On an Inequality of Kolmogorov Type for a Second-order Difference Expression

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In this paper we discuss an inequality of Kolmogorov type for the square of a second-order formally symmetric difference expression in the *limit-point* case. A connection between the existence of the inequality and the Hellinger–Nevanlinna $m(\lambda)$ function associated with the difference expression is established and it is shown that the best constant in the inequality is determined by the behaviour of the m-function. Analytical and computational results are obtained for specific classes of problems. Also necessary and sufficient conditions for the powers of the difference expression to be *partially separated* are given.

Keywords: Formally symmetric difference expressions; Kolmogorov type inequalities; Hellinger–Nevanlinna m-function; Limit-point condition; Partial separation

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1. INTRODUCTION

Landau [18] and Hadamard [12] established the inequality

$$\|f'\|^2 \le 4\|f\| \, \|f''\| \tag{1.1}$$

in the $L^\infty[0, \infty)$ setting, and showed that 4 is the best possible constant. Analogues of (1.1) in $L^2[0, \infty)$ and $L^2(-\infty, \infty)$ were given by Hardy and

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Littlewood [13]. Kolmogorov [17] extended (1.1) to inequalities of the form

$$\|f^{(m)}\| \le K_{n,m}\|f\|^{(n-m)/n}\|f^{(n)}\|^{m/n} \tag{1.2}$$

where $\|f\|$ is the $L^\infty(-\infty, \infty)$ norm and the best possible constants are given explicitly. Everitt [10] extended the Hardy–Littlewood inequalities to a general class of inequalities in $L_w^2(a, b)$ which involve the second-order formally symmetric differential expression

$$Nf := \frac{1}{w}[-(pf')' + qf], \tag{1.3}$$

namely,

$$\left(\int_a^b (p|f'|^2 + q|f|^2)\ \mathrm{d}x\right)^2 \le K \int_a^b |f|^2 w\ \mathrm{d}x \int_a^b |Nf|^2 w\ \mathrm{d}x; \tag{1.4}$$

these are generally called the HELP inequalities after Hardy, Everitt, Littlewood and Polya. In (1.3) p, q and w are real-valued functions on $[a, b)\ (-\infty < a < b \le \infty)$ and are assumed to satisfy minimal smoothness conditions in order for the equation

$$Nf = \lambda f, \quad \lambda \in \mathbb{C} \tag{1.5}$$

to be regular at a and also to satisfy the so-called *strong limit-point* condition at the singular end point b. The inequality (1.4) is required to hold on a domain of functions for which the right-hand side is defined and finite. Using the calculus of variations, Everitt proved that the existence of the inequality and the value of the best constant depend explicitly on the behaviour of the Titchmarsh–Weyl m-function for (1.5). Evans and Zettl [9] gave an alternative proof of Everitt's result by using the theory of linear operators in a Hilbert space.

The validity of Kolmogorov type inequalities for norms of powers of linear operators acting in a Hilbert space has been established by Ljubič [19]. Also, Phóng [21] gives necessary and sufficient conditions for the validity of the inequalities

$$\|N^m f\| \le C_{n,m}\|f\|^{(n-m)/n}\|N^n f\|^{m/n}, \quad n > m, \ n \text{ and } m \text{ integers}, \tag{1.6}$$

in which $\|\cdot\|$ is the L_w^2 norm over the interval $[0, \infty)$ and Nf is given by (1.3). Beynon [4] considers the inequality (1.6) in the special case when n is even and $m = n/2$;

$$\|N^{n/2}f\|^2 \leq K_n\|f\|\|N^nf\|, \quad n = 2, 4, 6, \ldots, \qquad (1.7)$$

on the maximal domain for which the right-hand side of (1.7) is finite in the $L_w^2(a, b)$ setting. He gives a criterion for a valid inequality in (1.7) in terms of the Titchmarsh–Weyl m-function, which is suitable for computational techniques. By using the von Neumann formula for the maximal operator generated by the differential expression Nf, he reduces the criterion to the positive definiteness of a certain real and symmetric $2n \times 2n$ matrix which involves the $m(\lambda)$ function and the spectral parameter λ in (1.5) with $\lambda \in \mathbb{C}\backslash\mathbb{R}$. He also gives a numerical method for computing the value of the best constant in the inequality, and applies it to certain different cases of the coefficients p, q and w for $n = 2, 4, 6$. The case $n = 2$,

$$\|Nf\|^2 \leq K\|f\|\|N^2f\|, \qquad (1.8)$$

with the $L_w^2[0, \infty)$ norm is discussed by Beynon *et al.* [5].

The discrete analogue of the inequality (1.1) in $\ell^2(0, \infty)$ was proved by Copson [8]: if $\{x_n\}_0^\infty$ is a sequence of real numbers such that $\sum_{n=0}^\infty x_n^2$ and $\sum_{n=0}^\infty (\Delta^2 x_n)^2$ are convergent, where $\Delta x_n = x_{n+1} - x_n$ and $\Delta^2 x_n = \Delta(\Delta x_n)$, then $\sum_{n=0}^\infty (\Delta x_n)^2$ is convergent and the inequality

$$\left(\sum_{n=0}^\infty (\Delta x_n)^2\right)^2 \leq 4 \sum_{n=0}^\infty x_n^2 \sum_{n=0}^\infty (\Delta^2 x_n)^2 \qquad (1.9)$$

holds with the constant 4 being best possible. Brown and Evans [6] extended Copson's inequality to the discrete HELP inequalities:

$$\left(\sum_{n=0}^\infty (p_n|\Delta x_n|^2 + q_n|x_n|^2) + p_{-1}|\Delta x_{-1}|^2\right)^2 \leq K \sum_{n=-1}^\infty |x_n|^2 w_n \sum_{n=0}^\infty |Mx_n|^2 w_n$$

$$(1.10)$$

where

$$Mx_n := \begin{cases} \frac{1}{w_n}[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n], & n \geq 0, \\ -\frac{p_{-1}}{w_{-1}}\Delta x_n, & n = -1, \end{cases} \tag{1.11}$$

where $\{p_n\}_{-1}^{\infty}$, $\{q_n\}_0^{\infty}$ and $\{w_n\}_{-1}^{\infty}$ are real valued with

$$p_n \neq 0 \quad \text{and} \quad w_n > 0 \qquad \forall n = -1, 0, 1, \ldots \tag{1.12}$$

The inequality is required to hold on the domain

$$D := \left\{ x = \{x_n\}_{-1}^{\infty}: \ x \in \ell_w^2 \text{ and } \sum_{n=0}^{\infty} |Mx_n|^2 w_n < \infty \right\} \tag{1.13}$$

and ℓ_w^2 is the Hilbert space of sequences $x = \{x_n\}_{-1}^{\infty}$ such that $\sum_{n=-1}^{\infty} |x_n|^2 w_n < \infty$. The criterion they obtain for the validity of (1.10) has a similar form to that of Everitt for (1.4), with the Titchmarsh–Weyl m-function now replaced by the Hellinger–Nevanlinna m-function for the difference equation

$$Mx_n = \lambda x_n, \quad n \geq 0, \quad \lambda \in \mathbb{C} \tag{1.14}$$

introduced simultaneously by Hellinger [14] and Nevanlinna [20]. For a full discussion of the construction and the properties of Hellinger–Nevanlinna m-function, Akhiezer [1, Chapter 1] and Atkinson [2, Chapter 5] may be consulted. Brown et al. [7] examined the inequality (1.10) when (1.14) is the recurrence relation for the classical orthogonal polynomials.

It is the discrete analogue of (1.8) that is going to be our main concern in this paper:

$$\sum_{n=0}^{\infty} |Mx_n|^2 w_n \leq K \left(\sum_{n=-1}^{\infty} |x_n|^2 w_n \right)^{1/2} \left(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n \right)^{1/2} \tag{1.15}$$

where Mx is defined by (1.11) and x belongs to a domain for which both right- and left-hand sides of (1.15) are finite. Our objective has been to

obtain a criterion which explicitly involves the Hellinger–Nevanlinna m-function for (1.14), and then to use the computational approach of Beynon *et al.* [5] for providing estimates of the best constant, and to give asymptotic results, whenever possible, to support the numerical results. In Section 2 we give the abstract results, while in Section 3 we give necessary and sufficient conditions for M^2 to be partially separated, the main theorem and the cases of equality. Finally in Section 4 we give some examples, including a norm inequality, and give estimates for the best constants together with the cases of equality.

2. PRELIMINARIES

Let M be defined by (1.11), namely

$$Mx_n := \begin{cases} \frac{1}{w_n}[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n], & n \geq 0, \\ -\frac{p_{-1}}{w_{-1}}\Delta x_n, & n = -1, \end{cases} \tag{2.1}$$

where $\Delta x_n = x_{n+1} - x_n$, and $\{p_n\}_{-1}^{\infty}$, $\{q_n\}_0^{\infty}$ and $\{w_n\}_{-1}^{\infty}$ are real valued with

$$p_n \neq 0 \quad \text{and} \quad w_n > 0 \quad \forall n = -1, 0, 1, \ldots \tag{2.2}$$

Also define the expression M^2 as

$$M^2 x_n := \begin{cases} \frac{1}{w_n}[-\Delta(p_{n-1}\Delta Mx_{n-1}) + q_n Mx_n], & n \geq 1, \\ \frac{1}{w_0}[-\Delta(p_{-1}\Delta Mx_{-1}) + q_0 Mx_0], & n = 0, \\ -\frac{p_{-1}}{w_{-1}}\Delta Mx_{-1}, & n = -1. \end{cases} \tag{2.3}$$

Let ℓ_w^2 denote the weighted ℓ^2 Hilbert space with the inner-product and norm

$$(x, y) = \sum_{n=-1}^{\infty} x_n \bar{y}_n w_n, \qquad \|x\| = (x, x)^{1/2}. \tag{2.4}$$

We denote by $\mathbb{Z}$ the set of all integers and $\mathbb{N}_0$ the set of all non-negative integers. By using the summation by parts formula, for $k \in \mathbb{N}_0$, $m \geq k$,

we have

$$\sum_{n=k}^{m} \overline{x_n} M y_n w_n = \sum_{n=k}^{m} (q_n y_n \overline{x_n} + p_n \Delta y_n \overline{\Delta x_n})$$
$$- p_m \Delta y_m \overline{x_{m+1}} + p_{k-1} \Delta y_{k-1} \overline{x_k}, \qquad (2.5)$$

$$\sum_{n=k}^{m} (\overline{x_n} M y_n - y_n \overline{M x_n}) w_n = p_m (y_m \overline{x_{m+1}} - \overline{x_m} y_{m+1})$$
$$- p_{k-1} (y_{k-1} \overline{x_k} - \overline{x_{k-1}} y_k). \qquad (2.6)$$

Also,

$$\sum_{n=k}^{m} (\overline{x_n} M^2 y_n - y_n \overline{M^2 x_n}) w_n$$
$$= p_m (\overline{x_{m+1}} M y_m - M y_{m+1} \overline{x_m} + \overline{M x_{m+1}} y_m - y_{m+1} \overline{M x_m})$$
$$- p_{k-1} (\overline{x_k} M y_{k-1} - M y_k \overline{x_{k-1}} + \overline{M x_k} y_{k-1} - y_k \overline{M x_{k-1}}). \qquad (2.7)$$

The identities (2.6) and (2.7) are the Green's formulas for M and M^2 respectively. We denote by $T_0'(M)$, $T_0(M)$, $T(M)$ the restriction of M respectively to the subspaces

$$D_{T_0'(M)} := \{x \in \ell_w^2 : x_{-1} = 0 \text{ and } x_n = 0$$
$$\text{for all but a finite number of values of } n\}, \qquad (2.8)$$

$$D_{T_0(M)} := \overline{D_{T_0'(M)}}, \text{ the closure in } \ell_w^2, \qquad (2.9)$$

$$D_{T(M)} := \left\{ x = \{x_n\}_{-1}^{\infty} \in \ell_w^2 : \sum_{n=0}^{\infty} |M x_n|^2 w_n < \infty \right\}. \qquad (2.10)$$

We will also denote by $T_0'(M^2)$, $T_0(M^2)$, $T(M^2)$ the restriction of M^2 to the subspaces

$$D_{T_0'(M^2)} := \{x \in \ell_w^2 : x_{-1} = x_0 = 0 \text{ and } x_n = 0$$
$$\text{for all but a finite number of values of } n\}, \qquad (2.11)$$

$$D_{T_0(M^2)} := \overline{D_{T_0'(M^2)}}, \qquad (2.12)$$

$$D_{T(M^2)} := \left\{ x = \{x_n\}_{-1}^{\infty} \in \ell_w^2 : \sum_{n=1}^{\infty} |M^2 x_n|^2 w_n < \infty \right\}. \qquad (2.13)$$

The expression M will be assumed to be in the *limit-point* (LP) case, i.e. for $\Im(\lambda) \neq 0$, there is a unique ℓ_w^2 (up to constant multiples) solution of

$$Mx_n = \lambda x_n, \quad n \geq 0. \tag{2.14}$$

Equivalently this means that [15, p. 435]

$$\lim_{m \to \infty} p_m(y_m \overline{x_{m+1}} - \overline{x_m} y_{m+1}) = 0, \quad \forall x, y \in D_{T(M)}. \tag{2.15}$$

If M is not limit-point, then it is said to be *limit-circle* (LC).

DEFINITION 2.1 We say that M^2 is *limit-point* (LP) if there exist precisely 2 linearly independent ℓ_w^2 solutions of

$$M^2 x_n = \lambda^2 x_n, \quad n \geq 1, \; \lambda \in \mathbb{C} \tag{2.16}$$

for $\lambda^2 \notin \mathbb{R}$.

Similar to (2.15) above, it can be shown that M^2 is LP if and only if

$$\lim_{m \to \infty} p_m(\overline{x_{m+1}} My_m - My_{m+1}\overline{x_m} + \overline{Mx_{m+1}}y_m - y_{m+1}\overline{Mx_m}) = 0,$$
$$\forall x, y \in D_{T(M^2)}. \tag{2.17}$$

DEFINITION 2.2 We say that M^n $(n \geq 2)$ is partially separated (PS) if $x, M^n x \in \ell_w^2$ imply $M^r x \in \ell_w^2$ for all $r = 1, 2, 3, \ldots, n-1$.

We start by giving necessary and sufficient conditions for M^2 to be partially separated in the following theorem, which is the analogue of Theorem 3.5 in [16, p. 66] for differential operators and which relates the deficiency indices of M and M^2 with partial separation. Defining our operators as above gives rise to a problem: the minimal operator does not have a dense domain, and hence an adjoint cannot be defined. So, we will define a new expression and new operators:

$$M_1 x_n := \begin{cases} \frac{1}{w_n}[-p_n x_{n+1} + (p_n + p_{n-1} + q_n)x_n - p_{n-1}x_{n-1}], & n \geq 2, \\[2mm] \frac{1}{w_1}[-p_1 x_2 + (p_1 + p_0 + q_1)x_1], & n = 1, \end{cases}$$
$$\tag{2.18}$$

where $\{p_n\}_{-1}^{\infty}, \{q_n\}_0^{\infty}, \{w_n\}_{-1}^{\infty}$ are real with $p_n \neq 0$, $w_n > 0$. Note that M_1 is defined to be M for $n \geq 1$ with $x_{-1} = x_0 = 0$ inserted.

We define the Hilbert space

$$\ell_{w,1}^2 := \left\{ x = \{x_n\}_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^2 w_n < \infty \right\}, \qquad (2.19)$$

with the inner-product and norm

$$(x, y)_1 := \sum_{n=1}^{\infty} x_n \overline{y_n} w_n, \qquad \|x\|_1 := \left(\sum_{n=1}^{\infty} |x_n|^2 w_n \right)^{1/2}. \qquad (2.20)$$

The maximal and the pre-minimal operators generated by M_1 are defined as M_1 restricted to

$$D_{T(M_1)} := \left\{ x \in \ell_{w,1}^2 : \sum_{n=1}^{\infty} |M_1 x_n|^2 w_n < \infty \right\},$$

$$D_{T_0'(M_1)} := \{ x \in D_{T(M_1)} : x_m = 0$$
$$\text{for all but a finite number of values of } m \},$$

respectively. The minimal operator is $T_0(M_1) := \overline{T_0'(M_1)}$ in $\ell_{w,1}^2$. Similarly, the maximal and the pre-minimal operators generated by M_1^2 are defined as M_1^2 restricted to

$$D_{T(M_1^2)} := \left\{ x \in \ell_{w,1}^2 : \sum_{n=1}^{\infty} |M_1^2 x_n|^2 w_n < \infty \right\},$$

$$D_{T_0'(M_1^2)} := \{ x \in D_{T(M_1^2)} : x_m = 0$$
$$\text{for all but a finite number of values of } m \},$$

respectively, and the minimal operator is defined to be $T_0(M_1^2) := \overline{T_0'(M_1^2)}$ in $\ell_{w,1}^2$. Defining our operators in this way, we see that $T(M_1^2) = T_0^*(M_1^2)$ and $T(M_1) = T_0^*(M_1)$.

The problem we investigate now is to find under which circumstances the equality $T_0(M_1^2) = [T_0(M_1)]^2$ holds. First of all, we see that

$$T_0'(M_1^2) = [T_0'(M_1)]^2. \qquad (2.21)$$

Now, let $N_1 = M_1 - w_1$ and $N_2 = M_1 - w_2$, where $w_1 = -w_2$, $w_1^2 = w_2^2 = i$.

THEOREM 2.3 *M_1^2 is PS if, and only if,* $\det[T_0(M_1^2) - i] = 2\det[T_0(M_1) - \sqrt{i}]$.

Proof We have

$$
\begin{aligned}
T_0(N_1 N_2) &= T_0[(M_1 - w_1)(M_1 - w_2)] \\
&= T_0(M_1^2) - i. \tag{2.22}
\end{aligned}
$$

The operators $T_0(N_1)$ and $T_0(N_2)$ are closed, densely defined and have closed range. Then,

$$
\begin{aligned}
T_0'(N_1 N_2) &= T_0'(N_1) T_0'(N_2) \subseteq T_0(N_1) T_0(N_2) \tag{2.23} \\
&\Rightarrow T_0(N_1 N_2) \subseteq T_0(N_1) T_0(N_2) \tag{2.24}
\end{aligned}
$$

since $T_0(N_1) T_0(N_2)$ is closed [11, Theorem A, p. 26], and since $T_0(N_1 N_2)$ is the minimal closed operator generated by $N_1 N_2$. Now,

$$
\begin{aligned}
M_1^2 \text{ is PS} &\Rightarrow D(T(N_1) T(N_2)) \supseteq D(T(N_1 N_2)) \\
&\Rightarrow T_0(N_1 N_2) \supseteq T_0(N_1) T_0(N_2)
\end{aligned}
$$

and so

$$
T_0(N_1 N_2) = T_0(N_1) T_0(N_2) \tag{2.25}
$$

by (2.24). By (2.22) and Glazman's Theorem again, we have

$$
\begin{aligned}
\det[T_0(M_1^2) - i] &= \det[T_0(N_1) T_0(N_2)] \\
&= \det[T_0(N_1)] + \det[T_0(N_2)] \\
&= 2\det[T_0(N_j)] \quad (j = 1, 2) \\
&= 2\det[T_0(M_1) - \sqrt{i}] \tag{2.26}
\end{aligned}
$$

since $\det[T_0(N_1)] = \det[T_0(N_2)] = \det[T_0(M_1) - \sqrt{i}]$. Since M is real, we have the same result when we replace i by $-i$. This gives the proof in one direction. Suppose now that,

$$
\det[T_0(M_1^2) - i] = 2\det[T_0(M_1) - \sqrt{-i}]. \tag{2.27}
$$

Then

$$\mathrm{def}[T_0^2(M_1) - \mathrm{i}] = 2\mathrm{def}[T_0(M_1) - \sqrt{\mathrm{i}}]$$

$$= \mathrm{def}[T_0(N_1)] + \mathrm{def}[T_0(N_2)]$$

$$= \mathrm{def}[T_0(N_1)T_0(N_2)]$$

$$= \mathrm{def}[T_0(M_1)^2 - \mathrm{i}] \qquad (2.28)$$

by Glazman's Theorem. Also

$$T_0(M_1)^2 - \mathrm{i} \subseteq T_0^2(M_1) - \mathrm{i}. \qquad (2.29)$$

But, by (2.28), the spaces $\mathrm{Range}\left[T_0^2(M_1) - \mathrm{i}\right]^{\perp}$, $\mathrm{Range}\left[T_0(M_1^2) - \mathrm{i}\right]^{\perp}$ have the same (finite) dimension and hence we must have equality in (2.29), i.e. $T_0(M_1^2) = T_0^2(M_1)$. On taking adjoints, we get $T(M_1^2) \subseteq T^2(M_1)$ and hence M_1^2 is partially separated.

The theorem is therefore proved.

As was mentioned earlier, the set $D_{T_0'(M^2)}$, (2.11), is not dense in ℓ_w^2 and so $T_0'(M^2)$ does not have an adjoint in ℓ_w^2. However, its graph

$$G_0' := \left\{ \langle x, T_0'(M^2)x \rangle \colon x \in D_{T_0'(M^2)} \right\},$$

where $\langle x, y \rangle$ denotes a pair in $\ell_w^2 \times \ell_w^2$, is a linear subset of $\ell_w^2 \times \ell_w^2$, a linear relation in the terminology of Bennewitz [3]. Hence, an adjoint is defined as a closed linear relation on ℓ_w^2 [3, p. 35]: for $E \subset \ell_w^2 \times \ell_w^2$ the adjoint relation E^* is defined as

$$\langle U, V \rangle \in E^* \iff (U, v) = (V, u) \quad \forall \langle u, v \rangle \in E. \qquad (2.30)$$

E is called symmetric if $E \subseteq E^*$.

If G_0 is the closure of G_0' then $G_0^* = (G_0')^*$. It is readily proved that

$$G_0^* = G := \left\{ \langle x, x^* \rangle \colon x_n^* = M^2 x_n \ (n \geq 1), \ x \in D_{T(M^2)} \right\}. \qquad (2.31)$$

LEMMA 2.4 *Let M^2 be limit-point. Then G_0 is a closed, symmetric relation on ℓ_w^2 and*

$$\langle y, y^* \rangle \in G_0 \iff \begin{cases} y_n^* = M^2 y_n, \ n \geq 1, \\ y_{-1} = y_0 = 0, \\ y_{-1}^* = M^2 y_{-1} = (p_0 p_{-1}/w_0 w_{-1}) y_1, \\ y_0^* = M^2 y_0 = (p_0 p_1/w_0 w_1) y_2 \\ \quad - (p_0/w_0)[(p_0 + p_1 + q_1)/w_1 \\ \quad + (p_{-1} + p_0 + q_0)/w_0] y_1. \end{cases} \tag{2.32}$$

Proof G_0 is closed by its definition, and it is symmetric since

$$G_0 = \overline{G_0'} \subset G = G_0^*. \tag{2.33}$$

Note that this also shows that G_0' is symmetric, since $G_0^* = (G_0')^*$. Now, for the necessity of (2.32), let $\langle y, y^* \rangle \in G_0 \subset G = G_0^*$. We then have $y_n^* = M^2 y_n$ $(n = 1, 2, 3, \ldots)$. Also let $\langle x, x^* \rangle \in G = G_0^*$ where $x = \{x_n\}_{-1}^\infty$ has only a finite number of non-zero x_n's but arbitrary, otherwise. Then, since $G_0^* = G$,

$$0 = (y^*, x) - (y, x^*)$$

$$= \sum_{n=1}^\infty (M^2 y_n \bar{x}_n - y_n \overline{M^2 x_n}) w_n$$

$$\quad + (y_{-1}^* \bar{x}_{-1} - y_{-1} \bar{x}_{-1}^*) w_{-1} + (y_0^* \bar{x}_0 - y_0 \bar{x}_0^*) w_0$$

$$= p_0 (\bar{x}_0 M y_1 - \bar{x}_1 M y_0 + \overline{M x_0} y_1 - \overline{M x_1} y_0)$$

$$\quad + y_{-1}^* \bar{x}_{-1} w_{-1} - y_{-1} \bar{x}_{-1}^* w_{-1} + y_0^* \bar{x}_0 w_0 - y_0 \bar{x}_0^* w_0$$

$$= \bar{x}_0 \left[y_0^* w_0 - \frac{p_0 p_1}{w_1} y_2 + \frac{p_0(p_0 + p_1 + q_1)}{w_1} y_1 + \frac{p_0(p_{-1} + p_0 + q_0)}{w_0} y_1 \right]$$

$$\quad + y_{-1} \left(\frac{p_{-1} p_0}{w_0} \bar{x}_1 - \bar{x}_{-1}^* w_{-1} \right) + \bar{x}_{-1} \left(y_{-1}^* w_{-1} - \frac{p_{-1} p_0}{w_0} y_1 \right)$$

$$\quad + y_0 \left[-\frac{p_0^2}{w_1} \bar{x}_0 - \frac{p_0(p_{-1} + p_0 + q_0)}{w_0} \bar{x}_1 - p_0 \overline{M x_1} - \bar{x}_0^* w_0 \right].$$

Now, since the x_n and x_n^* are arbitrary, it follows that

$$\begin{cases} y_{-1} & = y_0 = 0, \\ y_{-1}^* & = (p_0 p_{-1}/w_0 w_{-1})y_1 = M^2 y_{-1}, \\ y_0^* & = (p_0 p_1/w_0 w_1)y_2 - (p_0/w_0)[(p_0 + p_1 + q_1)/w_1 \\ & \quad + (p_{-1} + p_0 + q_0)/w_0]y_1 = M^2 y_0. \end{cases} \quad (2.34)$$

For the sufficiency, we need to show that the elements $\langle y, y^* \rangle \in G$ satisfying (2.34) are in G_0, i.e. in G^*. So, take an arbitrary $\langle x, x^* \rangle \in G$ and $\langle y, y^* \rangle \in G$ satisfying (2.34). Then, since M^2 is limit-point by the hypothesis,

$$(y^*, x) - (y, x^*)$$

$$= \sum_{n=1}^{\infty} (M^2 y_n \bar{x}_n - y_n \overline{M^2 x_n}) w_n$$

$$+ (y_{-1}^* \bar{x}_{-1} - y_{-1} \bar{x}_{-1}^*) w_{-1} + (y_0^* \bar{x}_0 - y_0 \bar{x}_0^*) w_0 = \frac{p_0 p_{-1}}{w_0 w_{-1}} y_1 \bar{x}_{-1} w_{-1}$$

$$+ \frac{p_0}{w_0} \left[\frac{p_1}{w_1} y_2 - \left(\frac{p_0 + p_1 + q_1}{w_1} + \frac{p_{-1} + p_0 + q_0}{w_0} \right) y_1 \right] \bar{x}_0 w_0$$

$$+ p_0 (\bar{x}_0 M y_1 - \bar{x}_1 M y_0 + \overline{M x_0} y_1 - \overline{M x_1} y_0)$$

$$= 0$$

since $y_{-1} = y_0 = 0$, and hence $\langle y, y^* \rangle \in G^* = G_0$. The lemma therefore follows.

We are now able to give the orthogonal direct sum of $G_0^* = G$ in terms of G_0 and the deficiency subspaces $N(\bar{\lambda}^2)$ and $N(\lambda^2)$ with respect to the graph inner product

$$(\langle a, a^* \rangle, \langle b, b^* \rangle)_{\lambda^2} := (a^* - \epsilon a, b^* - \epsilon b) + \eta^2(a, b), \quad \lambda^2 = \epsilon + i\eta \ (\eta \neq 0)$$
$$(2.35)$$

defined on G. In (2.35), $(.,.)$ is the ℓ_w^2 inner product and the norm on G is the norm produced by the graph inner product. The deficiency subspaces are defined as

$$N(\lambda^2) := \{ \langle x, \lambda^2 x \rangle \in G_0^* = G \}$$
$$= \{ \langle x, \lambda^2 x \rangle \in \ell_w^2 \times \ell_w^2 : M^2 x_n = \lambda^2 x_n, \ n = 1, 2, 3, \ldots \}, \quad (2.36)$$
$$N(\bar{\lambda}^2) := \{ \langle x, \bar{\lambda}^2 x \rangle \in \ell_w^2 \times \ell_w^2 : M^2 x_n = \bar{\lambda}^2 x_n, \ n = 1, 2, 3, \ldots \}.$$

We have the following lemma, which establishes the orthogonal decomposition of the maximal graph with respect to the graph inner-product defined by (2.35).

LEMMA 2.5   *Let $\lambda^2 = \epsilon + i\eta$, $\Im(\lambda^2) \neq 0$. Then,*

$$G = G_0 \oplus N(\lambda^2) \oplus N(\bar{\lambda}^2) \tag{2.37}$$

where $\oplus$ denotes the orthogonal sum with respect to the graph inner-product (2.35) defined on G.

Proof $G_0 \perp N(\lambda^2)$: Let $\langle y, y^* \rangle \in G_0$ and $\langle u, \lambda^2 u \rangle \in N(\lambda^2)$, then

$$
\begin{aligned}
(\langle y, y^* \rangle, \langle u, \lambda^2 u \rangle)_{\lambda^2} &= (y^* - \epsilon y, \lambda^2 u - \epsilon u) + \eta^2(y, u) \\
&= -i\eta(y^*, u) + i\epsilon\eta(y, u) + \eta^2(y, u) \\
&= -i\eta(y, \lambda^2 u) + i\epsilon\eta(y, u) + \eta^2(y, u) \\
&= 0
\end{aligned}
$$

since $\langle u, \lambda^2 u \rangle \in G_0^* = G$. Hence G_0 and $N(\lambda^2)$ are orthogonal; similarly for G_0 and $N(\bar{\lambda}^2)$. For $N(\lambda^2)$ and $N(\bar{\lambda}^2)$: let $\langle u, \lambda^2 u \rangle \in N(\lambda^2)$ and $\langle v, \bar{\lambda}^2 v \rangle \in N(\bar{\lambda}^2)$, then

$$
\begin{aligned}
(\langle u, \lambda^2 u \rangle, \langle v, \bar{\lambda}^2 v \rangle)_{\lambda^2} &= (i\eta u, -i\eta v) + \eta^2(u, v) \\
&= 0.
\end{aligned}
$$

The proof is therefore complete.

As another preliminary, we have the following lemma which establishes a basis for the solution spaces of $M^2\psi = \lambda^2\psi$ and $M^2\psi = \bar{\lambda}^2\psi$.

LEMMA 2.6   *Let M and M^2 be LP. Also, let $m(.)$ be the Hellinger–Nevanlinna function associated with M. For $\lambda, \lambda^2 \notin \mathbb{R}$, let $\theta(\lambda)$ and $\phi(\lambda)$ be the solutions of $Mx_n = \lambda x_n$ $(n \geq 0)$ defined by the initial conditions*

$$\theta_{-1} = 1, \quad \theta_0 = 0, \quad \phi_{-1} = 0, \quad p_{-1}\phi_0 = 1 \tag{2.38}$$

and set $\psi_n(\lambda) = \theta_n(\lambda) + m(\lambda)\phi_n(\lambda)$ for $n \geq -1$. Then $\psi(\lambda)$, $\psi(-\lambda)$ are linearly independent solutions of

$$M^2 x_n = \lambda^2 x_n, \quad n \geq 1 \tag{2.39}$$

and satisfy

$$\psi_{-1}(\pm\lambda) = 1, \qquad p_{-1}\psi_0(\pm\lambda) = m(\pm\lambda). \qquad (2.40)$$

Hence,

$$\{\langle\psi(\lambda), \lambda^2\psi(\lambda)\rangle, \langle\psi(-\lambda), \lambda^2\psi(-\lambda)\rangle\}$$

and

$$\left\{\langle\overline{\psi(\lambda)}, \overline{\lambda^2\psi(\lambda)}\rangle, \langle\overline{\psi(-\lambda)}, \overline{\lambda^2\psi(-\lambda)}\rangle\right\}$$

are bases for $N(\lambda^2)$ and $N(\bar{\lambda}^2)$ respectively.

Proof We see that (2.40) easily follows from (2.38). So we need only to specify the bases of the deficiency subspaces. But, since M^2 is LP, and $\langle u, \lambda^2 u\rangle \in N(\lambda^2)$ implies $M^2 u_n = \lambda^2 u_n$ ($n \geq 1$), it follows that u is a linear combination of $\psi(\lambda)$ and $\psi(-\lambda)$. Similarly for $N(\bar{\lambda}^2)$. The lemma is therefore proved.

LEMMA 2.7   *If M is LP, $\Im(\lambda) \neq 0$ and $\Re(\lambda) \neq 0$ then*

$$\|\psi(\pm\lambda)\|^2 = \frac{\Im[m(\pm\lambda)]}{\Im(\pm\lambda)} + w_{-1} \qquad (2.41)$$

and

$$(\psi(\lambda), \psi(-\lambda)) = \frac{m(\lambda) - \overline{m(-\lambda)}}{2\Re(\lambda)} + w_{-1}. \qquad (2.42)$$

Proof We insert $x_n = y_n = \psi_n(\lambda)$ in the Green's formula related to M. Since M is LP, we get

$$\sum_{n=0}^{\infty} \left[\overline{\psi_n(\lambda)}\lambda\psi_n(\lambda) - \psi_n(\lambda)\overline{\lambda\psi_n(\lambda)}\right] w_n$$

$$= p_{-1}\left[\overline{\psi_{-1}(\lambda)}\psi_0(\lambda) - \overline{\psi_0(\lambda)}\psi_{-1}(\lambda)\right] \Rightarrow \|\psi(\lambda)\|^2 = \frac{\Im(m(\lambda))}{\Im(\lambda)} + w_{-1}$$

and similarly taking $\psi(-\lambda)$ instead of $\psi(\lambda)$ we obtain (2.41). Also, by inserting $x_n = \psi_n(\lambda)$ and $y_n = \psi_n(-\lambda)$ in the Green's formula related to M we obtain (2.42), which completes the proof.

Note that, in order to prove our inequality, it is sufficient to consider only real sequences $x \in D_{T(M^2)} \subseteq D_{T(M)}$.

LEMMA 2.8   Let $\langle x, x^* \rangle \in G$ be real. Then, $\exists \langle y, y^* \rangle \in G_0$ and $A_1, A_2 \in \mathbb{C}$, such that

$$\langle x, x^* \rangle = \langle y, y^* \rangle + 2\Re \sum_{i=1}^{2} A_i \langle \psi^{(i)}, \lambda^2 \psi^{(i)} \rangle \quad \lambda^2 \notin \mathbb{R}, \qquad (2.43)$$

where $\psi^{(1)}$ is $\psi(\lambda)$ and $\psi^{(2)}$ is $\psi(-\lambda)$.

Proof   Using

$$G = G_0 \oplus N(\lambda^2) \oplus N(\bar{\lambda}^2), \quad \Im(\lambda) \neq 0$$

for all $\langle x, x^* \rangle \in G$, there exists $A_i, B_i \in \mathbb{C}, i = 1, 2$, and $\langle y, y^* \rangle \in G_0$ such that

$$\langle x, x^* \rangle = \langle y, y^* \rangle + \sum_{i=1}^{2} A_i \langle \psi^{(i)}, \lambda^2 \psi^{(i)} \rangle + \sum_{i=1}^{2} B_i \langle \overline{\psi^{(i)}}, \overline{\lambda^2 \psi^{(i)}} \rangle. \qquad (2.44)$$

Now, since $\langle x, x^* \rangle$ is real, (2.44) gives

$$\langle x, x^* \rangle = \overline{\langle y, y^* \rangle} + \sum_{i=1}^{2} \overline{B_i} \langle \psi^{(i)}, \lambda^2 \psi^{(i)} \rangle + \sum_{i=1}^{2} \overline{A_i} \langle \overline{\psi^{(i)}}, \overline{\lambda^2 \psi^{(i)}} \rangle. \qquad (2.45)$$

Hence, uniqueness in the orthogonal decomposition gives that $\langle y, y^* \rangle$ must be real and $A_i = \overline{B_i}, i = 1, 2$. This completes the proof.

LEMMA 2.9   For real $\langle x, x^* \rangle \in G$, satisfying (2.43), and $\lambda^2 = \epsilon + i\eta$ we have

$$\| \langle x, x^* \rangle \|_{\lambda^2}^2 = \| \langle y, y^* \rangle \|_{\lambda^2}^2$$
$$+ 4\eta^2 \Re \left\{ \sum_{i=1}^{2} \sum_{k=1}^{2} A_i \overline{A_k} (\psi^{(i)}, \psi^{(k)}) \right\}, \quad \lambda^2 \notin \mathbb{R}. \qquad (2.46)$$

Proof   By Lemma 2.8,

$$\langle x, x^* \rangle = \langle y, y^* \rangle + \sum_{i=1}^{2} A_i \langle \psi^{(i)}, \lambda^2 \psi^{(i)} \rangle + \sum_{i=1}^{2} \overline{A_i} \langle \overline{\psi^{(i)}}, \overline{\lambda^2 \psi^{(i)}} \rangle,$$

and on using the graph norm,

$$\|\langle x, x^* \rangle\|_{\lambda^2}^2 = \|\langle y, y^* \rangle\|_{\lambda^2}^2 + \left\| \sum_{i=1}^{2} A_i \langle \psi^{(i)}, \lambda^2 \psi^{(i)} \rangle \right\|_{\lambda^2}^2$$

$$+ \left\| \sum_{i=1}^{2} \overline{A_i} \langle \overline{\psi^{(i)}}, \overline{\lambda^2 \psi^{(i)}} \rangle \right\|_{\lambda^2}^2$$

$$= \|\langle y, y^* \rangle\|_{\lambda^2}^2 + 2 \left\| \sum_{i=1}^{2} A_i \langle \psi^{(i)}, \lambda^2 \psi^{(i)} \rangle \right\|_{\lambda^2}^2.$$

Now,

$$2 \left\| \sum_{i=1}^{2} A_i \langle \psi^{(i)}, \lambda^2 \psi^{(i)} \rangle \right\|_{\lambda^2}^2 = 2 \sum_{i=1}^{2} |A_i|^2 \|\langle \psi^{(i)}, \lambda^2 \psi^{(i)} \rangle\|_{\lambda^2}^2$$

$$+ 4 \Re\{ A_1 \overline{A_2} (\langle \psi^{(1)}, \lambda^2 \psi^{(1)} \rangle, \langle \psi^{(2)}, \lambda^2 \psi^{(2)} \rangle)_{\lambda^2} \}$$

and (2.46) follows. Therefore the lemma is proved.

LEMMA 2.10   *Let* $\lambda = \rho \exp(i\theta)$ $(\theta \in (0, \pi/4), \rho \in \mathbb{R} - \{0\})$, $\lambda^2 = \epsilon + i\eta$; *let* $\langle x, x^* \rangle \in G$ *be real, M LP and* M^2 *PS, and define*

$$Q_{\lambda^2}[x] := \|\langle x, x^* \rangle\|_{\lambda^2}^2 + 2\epsilon[x_0 w_0(x_0^* - M^2 x_0)$$
$$+ x_{-1}(p_{-1} M x_0 + x_{-1}^* w_{-1}) - p_{-1} x_0 M x_{-1}]. \qquad (2.47)$$

Then the following are equivalent:

(a) $2\epsilon \sum_{n=0}^{\infty} (M x_n)^2 w_n \leq \|x^*\|^2 + |\lambda^2|^2 \|x\|^2,$ \qquad (2.48)

(b) $Q_{\lambda^2}[x] \geq 0.$ \qquad (2.49)

Furthermore, there is equality in (a) *if and only if there is equality in* (b).

Proof   By using the graph norm $\|.\|_{\lambda^2}$, for real $\langle x, x^* \rangle \in G$,

$$\|\langle x, x^* \rangle\|_{\lambda^2}^2 = \|x^*\|^2 - 2\epsilon(x^*, x) + |\lambda^2|^2\|x\|^2$$

and

$$(x^*, x) = x_{-1}^* x_{-1} w_{-1} + x_0^* x_0 w_0 + \sum_{n=1}^{\infty} x_n M^2 x_n w_n$$

$$= x_{-1}^* x_{-1} w_{-1} + x_0^* x_0 w_0 - x_0 M^2 x_0 w_0 + \sum_{n=0}^{\infty} x_n M^2 x_n w_n$$

$$= x_{-1}^* x_{-1} w_{-1} + x_0^* x_0 w_0 - x_0 M^2 x_0 w_0 + \sum_{n=0}^{\infty} (M x_n)^2 w_n$$

$$+ p_{-1}(x_{-1} M x_0 - x_0 M x_{-1})$$

$$\Rightarrow \|x^*\|^2 + \rho^4\|x\|^2 - 2\epsilon \sum_{n=0}^{\infty} (M x_n)^2 w_n = Q_{\lambda^2}[x].$$

The proof follows from this fact.

Now, by using (2.40) and setting $n(\lambda) = m(-\lambda)$ *for* $\arg(\lambda) \in (0, \pi/4)$, we have the following for real $\langle x, x^* \rangle \in G$:

$$x_{-1} = 2\Re(A_1 + A_2),$$

$$x_0 = 2\Re\left(A_1 \frac{m(\lambda)}{p_{-1}} + A_2 \frac{n(\lambda)}{p_{-1}}\right),$$

$$M x_{-1} = 2\Re\left[A_1\left(-\frac{m(\lambda)}{w_{-1}} + \frac{p_{-1}}{w_{-1}}\right) + A_2\left(-\frac{n(\lambda)}{w_{-1}} + \frac{p_{-1}}{w_{-1}}\right)\right],$$

$$M x_0 = M y_0 + 2\Re\left(A_1 \lambda \frac{m(\lambda)}{p_{-1}} - A_2 \lambda \frac{n(\lambda)}{p_{-1}}\right),$$

$$x_{-1}^* = y_{-1}^* + 2\Re(A_1 \lambda^2 + A_2 \lambda^2),$$

$$x_0^* = y_0^* + 2\Re\left(A_1 \lambda^2 \frac{m(\lambda)}{p_{-1}} + A_2 \lambda^2 \frac{n(\lambda)}{p_{-1}}\right),$$

$$M^2 x_0 = M^2 y_0 + 2\Re\left\{A_1\left[\lambda^2 \frac{m(\lambda)}{p_{-1}} + \frac{p_{-1}}{w_0}\lambda + \frac{p_{-1}^2}{w_0 w_{-1}}\left(\frac{m(\lambda)}{p_{-1}} - 1\right)\right]\right.$$

$$\left. + A_2\left[\lambda^2 \frac{n(\lambda)}{p_{-1}} - \frac{p_{-1}}{w_0}\lambda + \frac{p_{-1}^2}{w_0 w_{-1}}\left(\frac{n(\lambda)}{p_{-1}} - 1\right)\right]\right\}.$$

Combining these, we get from (2.47) that

$$Q_{\lambda^2}[x] - \|\langle x, x^* \rangle\|_{\lambda^2}^2 = 8\epsilon\Re[A_1 m(\lambda) + A_2 n(\lambda)]\Re[A_1(-\lambda) + A_2\lambda]$$
$$+ 8\epsilon\Re(A_1 + A_2)\Re\{A_1[\lambda m(\lambda) + \lambda^2 w_{-1}]$$
$$+ A_2[-\lambda n(\lambda) + \lambda^2 w_{-1}]\}.$$

This implies, by using (2.46) and (2.42) with the notation $\lambda_1 := \lambda$ and $\lambda_2 := -\lambda$, and $m(\lambda_1) = m(\lambda)$ and $m(\lambda_2) = n(\lambda)$,

$$Q_{\lambda^2}[x] - \|\langle y, y^* \rangle\|_{\lambda^2}^2$$
$$= 4\eta^2\Re\sum_{i=1}^{2}\sum_{k=1}^{2} A_i\bar{A}_k\left[\frac{m(\lambda_i) - \bar{m}(\lambda_k)}{\lambda_i - \bar{\lambda}_k} + w_{-1}\right]$$
$$+ 8\epsilon\Re[A_1 m(\lambda) + A_2 n(\lambda)]\Re[A_1(-\lambda) + A_2\lambda]$$
$$+ 8\epsilon\Re(A_1 + A_2)\Re\{A_1[\lambda m(\lambda) + \lambda^2 w_{-1}] + A_2[-\lambda n(\lambda) + \lambda^2 w_{-1}]\}$$
$$(2.50)$$

Now, set $A_1 = a_1 + ib_1$, $A_2 = a_2 + ib_2$, $m(\lambda) = m_1 + im_2$, $n(\lambda) = n_1 + in_2$, $\lambda = \mu + i\nu$ and

$$f(\lambda) = \lambda m(\lambda) + \lambda^2 w_{-1} = f_1 + if_2,$$
$$g(\lambda) = -\lambda n(\lambda) + \lambda^2 w_{-1} = g_1 + ig_2.$$

Then

$$Q_{\lambda^2}[x] - \|\langle y, y^* \rangle\|_{\lambda^2}^2 = A^T P A \qquad (2.51)$$

where $A^T = (a_1, a_2, b_1, b_2)$ and P is a real and symmetric matrix of size 4×4 with the following entries:

$$P_{11} = 4\eta^2\left(\frac{m_2}{\nu} + w_{-1}\right) + 8\epsilon(f_1 - m_1\mu),$$

$$P_{12} = 4\eta^2\left(\frac{m_1 - n_1}{2\mu} + w_{-1}\right) + 8\epsilon\left(\frac{f_1 + g_1}{2} + \frac{m_1 - n_1}{2}\mu\right),$$

$$P_{13} = 8\epsilon\left(-\frac{f_2}{2} + \frac{m_1\nu + m_2\mu}{2}\right),$$

$$P_{14} = 4\eta^2\left(\frac{m_2 + n_2}{2\mu}\right) + 8\epsilon\left(-\frac{g_2}{2} + \frac{-m_1\nu + n_2\mu}{2}\right),$$

$$P_{22} = 4\eta^2\left(\frac{n_2}{-\nu} + w_{-1}\right) + 8\epsilon(g_1 + n_1\mu),$$

$$P_{23} = 4\eta^2 \left(\frac{m_2 + n_2}{-2\mu} \right) + 8\epsilon \left(-\frac{f_2}{2} + \frac{n_1\nu - m_2\mu}{2} \right),$$

$$P_{24} = 8\epsilon \left(-\frac{g_2}{2} - \frac{n_1\nu + n_2\mu}{2} \right),$$

$$P_{33} = 4\eta^2 \left(\frac{m_2}{\nu} + w_{-1} \right) + 8\epsilon(-m_2\nu),$$

$$P_{34} = 4\eta^2 \left(\frac{m_1 - n_1}{2\mu} + w_{-1} \right) + 8\epsilon \left(\frac{m_2 - n_2}{2} \nu \right),$$

$$P_{44} = 4\eta^2 \left(\frac{n_2}{-\nu} + w_{-1} \right) + 8\epsilon(n_2\nu).$$

We see that, by defining $D[x] := \sum_{n=0}^{\infty}(Mx_n)^2 w_n$, we have the equivalence of the following:

(i) $\quad \|x^*\|^2 + \rho^4\|x\|^2 - 2\rho^2\cos(2\theta)D[x] \geq 0 \ \forall \ \mathrm{real}\langle x, x^* \rangle \in G,$ (2.52)

(ii) $\quad \mathbf{P}(\rho, \theta) \geq 0$, i.e. $\mathbf{P}$ is non-negative definite, (2.53)

(iii) $\quad \sum_{n=1}^{\infty}(M^2 x_n)^2 w_n + \rho^4\|x\|^2 - 2\rho^2\cos(2\theta)D[x] \geq 0,$

$\quad\quad \forall \ \mathrm{real} \ x \in D_{T(M^2)}.$ (2.54)

3. MAIN RESULTS

LEMMA 3.1 *For any non-trivial, real* $x \in D_{T(M^2)}$ *and* $K = 2k^{-1}$, *the following are equivalent:*

(i) $D[x] \leq K\|x\|(\sum_{n=1}^{\infty}(M^2 x_n)^2 w_n)^{1/2}$,
(ii) $J_{\rho^2, k}[x] := \sum_{n=1}^{\infty}(M^2 x_n)^2 w_n + \rho^4\|x\|^2 - \rho^2 k D[x] \geq 0 \ \forall \rho \in \mathbb{R} \setminus \{0\}.$

Furthermore, there is equality in (ii) *for* $\rho^2 = \frac{1}{2}kD[x]/\|x\|^2$ *if and only if there is equality in* (i).

Proof   We see that

$$J_{\rho^2,k}[x] = \left(\rho^2 - \frac{k}{2} \frac{D[x]}{\|x\|^2} \right)^2 \|x\|^2$$

$$+ \frac{1}{\|x\|^2} \left(\|x\|^2 \sum_{n=1}^{\infty} (M^2 x_n)^2 w_n - \frac{k^2}{4} (D[x])^2 \right) \qquad (3.1)$$

and

$$J_{\rho^2,k}[x] \geq 0 \; \forall \rho \in \mathbb{R} - \{0\} \iff D[x] \leq 2k^{-1} \|x\| \left(\sum_{n=1}^{\infty} (M^2 x_n)^2 w_n \right)^{1/2}.$$

$$(3.2)$$

Therefore the lemma is proved.

Now, let $\chi(\rho, \theta)$ denote the least eigenvalue of the real, symmetric matrix $\mathbf{P}(\rho, \theta)$ with $\lambda = \rho \exp(i\theta)$ ($\theta \in (0, \pi/4)$, $\rho \in \mathbb{R}\backslash\{0\}$), and define

$$\theta_0 := \inf\left\{ \theta \colon \theta \in \left(0, \frac{\pi}{4} \right), \quad \chi(\rho, \theta) \geq 0 \; \forall \rho \in \mathbb{R}\backslash\{0\} \right\}. \qquad (3.3)$$

The main theorem is as follows.

THEOREM 3.2   *Suppose M is LP and M^2 is PS. Let $\lambda = \rho \exp(i\theta)$ with $\theta \in (0, \pi/4)$, $\rho \in \mathbb{R}\backslash\{0\}$. Then,*

$$\text{(A)} \; \theta_0 = \inf\left\{ \theta \colon \theta \in \left(0, \frac{\pi}{4} \right), \quad \chi(\rho, \phi) > 0 \; \forall \phi \in \left(\theta, \frac{\pi}{4} \right), \; \forall \rho \in \mathbb{R}\backslash\{0\} \right\},$$

$$(3.4)$$

(B)   *there exists a constant $K \in (1, \infty)$ such that (1.15) is valid if and only if $0 < \theta_0 < \pi/4$, in which case, the best constant is $K = \sec(2\theta_0)$.*

Proof   **(A)** Following (2.39)–(2.41) and Lemma 3.1, with $\lambda = \rho \exp(i\theta)$ the validity of (1.15) is equivalent to non-negativeness of $\mathbf{P}(\rho, \theta)$, i.e. non-negativeness of the least eigenvalue $\chi(\rho, \theta)$ of the matrix $\mathbf{P}(\rho, \theta)$. Suppose that $\chi(\rho, \theta) \geq 0 \; \forall \rho \in \mathbb{R}\backslash\{0\}$ and for some $\theta \in (0, \pi/4)$. Hence,

$$Q_{\lambda^2}[x] \geq 0, \qquad \lambda = \rho \exp(i\theta).$$

Since for $\phi \in (\theta, \pi/4)$ we have $\cos(2\phi) < \cos(2\theta)$, it follows from (2.41) that, as $2\epsilon/\rho^2 = 2\cos(2\theta)$,

$$2\cos(2\phi)D[x] \leq 2\cos(2\theta)D[x] \quad (\text{strict if } D[x] \neq 0)$$
$$\leq \rho^2\|x\|^2 + \frac{1}{\rho^2}\sum_{n=1}^{\infty}(M^2 x_n)^2 w_n \quad \forall \rho \in \mathbb{R}\backslash\{0\}.$$

The second inequality above, i.e.

$$2\cos(2\theta)D[x] \leq \rho^2\|x\|^2 + \frac{1}{\rho^2}\sum_{n=1}^{\infty}(M^2 x_n)^2 w_n \quad \forall \rho \in \mathbb{R}\backslash\{0\} \qquad (3.5)$$

remains strict if $D[x]=0$ and $x \neq 0$. Hence, from (2.39)–(2.41) and Lemma 3.1, $\chi(\rho, \phi) > 0 \, \forall \rho \in \mathbb{R}\backslash\{0\}$ and therefore (3.4) is established.
   (B) If (1.15) is satisfied for some $K \in (1, \infty)$, then on setting

$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{1}{K}\right) \in \left(0, \frac{\pi}{4}\right),$$

we see that (3.5) is satisfied. So, (2.40) and (2.41) now yield that when $\lambda = \rho \exp(i\theta)$

$$\chi(\rho, \theta) \geq 0 \quad \forall \rho \in \mathbb{R}\backslash\{0\}.$$

In particular, $\theta_0 < \pi/4$ and $\theta_0 = \frac{1}{2}\cos^{-1}(1/K)$ where K is the least constant in (1.15). Conversely, if $\theta_0 < \pi/4$ then with $\lambda = \rho\exp(i\theta)$, for any $\theta \in (\theta_0, \pi/4)$,

$$Q_{\lambda^2}[x] \geq 0 \quad \forall \rho \in \mathbb{R}\backslash\{0\} \text{ and } \forall \text{ real } x \in D_{T(M^2)},$$

and use of Lemma 3.1 again yields (1.15) with the constant $\sec(2\theta)$ and this implies that

$$K = \sec(2\theta_0).$$

The proof is therefore complete.

THEOREM 3.3 *Suppose that the hypothesis of Theorem 3.2 holds and $\theta_0 < \pi/4$ so that $K = \sec(2\theta_0)$ is the best constant. Then the sequences*

$x = \{x_n\}_{-1}^{\infty} \in D_{T(M^2)}$ *which give equality in* (1.15) *satisfy one of the following distinct conditions*:

(a) $x = 0$,
(b) $x \neq 0$, $Mx_n = 0$ ($n = 0, 1, 2, \dots$) *in which case, both sides of* (1.15) *are* 0,
(c) $x \neq 0$, $M^2 x_n \neq 0$ *for* $n = 1, 2, 3, \dots$ *and*

$$E := \{\rho: \rho \in \mathbb{R} \backslash \{0\}, \ \chi(\rho, \theta_0) = 0\} \neq \emptyset$$

in which case the equalising sequences $\{x_n\}_{-1}^{\infty}$ *are given by*

$$x = 2\Re \sum_{i=1}^{2} A_i \psi^{(i)}(\lambda_i), \tag{3.6}$$

where $A_i = a_i + \mathrm{i}b_i$ *and* $A^{\mathrm{T}} = (a_1, a_2, b_1, b_2)$ *satisfies*

$$P(\rho, \theta_0)A = 0 \ (\rho \in E).$$

Proof Cases (a) and (b) are obvious. As for (c), suppose $J_{\rho^2, k}[x] = 0$ with $\rho^2 = \frac{1}{2}kD[x]/\|x\|^2$. Then, if there is equality in (1.15), i.e.

$$D[x] = \sec(2\theta_0)\|x\| \left(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n \right)^{1/2},$$

we have, with $\lambda = \rho \exp(\mathrm{i}\theta_0)$ and $\epsilon = \Re(\lambda^2) = \rho^2 \cos(2\theta_0)$,

$$\frac{2\epsilon}{\rho^2} D[x] = \frac{1}{\rho^2} \sum_{n=1}^{\infty} |M^2 x_n|^2 w_n + \rho^2 \|x\|^2.$$

Hence, by Lemma 2.10, $Q_{\lambda^2}[x] = 0$. However, from

$$Q_{\lambda^2}[x] - \|\langle y, y^* \rangle\|_{\lambda^2}^2 = A^{\mathrm{T}} \mathbf{P}(\rho, \theta_0)A \geq 0, \tag{3.7}$$

it follows that $y = y^* = 0$ and $\mathbf{P}(\rho, \theta_0)A^{\mathrm{T}} = 0$. Then, $\rho \in E$ and, following (2.30), x is given by

$$x = 2\Re \sum_{i=1}^{2} A_i \psi^{(i)}(\lambda_i).$$

Conversely, suppose that $\rho \in E$ and x is given by (3.6). Then $x \neq 0$, $x \in D_{T(M^2)}$, $T(M^2)x = 2\Re\{\lambda^2 \sum_{i=1}^2 A_i \psi^{(i)}(\lambda_i)\}$, and with $y = y^* = 0$ in (3.7) we get $Q_{\lambda^2}[x] = 0$ in which $\lambda = \rho \exp(i\theta_0)$. It follows from Lemma 2.10 that

$$2\cos(2\theta_0)D[x] = \frac{1}{\rho^2}\sum_{n=1}^{\infty}|M^2 x_n|^2 w_n + \rho^2 \|x\|^2. \tag{3.8}$$

When $\rho^2 = \left(\sum_{n=1}^{\infty}|M^2 x_n|^2 w_n\right)^{1/2}/\|x\|$ the right-hand side of (3.8) attains its minimum, and hence

$$\cos(2\theta_0)D[x] \geq \left(\sum_{n=1}^{\infty}|M^2 x_n|^2 w_n\right)^{1/2}\|x\|$$
$$\geq \cos(2\theta_0)D[x].$$

Following the last argument there is therefore equality, and the theorem is proved.

THEOREM 3.4 *There is no valid inequality in (1.15) when $m(\lambda)$ has an isolated pole at the origin.*

Proof Following Theorem 3.2, if there exists a constant $K \in (1, \infty)$ such that (1.15) is valid, then for some $\theta \in (0, \pi/4)$ the least eigenvalue $\chi(\rho, \theta)$ of $P(\rho, \theta)$ must be non-negative for all $\rho \in (0, \infty)$. We shall show that under the conditions of the theorem, i.e when

$$m(\lambda) = \frac{B}{\lambda} + O(1), \quad B \in \mathbb{R}, \tag{3.9}$$

this is not possible. On setting $\lambda = \rho \exp(i\theta)$ $(\theta \in (0, \pi/4), \ \rho \in (0, \infty))$, we have

$$m(\lambda) = m_1 + im_2 = \left[\frac{B}{\rho}\cos\theta + O(1)\right] + i\left[-\frac{B}{\rho}\sin\theta + O(1)\right],$$
$$n(\lambda) = n_1 + in_2 = \left[-\frac{B}{\rho}\cos\theta + O(1)\right] + i\left[\frac{B}{\rho}\sin\theta + O(1)\right].$$

Substituting these values into the entries of (2.51), we get the real and symmetric matrix

$$P(\rho, \theta) = 8\rho^2 B \begin{pmatrix} -\sin^2\theta & 3\cos^2\theta - 1 & 0 & 0 \\ . & -\sin^2\theta & 0 & 0 \\ . & . & -\sin^2\theta & \sin^2\theta \\ . & . & . & -\sin^2\theta \end{pmatrix} + O(\rho^3).$$

Thus, for small enough ρ and $\theta \in (0, \pi/4)$, the least eigenvalue of $P(\rho, \theta)$ is either $\chi(\rho, \theta) = -16B\rho^2\cos^2\theta$ $(B > 0)$, or $\chi(\rho, \theta) = 16B\rho^2\cos(2\theta)$ $(B < 0)$. In both cases the least eigenvalue $\chi(\rho, \theta)$ of $P(\rho, \theta)$ remains negative. Hence, there is no valid inequality in (1.15) and the theorem is proved.

4. EXAMPLES

In the following two examples we give information on θ_0 (hence the best constant K) and E (hence the equalising sequences). The difficulty in handling the least eigenvalue of $P(\rho, \theta)$ forces us to use a computational approach. The numerical algorithm we use is the same as that in [5]: in order to estimate K we solve numerically the equation $\chi(\rho, \theta) = 0$ $(\theta \in (0, \pi/4))$ and plot the graph of θ against ρ for ρ less than some empirically determined R. Since, from (3.4),

$$\theta_0 = \sup\left\{\theta \colon \theta \in \left(0, \frac{\pi}{4}\right), \chi(\rho, \theta) = 0 \text{ for some } \rho \in (0, \infty)\right\}, \quad (4.1)$$

the value of θ_0 is the maximum of this graph and $K = \sec(2\theta_0)$. We recall from Theorem 3.2 that the condition for a valid inequality requires the least eigenvalue $\chi(\rho, \theta)$ of $P(\rho, \theta)$ to be non-negative for some $\theta \in (0, \pi/4)$ and all $\rho \in (0, \infty)$.

(i) The case where $p_n = w_n = 1$ and $q_n = -\tau$ $(\tau \in \mathbb{R})$

In this case

$$Mx_n = -\Delta^2 x_{n-1} - \tau x_n \quad (n \geq 0) \quad (4.2)$$

and the recurrence relation is given by

$$-\Delta^2 x_{n-1} = (\lambda + \tau)x_n \quad (n \geq 0), \quad \lambda \in \mathbb{C}\backslash\mathbb{R}, \; \tau \in \mathbb{R}.$$

The inequality to be established is

$$\sum_{n=0}^{\infty} |-\Delta^2 x_{n-1} - \tau x_n|^2 \leq K \left(\sum_{n=-1}^{\infty} |x_n|^2 \right)^{1/2}$$
$$\times \left(\sum_{n=1}^{\infty} |\Delta^4 x_{n-2} + 2\tau\Delta^2 x_{n-1} + \tau^2 x_n|^2 \right)^{1/2}. \tag{4.3}$$

The expression M^2 is partially separated since the mapping $\{x_n\}_{-1}^{\infty} \mapsto \{Mx_n\}_{-1}^{\infty}$ is bounded. Also M is limit-point following the limit-point criterion of Hinton and Lewis [15, p. 435], which is that $\sum_{n=0}^{\infty}[(w_n w_{n+1})^{1/2}/|p_n|] = \infty$. The Hellinger–Nevanlinna $m(\lambda)$ functions, which we now denote by $m_\tau(\lambda)$ and $n_\tau(\lambda)$ to indicate their dependence on τ, are

$$m_\tau(\lambda) := m(\lambda + \tau)$$
$$= 1 - \frac{\lambda + \tau}{2} + \frac{1}{2}\sqrt{(\lambda + \tau)(\lambda + \tau - 4)}, \quad \lambda \in \mathbb{C}_+,$$
$$n_\tau(\lambda) := m(-\lambda + \tau)$$
$$= 1 - \frac{-\lambda + \tau}{2} + \frac{1}{2}\sqrt{(-\lambda + \tau)(-\lambda + \tau - 4)}, \quad \lambda \in \mathbb{C}_+$$

(see [5]). First of all, with $\lambda = \rho\exp(i\theta)$, we present an analytical result proving that for large values of ρ, $\theta = 0$ in the graph $\{(\rho, \theta): \chi(\rho, \theta) = 0\}$ for any $\tau \in \mathbb{R}$: as $\rho \to \infty$ the Hellinger–Nevanlinna m-functions satisfy

$$m_\tau(\lambda) = \frac{-1}{\lambda} + O\left(\frac{1}{\rho^2}\right), \quad n_\tau(\lambda) = \frac{1}{\lambda} + O\left(\frac{1}{\rho^2}\right),$$

which are independent of the choice of τ. On inserting these values in $P(\rho, \theta)$, we find that the least eigenvalue of the leading matrix is 0 for $\theta \in (0, \pi/4)$. Hence, in the graph $\{(\rho, \theta): \chi(\rho, \theta) = 0\}$, $\theta = 0$ for all $\tau \in \mathbb{R}$ and large enough values of ρ. This also can be seen from the numerical results presented in Figs. 1 and 2.
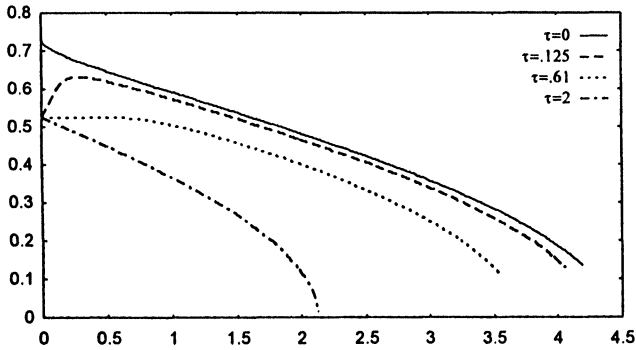
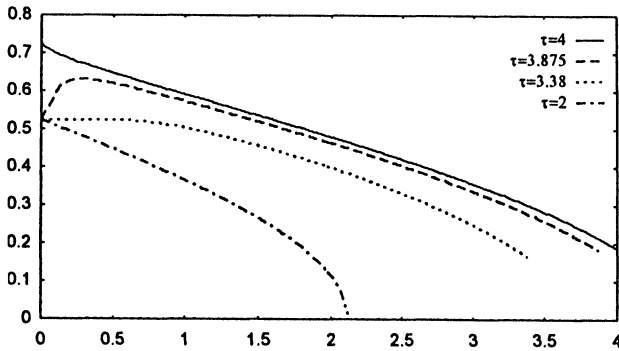FIGURE 1   The graph $\{(\rho,\theta): \chi(\rho,\theta)=0\}$, $q_n = -\tau$, $\tau \in [0,2]$.

FIGURE 2   The graph $\{(\rho,\theta): \chi(\rho,\theta)=0\}$, $q_n = -\tau$, $\tau \in [2,4]$.

So, in the investigation of the value of θ_0, we need to examine the case $\rho \to 0$. In the case $\tau \in \mathbb{R} \backslash [0,4]$, the Hellinger–Nevanlinna m-functions are $m_\tau(\lambda) = 1 - \tau/2 + \frac{1}{2}\sqrt{\tau(\tau-4)} + O(\rho)$ and $n_\tau(\lambda) = 1 - \tau/2 - \frac{1}{2}\sqrt{\tau(\tau-4)} + O(\rho)$, and inserting these values in the entries of $P(\rho,\theta)$ we see that the least eigenvalue of the leading matrix remains strictly negative. We conclude that there is no valid inequality in (4.3). In the case $\tau = 0, 4$, we do not have analytical results but numerical results show that the graph $\{(\rho,\theta): \chi(\rho,\theta)=0\}$ starts at $\theta = \theta_0$ in the interval $(0.72896067, 0.72896068)$ when $\rho \to 0$ and decreases monotonically.

Hence we predict that the value of the best constant satisfies $8.878200 < K < 8.878201$ and $E = \emptyset$, i.e. there are no non-trivial equalising sequences. Note that, the case $\tau = 0$ gives the norm inequality

$$\|\Delta^2 x\|^2 \leq K \|x\| \|\Delta^4 x\|, \qquad (4.4)$$

where the norm is the ℓ^2 norm with the set $E = \emptyset$ and the best possible constant K satisfying $8.878200 < K < 8.878201$. The interval for K is the same as in the analogous problem in [5]. An interesting point is that when $\tau = 0, 4$ the leading term in the asymptotic value of $P(\rho, \theta)$ becomes zero, and one has to examine the next term. This accounts for the discontinuous behaviour of the graphs at $\rho = 0$ in Figs. 1 and 2.

Finally in the case $\tau \in (0, 4)$, as $\rho \to 0$ we have $m_\tau(\lambda) = 1 - \tau/2 + (i/2) \sqrt{\tau(4 - \tau)} + O(\rho)$ and $n_\tau(\lambda) = 1 - \tau/2 - (i/2)\sqrt{\tau(4 - \tau)} + O(\rho)$. With $\lambda = \rho \exp(i\theta)$, the least eigenvalue of $P(\rho, \theta)$ is $\chi(\rho, \theta) = 4\rho^3 \sqrt{\tau(4 - \tau)}[\sin \theta - \cos(2\theta)]$ and it remains non-negative for $\theta \geq \pi/6$ which proves that as $\rho \to 0$, θ tends to $\pi/6$ (see also Figs. 1 and 2 and Table I). We see from Figs. 1 and 2 that the graph $\{(\rho, \theta): \chi(\rho, \theta) = 0\}$ always starts from $\pi/6$ and either decreases monotonically or increases to a value (θ_0, ρ_0) and decreases again. In the first case, the best constant in

TABLE I $q_n = -\tau, \tau \in [0,4]$

τ	θ_0	K	$E = \{\rho_0\}$
0.000000	0.72896068	8.8782005	$\emptyset$
0.001000	0.71436994	7.0631870	{0.01330}
0.010000	0.69459334	5.5367007	{0.05913}
0.067500	0.65427425	3.8572481	{0.19380}
0.125000	0.63115740	3.2936753	{0.27880}
0.250000	0.59459489	2.6851985	{0.40915}
0.500000	0.54178273	2.1359270	{0.54807}
0.610962	0.52359888	2.0000007	{0.51756}
1.000000	$\pi/6$	2.0000000	$\emptyset$
2.000000	$\pi/6$	2.0000000	$\emptyset$
3.000000	$\pi/6$	2.0000000	$\emptyset$
3.389038	0.52359888	2.0000007	{0.51756}
3.500000	0.54178273	2.1359270	{0.54807}
3.750000	0.59459489	2.6851985	{0.40916}
3.875000	0.63115740	3.2936753	{0.27880}
3.932500	0.65427425	3.8572481	{0.19380}
3.990000	0.69459334	5.5367007	{0.05913}
3.999000	0.71436994	7.0631870	{0.01330}
4.000000	0.72896068	8.8782005	$\emptyset$

the inequality is $K = 2$ and the unique equalising sequence from $D_{T(M^2)}$ is the null sequence. The second case occurs for $0 < \tau < \tau_0$ and $4 - \tau_0 < \tau < 4$, say, where $\tau_0 \in (0.610962, 0.610963)$. In the second case the best constant is $K = \sec(2\theta_0)$, for some $\theta_0 > \pi/6$, and E is a singleton set $E = \{\rho_0\}$. Table I gives a summary of what we have discussed.

(ii) *Legendre problem with $-\tau$ shift $(\tau \in \mathbb{R})$*

The Legendre polynomials $P_n(\lambda), n \geq -1$, satisfy the recurrence relation

$$(n + 2)P_{n+1} + (n + 1)P_{n-1} = \lambda(2n + 3)P_n, \quad n \geq 0 \qquad (4.5)$$

with

$$P_{-1} = 1, \qquad P_0 = \lambda. \qquad (4.6)$$

In our notation the difference expression we are dealing with is defined by

$$Mx_n$$
$$:= \begin{cases} \dfrac{1}{w_n}[-p_n x_{n+1} + (p_{n-1} + p_n + q_n)x_n - p_{n-1}x_{n-1}] = \lambda x_n, & n \geq 0, \\ \dfrac{(x_0 - x_{-1})}{w_{-1}}, & n = -1, \end{cases}$$
$$(4.7)$$

in which $p_n = -(n + 2)$, $q_n = w_n = 2n + 3$ and $p_{n-1} + p_n + q_n = 0$. Note that the Legendre problem now becomes: to solve the equation

$$Mx_n = \lambda x_n, \quad n \geq 0, \quad \lambda \in \mathbb{C}\backslash\mathbb{R} \qquad (4.8)$$

with the initial conditions $x_{-1} = 1$, $x_0 = \lambda$ (see [7]). The Hellinger–Nevanlinna functions for (4.8) are

$$m(\lambda) = -\frac{1}{\mathcal{M}(\lambda)} - \lambda, \quad n(\lambda) = -\frac{1}{\mathcal{M}(-\lambda)} + \lambda, \qquad \lambda \in \mathbb{C}_+,$$

where $\mathcal{M}(\lambda) = \frac{1}{2}\log[(\lambda - 1)/(\lambda+1)]$, $\lambda \in \mathbb{C}_+$ (see [7]).

The shifted Legendre problem is the case where $p_n = -(n+2)$, $w_n = 2n+3$ and $q_n = 2n+3-\tau$ $(\tau \in \mathbb{R})$ in (4.7), i.e.

$$
Mx_n := \begin{cases} \dfrac{1}{2n+3}[(n+2)x_{n+1} - \tau x_n + (n+1)x_{n-1}] = \lambda x_n, & n \geq 0, \\[3mm] \dfrac{(x_0 - x_{-1})}{w_{-1}}, & n = -1. \end{cases}
$$

$$(4.9)$$

The inequality to be established is

$$
\sum_{n=0}^{\infty} \left| \frac{1}{2n+3}[(n+2)x_{n+1} - \tau x_n + (n+1)x_{n-1}] \right|^2 (2n+3)
$$

$$
\leq K \left(\sum_{n=-1}^{\infty} |x_n|^2 (2n+3) \right)^{1/2}
$$

$$
\times \left(\sum_{n=1}^{\infty} \left| \frac{1}{2n+3}[(n+2)Mx_{n+1} - \tau Mx_n + (n+1)Mx_{n-1}] \right|^2 (2n+3) \right)^{1/2},
$$

$$(4.10)$$

where Mx_n is defined by (4.9).

Using the criterion of Hinton and Lewis [15], we see that M satisfies the required LP condition. Also M^2 satisfies the PS condition since, given $x, M^2x \in \ell_w^2$, it is easy to show that $Mx \in \ell_w^2$. The $m_\tau(\lambda)$ and $n_\tau(\lambda)$ functions related to this problem are now given by the formulas $m_\tau(\lambda) = m(\lambda + \tau)$ and $n_\tau(\lambda) = m(-\lambda + \tau)$. We first prove that for large enough values of ρ, $\theta = 0$ in the graph $\{(\rho, \theta): \chi(\rho, \theta) = 0\}$: as $\rho \to \infty$ the m-functions are

$$
m_\tau(\lambda) = -\frac{1}{3\lambda} + O\left(\frac{1}{\rho^2}\right), \quad n_\tau(\lambda) = \frac{1}{3\lambda} + O\left(\frac{1}{\rho^2}\right), \qquad \lambda \in \mathbb{C}_+.
$$

In this case the least eigenvalue of the leading matrix of $P(\rho, \theta)$ is zero for $\theta \in (0, \pi/6)$ as $\rho \to \infty$. Hence, $\theta = 0$ in the graph $\{(\rho, \theta): \chi(\rho, \theta) = 0\}$ for all large enough ρ, and for all $\tau \in \mathbb{R}$. So, to obtain the value of θ_0, we will have to examine the asymptotics as $\rho \to 0$. We present these results according as $\tau \in \mathbb{R} \setminus [-1, 1]$, $\tau = \pm 1$, $\tau \in (-1, 1)$ below.

In the first case, we have no analytical results but numerical estimates show that the graph $\{(\rho, \theta): \chi(\rho, \theta) = 0\}$, as presented in Figs. 4 and 3,

starts from $\pi/4$ and decreases monotonically. This gives no valid inequality in (4.10). In the case $\tau = \pm 1$, the graph starts from a θ_0, $0.71552 < \theta_0 < 0.71553$, with ρ near 0 and decreases monotonically. This means that the best constant satisfies $K \in (7.1786, 7.1787)$ and $E = \emptyset$. For the case $\tau \in (-1, 1)$, we have the following asymptotic results as $\rho \to 0$: for $\Im(\lambda) > 0$ and $\lambda = \rho \exp(i\theta)$, the m-functions are

$$m_\tau(\lambda) = \frac{-2}{\log((1-\tau)/(1+\tau)) + \pi i} - \tau + O(\rho),$$

$$n_\tau(\lambda) = \frac{-2}{\log((1-\tau)/(1+\tau)) - \pi i} - \tau + O(\rho).$$

Then, the least eigenvalue of the leading matrix of $P(\rho, \theta)$ is asymptotic to

$$\chi(\rho, \theta) = \frac{16\rho^3\pi}{\log^2((1-\tau)/(1+\tau)) + \pi^2} \left[\sin\theta - \cos(2\theta) \right],$$

giving $\theta_0 = \inf\{\theta \in (0, \pi/4): \chi(\rho, \theta) \geq 0\} = \pi/6$. Hence, $\theta_1 = \pi/6$ for all $\tau \in (-1, 1)$ as $\rho \to 0$. These are in agreement with the numerical results: when $\tau \in (-1,1) \backslash (-\tau_0, \tau_0)$, $0.66118628 \leq \tau_0 < 0.66118629$, the graph $\{(\rho, \theta): \chi(\rho, \theta) = 0\}$ starts from $\pi/6$ and increases to a point (ρ_0, θ_0) $(\theta_0 < 0.7155203)$, giving the best constant $K = \sec(2\theta_0) < 7.178688$ and a singleton set $E = \{\rho_0\}$. When $\tau \in (-\tau_0, \tau_0)$, the graph $\{(\rho, \theta): \chi(\rho, \theta) = 0\}$ starts from $\pi/6$ with ρ near 0 and decreases monotonically, giving the best constant $K = 2$ and $E = \emptyset$. These results are presented in Figs. 3 and 4. See also Table II for summarised information. Note that similar to the example (i) above, when $\tau = \pm 1$ in the leading matrix, the leading term
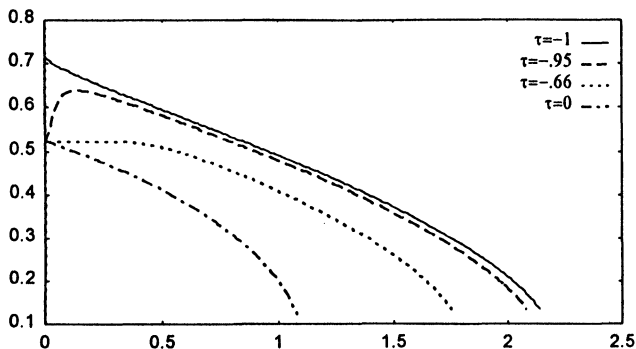

FIGURE 3   The graph $\{(\rho, \theta): \chi(\rho, \theta) = 0\}$, shifted Legendre problem, $\tau \in [-1, 0]$.
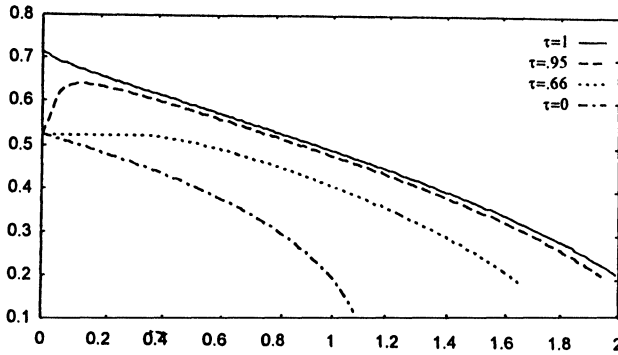
FIGURE 4   The graph $\{(\rho, \theta): \chi(\rho, \theta) = 0\}$, shifted Legendre problem, $\tau \in [0, 1]$.

TABLE II   Shifted Legendre problem, $q_n = 2n + 3 - \tau$, $\tau \in [-1, 1]$

τ	θ_0	K	$E = \{\rho_0\}$
-1.00000	0.7155203	7.178688	$\emptyset$
-0.99900	0.7021850	6.036489	$\{0.013327\}$
-0.93250	0.6280650	3.230251	$\{0.155717\}$
-0.87500	0.5977690	2.728417	$\{0.216576\}$
-0.75000	0.5494347	2.199719	$\{0.293944\}$
-0.66119	0.5235988	2.00000003	$\{0.276030\}$
-0.50000	$\pi/6$	2.000000	$\emptyset$
0.000000	$\pi/6$	2.000000	$\emptyset$
0.500000	$\pi/6$	2.000000	$\emptyset$
0.661186	0.5235988	2.00000003	$\{0.276030\}$
0.750000	0.5494347	2.199719	$\{0.293944\}$
0.875000	0.5977690	2.728417	$\{0.216576\}$
0.932500	0.6280650	3.230251	$\{0.155717\}$
0.999000	0.7021850	6.036489	$\{0.013327\}$
1.000000	0.7155203	7.178688	$\emptyset$

becomes zero, and so, one has to examine the next term. This again is caused by the discontinuous behaviour of the graphs at $\rho = 0$.

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