y)=f(x_0,\bar{y})$ for some $\bar{y}\in Y$ and

$f(x_\alpha, \bar{y}) > g(y_\alpha) - 1/\alpha$ for all α, and g is lower semicontinuous, it follows that

$$\begin{aligned} f(x_0, y_0) & \\ &\geq \min_{y\in Y} f(x_0, y) = f(x_0, \bar{y}) \geq \limsup_{n\to\infty} f(x_{k_n}, \bar{y}) \geq \limsup_{\alpha} f(x_\alpha, \bar{y}) \\ &\geq \liminf_{\alpha} (g(y_\alpha) - 1/\alpha) \geq g(y_0) \geq f(x_0, y_0) \end{aligned}$$

and hence $f(x_0, y_0) = \min_{y\in Y} f(x_0, y) = g(y_0) = \sup_{x\in X} f(x, y_0)$. This completes the proof.

The following saddle point theorem is a generalization of [3, Theorem 3] to G-convex spaces.

THEOREM 3 *Let $(X;\Gamma)$ be a G-convex space and Y a Hausdorff compact connected space. Let $f: X \times Y \to \mathbb{R}$ be a lower semicontinuous real function and* $\min_{y\in Y}\sup_{x\in X} f(x, y) < +\infty$ *such that*

(1) *for each $y \in Y, f(\cdot, y)$ is upper semicontinuous on X;*
(2) *for each $y \in Y$ and $t \in \mathbb{R}$, the set $\{x \in X: f(x, y) > t\}$ is Γ-convex;*
(3) *for each $x \in X$, the set $\{y \in Y: f(x, y) = \min_{y\in Y} f(x, y)\}$ is acyclic; and*
(4) *$\{f(x, \cdot): x \in X\}$ is equicontinuous and closed in $C(Y)$, where $C(Y)$ is the Banach space of all continuous real functions defined on Y equipped with the supremum norm.*

Then f has a saddle point (x_0, y_0) in $X \times Y$.

Proof I. Since $\{f(x, \cdot): x \in X\}$ is equicontinuous, Y is compact and connected, and $\min_{y\in Y}\sup_{x\in X} f(x, y) < +\infty$, it follows that f is bounded from above (see the proof of [3, Theorem 3]); that is, there is a real number M such that

$$f(x, y) \leq M \quad \text{for all } (x, y) \in X \times Y.$$

II. Define a multimap $A: X \multimap Y$ by

$$Ax := \{y \in Y: \ f(x, y) = \min_{y\in Y} f(x, y)\}.$$

Then A is upper semicontinuous by (1) and Lemma, and A has closed acyclic values by (3). A function $g: Y \to \mathbb{R}$ defined by

$$g(y) := \sup_{x\in X} f(x, y)$$

is continuous by the equicontinuity of $\{f(x,\cdot)\colon x\in X\}$. For any $k\in\mathbb{N}$, let a multimap $B_k\colon Y \multimap X$ be defined by

$$B_k y := \{x\in X\colon\ f(x,y) > g(y) - 1/k\}.$$

Then B_k has Γ-convex values by (2) and open fibers since $f(x,\cdot)-g(\cdot)$ is continuous. By Theorem 1, there exist points $x_k\in X$ and $y_k\in Y$ such that $y_k\in Ax_k$ and $x_k\in B_k y_k$. Hence we have

$$M\geq f(x_k,y)\geq f(x_k,y_k) > g(y_k)-1/k \quad \text{for all } k\in\mathbb{N} \text{ and all } y\in Y.$$

Since g is continuous on the compact set Y, $\{g(y_k)-1/k\colon k\in\mathbb{N}\}$ is bounded (from below). Therefore, $\{f(x_k,\cdot)\colon k\in\mathbb{N}\}$ is bounded in $C(Y)$.

III. Since $\{f(x_k,\cdot)\colon k\in\mathbb{N}\}$ is equicontinuous and bounded in $C(Y)$, by the Arzelà–Ascoli Theorem, $\{f(x_k,\cdot)\colon k\in\mathbb{N}\}$ is relatively compact in $C(Y)$. We may suppose that $\{f(x_k,\cdot)\}_{k\in\mathbb{N}}$ converges uniformly to $f(x_0,\cdot)$ for some $x_0\in X$ and $\{y_k\}_{k\in\mathbb{N}}$ converges to a point $y_0\in Y$ (because $\{f(x,\cdot)\colon x\in X\}$ is closed in $C(Y)$ and Y is compact).

Since $\min_{y\in Y} f(x_0,y)=f(x_0,\bar{y})$ for some $\bar{y}\in Y$ and the inequality $f(x_k,\bar{y}) > g(y_k)-1/k$ for all $k\in\mathbb{N}$, it follows that

$$\begin{aligned} f(x_0,y_0) &\geq \min_{y\in Y} f(x_0,y) = f(x_0,\bar{y}) = \lim_{k\to\infty} f(x_k,\bar{y}) \\ &\geq \lim_{k\to\infty} (g(y_k)-1/k) = g(y_0)\geq f(x_0,y_0). \end{aligned}$$

Therefore, (x_0,y_0) is a saddle point of f. This completes the proof.

COROLLARY *Let X be a convex space and Y a Hausdorff compact connected space. Let $f\colon X\times Y\to\mathbb{R}$ be a continuous real function which is quasiconcave in its first variable and quasiconvex in its second variable and satisfies* $\min_{y\in Y}\sup_{x\in X} f(x,y) < +\infty$. *Let the family $\{f(x,\cdot)\colon x\in X\}$ be equicontinuous and closed in $C(Y)$. Then f has a saddle point in $X\times Y$.*

Note that Theorem 3 generalizes the result [3, Theorem 3] for convex sets in topological vector spaces in several points of view. Here, convexity of the involved sets can be replaced by Γ-convexity or acyclicity, and continuity of the involved functions by upper semicontinuity in some variable or lower semicontinuity. Moreover, Corollary is a special case of Theorem 3 for convex spaces due to Lassonde.

If condition (4) of Theorem 3 is replaced by

(4′) $\{f(x,\cdot): x \in X\} \subset C(Y)$ is sequentially compact,

then Theorem 3 can be deduced from Theorem 2.

THEOREM 4 *Under the hypotheses of Theorem* 2, *we have the minimax theorem*

$$\min_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\min_{y\in Y} f(x,y).$$

Proof Theorem 2 implies that there exists a point $(x_0, y_0) \in X \times Y$ such that

$$\sup_{x\in X} f(x,y_0) = f(x_0,y_0) = \min_{y\in Y} f(x_0,y).$$

As in the proof of Theorem 2, $\sup_{x\in X} f(x,\cdot)$ is lower semicontinuous on the compact set Y and $\min_{y\in Y} f(\cdot,y)$ is upper semicontinuous on X, and hence we conclude that

$$\min_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X} f(x,y_0) = \min_{y\in Y} f(x_0,y) \le \sup_{x\in X}\min_{y\in Y} f(x,y).$$

The inequality $\min_{y\in Y}\sup_{x\in X} f(x,y) \ge \sup_{x\in X}\min_{y\in Y} f(x,y)$ is obvious. This completes the proof.

We have seen that the minimax theorem can be deduced from the saddle point theorem. For minimax theorems on convex sets in topological vector spaces, see [2, Theorem 4].

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