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On Generalized Wazewski and Lozinskii Inequalities for Semilinear Abstract Differential-Delay Equations

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A semilinear abstract differential-delay equation with a nonautonomous linear part is considered. Solution estimates are derived. They generalize Wazewski and Lozinskii inequalities. Conditions for global stability are established.

Keywords: Abstract differential-delay equations; Retarded parabolic systems; Inequalities for solutions; Stability

AMS (MOS) Subject Classification: 34G20; 34K20; 34K99

1 INTRODUCTION

1.1

It is well-known that quite a few problems of applied mathematics lead to abstract semilinear functional differential equations with delay, for example problems in biochemical control pathways in cells by a negative feedback mechanism (Boodwin, 1963), in distributed retarded systems (Henriquez, 1994), in epidemiology, in modelling of viscoelasticity, in theory of heat flow in materials with memory (Kolmanovskii and Myshkis, 1992), in population dynamics (Prüss, 1993), etc.

In recent years abstract nonlinear functional differential equations have been the object of intensive studies. The theory of these equations centers around the concepts of existence and uniqueness (see e.g. Jong Son Shin, 1994; Pao, 1992; Bahuguna and Raghavendra, 1989; Vrabie, 1987; and references therein).

Stability of scalar functional parabolic equations was investigated by Satoru Murokami (1995), Burton and Zhang (1992) and many other specialists while only few papers are devoted to stability of nonlinear abstract equations with delay and coupled parabolic systems with delay. The paper by Pao and Mahaffy (1985) should be mentioned (see also Pao, 1992). In that paper, exact conditions are established for global stability of parabolic systems containing autonomous linear parts without delays and quasimonotone nonlinearities with one delay. In addition, in the paper (Ruess and Summers, 1996), the linearized stability was established for a wide class of abstract differential-delay equations.

1.2

Consider in a Euclidean space $\mathbf{C}^n$ the equation

$$\dot{x} = Q(t)x \quad (\dot{x} = \mathrm{d}x/\mathrm{d}t,\ t \geq 0) \tag{1.1}$$

with a variable matrix $Q(t)$. As it is well-known for any solution $x(t)$ of (1.1) Wazewski established the inequality

$$|x(t)|_2 \leq |x(\tau)|_2 \exp\left[\int_\tau^t \alpha(Q_R(s))\,\mathrm{d}s\right] \quad (0 \leq \tau \leq t < \infty), \tag{1.2}$$

where $|\cdot|_2$ is the Euclidean norm, and

$$\alpha(Q_R(s)) = \max_{h \in \mathbf{C}^n} \frac{\mathrm{Re}(Q(s)h, h)_{\mathbf{C}^n}}{(h, h)_{\mathbf{C}^n}}$$

(see for instance Izobov, 1974; Winter, 1946). Here and below the scalar product in a Hilbert or Euclidean space Y is denoted by $(h, h)_Y$.

Denote by $\kappa(x)$ the (upper) Lyapunov exponent of a solution $x(t)$ of Eq. (1.1). That is,

$$\kappa(x) = \overline{\lim_{t \to \infty}} \frac{\ln |x(t)|_{\mathbf{C}^n}}{t}.$$

The set Σ of Lyapunov exponents of all possible solutions of Eq. (1.1) is called the (upper) Lyapunov spectrum of Eq. (1.1). Since Eq. (1.1) has n linearly independent solutions, the Lyapunov spectrum Σ consists of no more that n exponents. By the upper Lyapunov exponent of Eq. (1.1) is meant the quantity

$$\kappa_Q = \max_{\kappa \in \Sigma} \kappa.$$

With the notation

$$\bar{p} = \overline{\lim_{t \to \infty}} \frac{1}{t} \int_0^t p(s)\,\mathrm{d}s,$$

the estimate $\kappa_Q \le \overline{\alpha}(Q_R)$ is due to (1.2).

Now let $|\cdot|_{\mathbf{C}^n}$ be an arbitrary norm in $\mathbf{C}^n$. Lozinskii introduced the logarithm norm

$$L(t) \equiv \overline{\lim_{h \to 0}} \frac{1}{h} [|I_{\mathbf{C}^n} + hQ(t)|_{\mathbf{C}^n} - 1],$$

see e.g. (Vidyasagar, 1993, p. 22). Here and below I_Y is the unit operator in a space Y. For any solution $x(t)$ of Eq. (1.1), by the logarithm norm, the estimate

$$|x(t)|_{\mathbf{C}^n} \le |x(\tau)|_{\mathbf{C}^n} \exp\left[\int_\tau^t L(s)\,\mathrm{d}s\right] \quad (0 \le \tau \le t < \infty) \tag{1.3}$$

was derived (cf. Vidyasagar, 1993, p. 22). This inequality implies $\kappa_Q \le \bar{L}$. Kolmanovskii (1995) extended the latter inequality for the Lyapunov exponent to a wide class of (ordinary) functional differential equations. Besides, he established effective stability conditions.

1.3

Now let $A(t)$ be a linear closed operators in a Banach space X, with a norm $|\cdot|_X$. Besides, the domain $D(A(t)) \equiv D_A$ of $A(t)$ is constant and dense in X. Consider in X the equation

$$\dot{x} = A(t)x + F(t, x(h(t))) \quad (t \ge 0), \tag{1.4}$$

where $h(t)$ is an increasing differentiable scalar-valued function defined on $R_+ := [0, \infty)$ and satisfying the inequalities

$$-\eta \leq h(t) \leq t \quad (t \geq 0;\ 0 < \eta = \text{const.} < \infty), \tag{1.5}$$

and F maps $R_+ \times C([-\eta, 0], X)$ into X. As usually, $C([a, b], X)$ denotes the space of continuous functions acting from the segment $[a, b]$ into X and equipped with the sup-norm.

The present paper is devoted to an extension of the inequalities (1.2) and (1.3) to Eq. (1.4). By the obtained inequalities, stability conditions for Eq. (1.4) are established. Besides, some results from Gil' (1994; 1996a) are generalized.

Take the initial condition

$$x(t) = \Phi(t) \quad \text{for } -\eta \leq t \leq 0, \tag{1.6}$$

where $\Phi(t) : [-\eta, 0] \to X$ is a given continuous function. Further, suppose the Cauchy problem for the "shortened" equation

$$\dot{v} = A(t)v \quad (t \geq 0). \tag{1.7}$$

is well posed (Tanabe, 1979). Then Eq. (1.7) has the evolution operator (fundamental solution) $U(t, s)$ acting in X and defined by the equality $U(t, s)v(s) = v(t)$, where $v(t)$ is the solution of (1.7) (Tanabe, 1979, p. 89). Following Browder's terminology (cf. Vrabie, 1987, Chapter 5; Henry, 1981, p. 55) let us introduce:

DEFINITION 1.1 Suppose the Cauchy problem for Eq. (1.7) is well posed. A continuous function $x(t) : [-\eta, \infty) \to X$ satisfying the equation

$$x(t) = U(t, 0)x(0) + \int_0^t U(t, s)F(s, x(h(s)))\, \mathrm{d}s \tag{1.8}$$

and condition (1.6) will be called the mild solution of (1.4) with the initial function Φ.

In particular, (1.4) takes the form

$$\dot{x} = A(t)x + B(t)x(h(t)) \quad (t \geq 0),$$

where $B(t)$ is a bounded Bochner integrable operator.

Note that our results below can be easily extended to equations of the form

$$\dot{x} = A(t)x + F(t, x(h_1(t))), \ldots, x(h_m(t)) \quad (t \geq 0),$$

where $h_k(t)$ $(k = 1, \ldots, m)$ are similar to $h(t)$.

2 THE MAIN RESULT

Our main assumptions are as follows

(A1): *The operator $A(t)$ is continuously differentiable on D_A.* That is, $A(t)v$ is a strongly continuously differentiable function for any $v \in D_A$.

(A2): *F has the Lipschitz property*;

$$|F(t,h) - F(t,g)|_X \leq \mu |h - g|_X \quad (\mu = \text{const.};\ h, g \in X;\ t \geq 0).$$

Further, suppose that the operator $I_X - \delta A(t)$ is invertible for every sufficiently small $\delta > 0$, and there are bounded measurable scalar-valued functions $a(t)$ and $w(t)$ such that the inequalities

$$|(I_X - \delta A(t))^{-1}|_X \leq 1 + a(t)\delta \quad (t \geq 0) \tag{2.1}$$

and

$$|F(t,h)|_X \leq w(t)|h|_X \quad (h \in X,\ t \geq 0) \tag{2.2}$$

hold. Denote by $\psi(z)$ the function inverse to $h(s)$. That is, if $\tau = h(s)$, then $s = \psi(\tau)$. Put

$$q(s) = w(s) \exp\left[-\int_{h(s)}^{s} a(\tau)\, \mathrm{d}\tau\right].$$

Now we are in a position to formulate the main result of the paper.

THEOREM 2.1 *Under assumptions* (A1) *and* (A2), *let inequalities* (2.1) *and* (2.2) *hold. Then for any initial function* $\Phi \in C([-\eta, 0], X)$, *Eq.* (1.4) *has a unique mild solution* $x(t)$. *Moreover, the inequality*

$$|x(t)|_X \leq z(t) \exp\left[\int_0^t a(\tau)\, \mathrm{d}\tau\right] \quad (t \geq \psi(0)) \tag{2.3}$$

is valid, where $z(t)$ *is a solution of the scalar equation*

$$z(t) = e_0(\Phi) + \int_{\psi(0)}^{t} q(s)z(h(s))\,\mathrm{d}s \quad (t \geq \psi(0)) \tag{2.4}$$

with the notation

$$e_0(\Phi) = |\Phi(0)|_X + \int_0^{\psi(0)} \exp\left[\int_s^0 \alpha(\tau)\,\mathrm{d}\tau\right] w(s)|\Phi(h(s))|_X\,\mathrm{d}s.$$

Further, put

$$\Lambda(t) = \overline{\lim_{h\to+0}} \frac{1}{h}[|(I_X - hA(t))^{-1}|_X - 1].$$

COROLLARY 2.2 *Under assumptions* (A1) *and* (A2), *let* $\Lambda(t)$ *be a Riemann integrable function, and let condition* (2.2) *hold. Then for any initial function* $\Phi \in C([-\eta, 0], X)$, *Eq.* (1.4) *has a unique mild solution* $x(t)$. *Moreover, inequality* (2.3) *holds with* $a(t) = \Lambda(t)$.

COROLLARY 2.3 *Let* $X = H$ *be a Hilbert space. In addition, under assumptions* (A1) *and* (A2), *let the condition* (2.2), *and*

$$\mathrm{Re}(A^*(t)g, g) < a(t)|g|_H^2 \quad and \quad \mathrm{Re}(A(t)h, h)_H \leq a(t)|h|_H^2$$
$$g \in D(A^*(t)), \quad h \in D_A; \qquad t \geq 0$$

be fulfilled. Then for any initial function $\Phi \in C([-\eta, 0], X)$, *Eq.* (1.4) *has a unique mild solution* $x(t)$, *and estimate* (2.3) *holds.*

Indeed, we have

$$\begin{aligned}|(I_H - \delta A(t))h|_H^2 &= 1 - 2\,\mathrm{Re}(A(t)h, h)_H\delta + (A(t)h, A(t)h)_H\delta^2 \\ &\geq 1 - 2a(t)\delta \quad h \in D_A, \; \|h\|_H = 1.\end{aligned}$$

Under consideration the operator $I_H - \delta A(t)$ is clearly invertible. Thus omitting simple calculations we easily get inequality (2.1). Now the result is due to Theorem 2.1.

According to (1.2), Corollary 2.3 extends the Wazewski inequality to Eq. (1.4). If $A(t)$ is a bounded, then for small enough $\delta > 0$,

$$(I_X - \delta A(t))^{-1} = \sum_{k=0}^{\infty} \delta^k A^k(t).$$

So

$$(I_X - \delta A(t))^{-1} = I_X + \delta A(t) + o(\delta) \quad (\delta \downarrow 0).$$

Hence,

$$\Lambda(t) = \overline{\lim_{h\to+0}} \frac{1}{h}[|(I_X + hA(t))^{-1}|_X - 1] = \overline{\lim_{h\to+0}} \frac{1}{h}[|I_X + hA(t)|_X - 1].$$

According to (1.3) Corollary 2.2 generalizes the Lozinskii inequality.

Note that the extension of the Lozinskii inequality to Eq. (1.7) is proved in Gil' (1996b).

3 PROOF OF THEOREM 2.1

Take a fixed positive $T < \infty$. We need the following trivial:

LEMMA 3.1 *Let $\Psi(x)$ be a continuous mapping of a closed subset $C_1 \subseteq C([0,T], X)$ into itself, satisfying the inequality*

$$|(\Psi x)(t) - (\Psi y)(t)|_X \le (M|x-y|_X)(t) \quad (t \in [0,T];\ x, y \in C_1),$$

where M is a bounded linear positive operator acting in the space of real continuous scalar-valued functions $C[0,T]$. If, in addition, the spectral radius $d(M)$ of M is less than one: $d(M) < 1$, then Ψ has a unique fixed point $\overline{x} \in C_1$. Moreover, that point can be found by the method of successive approximations.

Proof Take an arbitrary $x_0 \in \Phi$ and define the successive approximations $x_k = \Psi(x_{k-1})$ $(k = 1, 2, \ldots)$. Hence,

$$\begin{aligned}|x_{k+1}(t) - x_k(t)|_X &= |(\Psi x_k)(t) - (\Psi x_{k-1})(t)|_X \le (M|x_k - x_{k-1}|_X)(t)\\ &\le \cdots \le (M^k|x_1 - x_0|_X)(t) \quad (0 \le t \le T).\end{aligned}$$

Inasmuch as $d(M) < 1$, $|M^k|_{C(0,T)} \to 0$ $(k \to \infty)$. Thus some iteration of Ψ is a contraction in $C([0,T],X)$. This proves the result.

LEMMA 3.2 *Let the Cauchy problem for the Eq.* (1.7) *be well posed. Let F have the Lipschitz property. Then for any initial function $\Phi \in C([-\eta,0],X)$, Eq.* (1.4) *has a unique mild solution $x(t) \in C([-\eta,\infty],X)$.*

Proof For a finite $T > \psi(0)$ define in $C([0,T],X)$ a mapping W by the relations

$$(Wu)(t) = \begin{cases} f(t) & \text{if } 0 \le t < \psi(0), \\ f(t) + \int_{\psi(0)}^{t} U(t,s)F(s,u(h(s)))\,\mathrm{d}s & \text{if } \psi(0) \le t \le T \end{cases}$$

$$u \in C([0,T],X), \tag{3.1}$$

where

$$f(t) = U(t,0)\Phi(0) + \int_0^{\psi(0)} U(t,s)F(s,\Phi(h(s)))\,\mathrm{d}s.$$

Then (1.8) takes the form

$$v = Wv. \tag{3.2}$$

Let μ be the Lipschitz constant for F, then the following inequalities hold:

$$|(Wx)(t) - (Wy)(t)|_X = 0 \quad \text{if } 0 \le t < \psi(0),$$

and

$$|(Wx)(t) - (Wy)(t)|_X \le m \int_{\psi(0)}^{t} |x(h(s)) - y(h(s))|_X\,\mathrm{d}s \text{ if } \psi(0) < t \le T$$

with

$$m = \sup_{t,s \in [0,T]} |U(t,s)|_X \mu,$$

and every $x, y \in C([0,T],X)$. Taking into account that $h(t) \le t$ we get

$$|(Wx)(t) - (Wy)(t)|_X \le m \int_0^t \dot{\psi}(z)|x(z) - y(z)|_X\,\mathrm{d}z \ (x, y \in C([0,T],X)).$$

It can be written

$$|(Wx)(t) - (Wy)(t)|_X \leq (V|x-y|_X)(t) \quad (x, y \in C([0,T], X)),$$

where V is a Volterra operator in $C(0,T)$ with the kernel $m\dot{\psi}(s)$. Since $\psi(s)$ is a bounded integrable function, the spectral radius $d(V)$ of V equals zero. Consequently, due to Lemma 3.1 Eq. (3.2) has a solution, as claimed.

LEMMA 3.3 *Let operator $A(t)$ be continuously differentiable on D_A. In addition, with any small enough $\delta > 0$ let the operator $I_X - \delta A(t)$ be invertible, and inequality* (2.1) *hold. Then the Cauchy problem for Eq.* (1.7) *is well posed. Moreover, its evolution operator $U(t,\tau)$ satisfies the inequality*

$$|U(t,s)|_X \leq \exp\left[\int_s^t a(\tau)\,\mathrm{d}\tau\right] \quad (s, t \geq 0). \tag{3.3}$$

Proof For some partitioning of a segment $[0,t]: 0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = t$ let us denote

$$U_{n,k} = (I - A(t_n^{(n)})\delta_n)^{-1}(I - A(t_{n-1}^{(n)})\delta_{n-1})^{-1} \cdots (I - A(t_{k+1}^{(n)})\delta_{k+1})^{-1},$$

for $k < n$, and $U_{n,n} = I$. Here $\delta_k = \delta_k^{(n)} = t_k^{(n)} - t_{k-1}^{(n)}$, $(k = 1, \ldots, n)$. As proved in Gil' (1996b, formula (2.5)), it can be written

$$u(t) = \lim U_{n0}u(s) \quad \text{as} \ \max_k \delta_k^{(n)} \to 0 \tag{3.4}$$

in the sense of the norm of space X. Due to (2.1)

$$\|U_{n,0}\| \leq \prod_{1 \leq k \leq n} (1 + a(t_k)\delta_k^{(n)}) \leq \exp\left[\sum_{k=1}^n a(t_k)\delta_k^{(n)}\right].$$

Now the desired assertion follows from equality (3.4).

Note that the previous lemma is a variant of formula (2.7) from Gil' (1996b) but in the form convenient for us.

Proof of Theorem 2.1

The existence of the unique mild solution is due to Lemmas 3.2 and 3.3. Due to relations (2.2) and (3.3) Eq. (1.8) yields

$$
\begin{aligned}
|x(t)|_X &\le \exp\left[\int_0^t \alpha(\tau)\,\mathrm{d}\tau\right]|x(0)|_X + \int_0^t \exp\left[\int_s^t \alpha(\tau)\,\mathrm{d}\tau\right] w(s)|x(h(s))|_X\,\mathrm{d}s \\
&= \exp\left[\int_0^t \alpha(\tau)\,\mathrm{d}\tau\right]\left(|\Phi(0)|_X + \int_0^{\psi(0)} \exp\left[\int_s^0 \alpha(\tau)\,\mathrm{d}\tau\right]\right. \\
&\quad \left.\times\, w(s)|\Phi(h(s))|_X\,\mathrm{d}s + \int_{\psi(0)}^t \exp\left[\int_s^0 \alpha(\tau)\mathrm{d}\tau\right] w(s)|x(h(s))|_X\,\mathrm{d}s\right) \\
&= \exp\left[\int_0^t \alpha(\tau)\,\mathrm{d}\tau\right]\left(e_0(\Phi) + \int_{\psi(0)}^t \exp\left[\int_s^0 \alpha(\tau)\,\mathrm{d}\tau\right]\right. \\
&\quad \left.\times\, w(s)|x(h(s))|_X\mathrm{d}s\right) \quad (t \ge \psi(0)).
\end{aligned}
$$

Put

$$
y(t) = |x(t)|_X \exp\left[-\int_0^t \alpha(\tau)\,\mathrm{d}\tau\right]. \tag{3.5}
$$

Then

$$
\begin{aligned}
y(t) &\le e_0(\Phi) + \int_{\psi(0)}^t \exp\left[\int_s^0 \alpha(\tau)\,\mathrm{d}\tau\right] w(s)y(h(s)) \exp\left[\int_0^{h(s)} \alpha(\tau)\,\mathrm{d}\tau\right]\mathrm{d}s \\
&= e_0(\Phi) + \int_{\psi(0)}^t q(s)y(h(s))\,\mathrm{d}s,
\end{aligned}
$$

or

$$
y(t) \le e_0(\Phi) + \int_0^{h(t)} \dot{\psi}(s_1)q(\psi(s_1))y(s_1)\,\mathrm{d}s_1.
$$

Set

$$
K(t,s) = \begin{cases} \dot{\psi}(z)q(\psi(z)) & \text{for } 0 \le s \le h(t), \\ 0 & \text{for } h(t) < s \le t. \end{cases}
$$

Then

$$y(t) \le e_0(\Phi) + \int_0^t K(t,s)y(s)\,ds \quad (t \ge 0).$$

By Daleckii and Krein (1974, Theorem 1.9.3) we get $y(t) \le z_1(t)$, where $z_1(t)$ is a solution of the equation

$$\begin{aligned} z_1(t) &= e_0(\Phi) + \int_0^t K(t,s)z_1(s)\,ds \\ &= e_0(\Phi) + \int_0^{h(t)} \dot{\psi}(s_1)q(\psi(s_1))z_1(s_1)\,ds_1. \end{aligned} \tag{3.6}$$

Clearly, this equation is equivalent to (2.4). Now (3.5) implies the required estimate (2.3).

STABILITY CONDITIONS

DEFINITION 4.1 We will say that the zero solution of Eq. (1.4) is absolutely stable in the class of nonlinearities (2.2) if for every initial function $\Phi \in C([-\eta, 0], X)$, Eq. (1.4) has under (2.2) a unique mild solution $x(t) \in C([-\eta, \infty), X)$, and there is a positive constant N independent of the specific form of the function F (but depending on $w(t)$) such that

$$\|x\|_{C([0,\infty),X)} \le N\|\Phi\|_{C([-\eta,0],X)}.$$

Since $h(t) \le t$, we get by Eq. (3.6) the inequality

$$z(t) \le e_0(\Phi) + \int_0^t \dot{\psi}(s_1)q(\psi(s_1))z(s_1)\,ds_1.$$

Due to the Gronwall inequality,

$$z(t) \le e_0(\Phi)\exp\left\{\int_0^t \dot{\psi}(s_1)q(\psi(s_1))\,ds_1\right\}.$$

Thus Theorem 2.1 gives the estimate

$$
\begin{aligned}
|x(t)|_X &\le e_0(\Phi)\exp\left\{\int_0^t [a(\tau) + \dot{\psi}(\tau)q(\psi(\tau))]\,\mathrm{d}\tau\right\} \\
&= e_0(\Phi)\exp\left\{\int_0^t a(\tau)\,\mathrm{d}\tau + \int_{\psi(0)}^{\psi(t)} q(s)\,\mathrm{d}s\right\} \quad (t \ge \psi(0)). \qquad (3.7)
\end{aligned}
$$

We thus have derived.

COROLLARY 4.2 *Under the hypothesis of Theorem* 2.1, *let the condition*

$$\overline{\lim_{t\to\infty}}\left\{\int_0^t a(\tau)\,\mathrm{d}\tau + \int_0^{\psi(t)} q(s)\,\mathrm{d}s\right\} < \infty$$

be fulfilled. Then the zero solution of Eq. (1.4) *is absolutely stable in the class of nonlinearities* (2.2).

Let the relation

$$\Lambda_1 \equiv \overline{\lim_{t\to\infty}}\, t^{-1}\left\{\int_0^t a(\tau)\,\mathrm{d}\tau + \int_0^{\psi(t)} q(s)\,\mathrm{d}s\right\} < \infty$$

hold. Then due to (3.7) for the Lyapunov exponent of a solution $x(t)$ of Eq. (1.4) we have the inequality

$$\overline{\lim_{t\to\infty}}\frac{\ln|x(t)|_X}{t} \le \Lambda_1.$$

In particular, the condition $\Lambda_1 < 0$ provides the global asymptotic stability.

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