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Operator Inequalities Associated with Hölder–McCarthy and Kantorovich Inequalities*

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We discuss operator inequalities associated with Hölder–McCarthy and Kantorovich inequalities. We give a complementary inequality of Hölder–McCarthy one as an extension of [2] and also we give an application to the order preserving power inequality.

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1 OPERATOR INEQUALITIES ASSOCIATED WITH HÖLDER–McCARTHY AND KANTOROVICH INEQUALITIES

An operator means a bounded linear operator on a Hilbert space H. The celebrated Kantorovich inequality asserts that if A is an operator on H such that $M \geq A \geq m > 0$, then $(A^{-1}x, x)(Ax, x) \leq (m+M)^2/4mM$ holds for every unit vector x in H. Many authors investigated a lot of papers on Kantorovich inequality, among others, there is a long research series of Mond–Pecaric, some of them are [6,7]. At first we give some operator inequalities associated with extension of Kantorovich inequality.

* Dedicated to Professor Yoji Hatakeyama on his 65th birthday with respect and affection.

THEOREM 1.1 *Let A be positive operator on a Hilbert space H satisfying $M \geq A \geq m > 0$. Also let $f(t)$ be a real valued continuous convex function on $[m, M]$. Then the following inequality holds for every unit vector x and for any real number q depending on* (i) *or* (ii);

$$(f(A)x, x) \leq \frac{(mf(M) - Mf(m))}{(q-1)(M-m)} \left(\frac{(q-1)(f(M) - f(m))}{q(mf(M) - Mf(m))} \right)^q (Ax, x)^q \tag{1.0}$$

under any one of the following conditions (i) *and* (ii) *respectively*:

(i) $f(M) > f(m), \ \dfrac{f(M)}{M} > \dfrac{f(m)}{m}$

and $\dfrac{f(m)}{m} q \leq \dfrac{f(M) - f(m)}{M - m} \leq \dfrac{f(M)}{M} q$

holds for a real number $q > 1$,

(ii) $f(M) < f(m), \ \dfrac{f(M)}{M} < \dfrac{f(m)}{m}$

and $\dfrac{f(m)}{m} q \leq \dfrac{f(M) - f(m)}{M - m} \leq \dfrac{f(M)}{M} q$

holds for a real number $q < 0$.

COROLLARY 1.2 *Let A be self-adjoint operator on a Hilbert space H satisfying $M \geq A \geq m > 0$. Then the following inequality holds for every unit vector x and for any real numbers p and q depending on* (i) *or* (ii);

$$(A^p x, x) \leq \frac{(mM^p - Mm^p)}{(q-1)(M-m)} \left(\frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)} \right)^q (Ax, x)^q \tag{1.1}$$

under any one of the following conditions (i) *and* (ii) *respectively*:

(i) $m^{p-1} q \leq \dfrac{M^p - m^p}{M - m} \leq M^{p-1} q$

holds for real numbers $p > 1$ *and* $q > 1$,

(ii) $\quad m^{p-1}q \leq \dfrac{M^p - m^p}{M - m} \leq M^{p-1}q$

holds for real numbers $p < 0$ *and* $q < 0$.

We cite the following lemma to give a proof of Theorem 1.1.

LEMMA 1.3 *Let* $h(t)$ *be defined by* (1.2) *on* $[m, M]$ $(M > m > 0)$ *and any real number* q *such that* $q \neq 0, 1$ *and any real numbers* K *and* k;

$$h(t) = \frac{1}{t^q}\left(k + \frac{K-k}{M-m}(t-m)\right). \tag{1.2}$$

Then $h(t)$ *has the following upper bound* (2.1) *on* $[m, M]$:

$$\frac{(mK - Mk)}{(q-1)(M-m)}\left(\frac{(q-1)(K-k)}{q(mK-Mk)}\right)^q, \tag{1.3}$$

where m, M, k, K *and* q *in* (1.3) *satisfy any one of the following conditions* (i) *and* (ii) *respectively*:

(i) $\quad K > k, \ \dfrac{K}{M} > \dfrac{k}{m}$ *and* $\dfrac{k}{m}q \leq \dfrac{K-k}{M-m} \leq \dfrac{K}{M}q$

holds for a real number $q > 1$,

(ii) $\quad K < k, \ \dfrac{K}{M} < \dfrac{k}{m}$ *and* $\dfrac{k}{m}q \leq \dfrac{K-k}{M-m} \leq \dfrac{K}{M}q$

holds for a real number $q < 0$.

Proof By an easy differential calculus, $h'(t_1) = 0$ when

$$t_1 = \frac{q}{(q-1)}\,\frac{(mK-Mk)}{(K-k)}$$

and it turns out that t_1 satisfies the required condition $t_1 \in [m, M]$ and also t_1 gives the upper bound (1.3) of $h(t)$ on $[m, M]$ under any one of the conditions (i) and (ii) respectively.

Proof of Theorem 1.1 As $f(t)$ is a real valued continuous convex function on $[m, M]$, we have

$$f(t) \leq f(m) + \frac{f(M) - f(m)}{M-m}(t-m) \quad \text{for any } t \in [m, M]. \tag{1.4}$$

By applying the standard operational calculus of positive operator A to (1.4) since $M \geq (Ax,x) \geq m$, we obtain for every unit vector x

$$(f(A)x,x) \leq f(m) + \frac{f(M)-f(m)}{M-m}((Ax,x)-m). \tag{1.5}$$

Multiplying $(Ax,x)^{-q}$ on both sides of (1.5), we have

$$(Ax,x)^{-q}(f(A)x,x) \leq h(t), \tag{1.6}$$

where

$$h(t) = (Ax,x)^{-q}\left(f(m) + \frac{f(M)-f(m)}{M-m}((Ax,x)-m)\right).$$

Then we obtain

$$(f(A)x,x) \leq \left[\max_{m \leq t \leq M} h(t)\right](Ax,x)^q. \tag{1.7}$$

Putting $K = f(M)$ and $k = f(m)$ in Theorem 1.1, then (i) and (ii) in Theorem 1.1 just correspond to (i) and (ii) in Lemma 1.3, so the proof is complete by (1.7) and Lemma 1.3.

Proof of Corollary 1.2 Put $f(t) = t^p$ for $p \notin [0,1]$ in Theorem 1.1. As $f(t)$ is a real valued continuous convex function on $[m,M]$, $M^p > m^p$ and $M^{p-1} > m^{p-1}$ hold for any $p > 1$, that is, $f(M) > f(m)$ and $f(M)/M > f(m)/m$ for any $p > 1$ and also $M^p < m^p$ and $M^{p-1} < m^{p-1}$ hold for any $p < 0$, that is, $f(M) < f(m)$ and $f(M)/M < f(m)/m$ for any $p < 0$ respectively. Whence the proof of Corollary 1.2 is complete by Theorem 1.1.

Remark 1.4 In case $q = p$ and every integer $p \neq 0, 1$, Corollary 1.2 is shown in [1].

Next we state the following result associated with Hölder–McCarthy and Kantorovich inequalities.

THEOREM 1.5 *Let A be positive operator on a Hilbert space H satisfying $M \geq A \geq m > 0$. Then the following inequality holds for every unit vector x:*

(i) *In case $p > 1$:* $\quad (Ax,x)^p \leq (A^p x,x) \leq K_+(m,M)(Ax,x)^p$, *where*

$$K_+(m,M)=\frac{(p-1)^{p-1}}{p^p}\frac{(M^p-m^p)^p}{(M-m)(mM^p-Mm^p)^{p-1}}.$$

(ii) *In case* $p<0$: $\quad (Ax,x)^p \le (A^p x,x) \le K_-(m,M)(Ax,x)^p$,
where

$$K_-(m,M)=\frac{(mM^p-Mm^p)}{(p-1)(M-m)}\left(\frac{(p-1)(M^p-m^p)}{p(mM^p-Mm^p)}\right)^p.$$

Proof As $f(t)=t^p$ is convex function for $p\notin[0,1]$, (i) and (ii) in Corollary 1.2 hold in case $p\notin[0,1]$ and $q=p$, so that the inequalities of the right-hand sides of (i) and (ii) hold by Corollary 1.2 and ones of the left-hand sides of (i) and (ii) follow by Hölder–McCarthy inequality [5].

COROLLARY 1.6 *Let A be positive operator on a Hilbert space H such that $M\ge A\ge m>0$. Then the following inequalities hold for every unit vector x in H:*

(i) $$(Ax,x)^p(A^{-1}x,x)\le\frac{p^p}{(p+1)^{p+1}}\frac{(m+M)^{p+1}}{mM},$$

(ii) $$(A^2x,x)\le\frac{P^p}{(p+1)^{p+1}}\frac{(m+M)^{p+1}}{(mM)^p}(Ax,x)^{p+1}$$

for any p such that $m/M\le p\le M/m$.

Proof (i) In (ii) of Corollary 1.2 we have only to put $p=-1$ and replacing q by $-p$ for $p>0$.

(ii) In (i) of Corollary 1.2 we have only to put $p=2$ and replacing q by $p+1$ for $p>0$.

When $p=1$ (i) in Corollary 1.6 becomes the Kantorovich inequality.

Recently the following interesting complementary inequality of Hölder–McCarthy inequality is shown in [2].

THEOREM A ([2]) *Let A and B be positive operators on a Hilbert space H satisfying $M_1\ge A\ge m_1>0$ and $M_2\ge B\ge m_2>0$. Let p and q be $p>1$ with $1/p+1/q=1$. Then the following inequality holds for every vector x.*

$$\begin{aligned}(B^q\sharp_{1/p}A^px,x)&\le(A^px,x)^{1/p}(B^qx,x)^{1/q}\\&\le\lambda(p,m_1/M_2^{q-1},M_1/m_2^{q-1})^{1/p}(B^q\sharp_{1/p}A^px,x),\end{aligned}$$

where

$$\lambda(p,m,M)=\left\{\frac{1}{p^{1/p}q^{1/q}}\frac{M^p-m^p}{(M-m)^{1/p}(mM^p-Mm^p)^{1/q}}\right\}^p.$$

We give the following extension of Theorem A by considering the case $p<0$ and $1>q>0$.

THEOREM 1.7 *Let A and B be positive operators on a Hilbert space H satisfying $M_1\geq A\geq m_1>0$ and $M_2\geq B\geq m_2>0$. Let p and q be conjugate real numbers with $1/p+1/q=1$. Then the following inequalities hold for every vector x and real numbers r and s:*

(i) *In case $p>1$, $q>1$, $r\geq 0$ and $s\geq 0$:*

$$\begin{aligned}(B^r\sharp_{1/p}A^s x,x)&\leq (A^s x,x)^{1/p}(B^r x,x)^{1/q}\\ &\leq K_+\left(\frac{m_1^{s/p}}{M_2^{r/p}},\frac{M_1^{s/p}}{m_2^{r/p}}\right)^{1/p}(B^r\sharp_{1/p}A^s x,x).\end{aligned}\tag{1.8}$$

(ii) *In case $p<0$, $1>q>0$, $r\geq 0$ and $s\leq 0$:*

$$\begin{aligned}(B^r\sharp_{1/p}A^s x,x)&\geq (A^s x,x)^{1/p}(B^r x,x)^{1/q}\\ &\geq K_-\left(\frac{m_1^{s/p}}{m_2^{r/p}},\frac{M_1^{s/p}}{M_2^{r/p}}\right)^{1/p}(B^r\sharp_{1/p}A^s x,x),\end{aligned}\tag{1.9}$$

where $K_+(,)$ and $K_-(,)$ are the same as defined in Theorem 1.5. In particular:

(i) *In case $p>1$ and $q>1$:*

$$\begin{aligned}(B^q\sharp_{1/p}A^p x,x)&\leq (A^p x,x)^{1/p}(B^q x,x)^{1/q}\\ &\leq K_+\left(\frac{m_1}{M_2^{q-1}},\frac{M_1}{m_2^{q-1}}\right)^{1/p}(B^q\sharp_{1/p}A^p x,x).\end{aligned}\tag{1.10}$$

(ii) *In case $p<0$ and $1>q>0$:*

$$\begin{aligned}(B^q\sharp_{1/p}A^p x,x)&\geq (A^p x,x)^{1/p}(B^q x,x)^{1/q}\\ &\geq K_-\left(\frac{m_1}{m_2^{q-1}},\frac{M_1}{M_2^{q-1}}\right)^{1/p}(B^q\sharp_{1/p}A^p x,x).\end{aligned}\tag{1.11}$$

Proof (i) In case $p>1$, $q>1$, $r\geq 0$ and $s\geq 0$. Theorem 1.5 ensures the following (1.12)

$$(Ax,x)\leq (A^p x,x)^{1/p}(x,x)^{1/q}\leq K_+^{1/p}(m,M)(Ax,x) \tag{1.12}$$

holds for every vector x. As $r\geq 0$ and $s\geq 0$, we have

$$M_2^{-r}m_1^s\leq m_1^sB^{-r}\leq B^{-r/2}A^sB^{-r/2}\leq M_1^sB^{-r}\leq M_1^sm_2^{-r},$$

that is, we have for $p>1$

$$M_2^{-r/p}m_1^{s/p}\leq (B^{-r/2}A^sB^{-r/2})^{1/p}\leq M_1^{s/p}m_2^{-r/p}.$$

Replacing A by $(B^{-r/2}A^sB^{-r/2})^{1/p}$ and also x by $B^{r/2}x$ in (1.12), we have

$$\begin{aligned}(B^r\sharp_{1/p}A^sx,x)&\leq (A^sx,x)^{1/p}(B^rx,x)^{1/q}\\ &\leq K_+\left(\frac{m_1^{s/p}}{M_2^{r/p}},\frac{M_1^{s/p}}{m_2^{r/p}}\right)^{1/p}(B^r\sharp_{1/p}A^sx,x).\end{aligned}$$

(ii) In case $p<0$, $1>q>0$, $r\geq 0$ and $s\leq 0$. Theorem 1.5 ensures the following (1.13)

$$(Ax,x)\geq (A^p x,x)^{1/p}(x,x)^{1/q}\geq K_-^{1/p}(m,M)(Ax,x) \tag{1.13}$$

holds for every vector x. As $p<0$ and $s\leq 0$ we have

$$M_2^{-r}M_1^s\leq M_1^sB^{-r}\leq B^{-r/2}A^sB^{-r/2}\leq m_1^sB^{-r}\leq m_1^sm_2^{-r},$$

that is, we have for $p<0$

$$M_2^{-r/p}M_1^{s/p}\geq (B^{-r/2}A^sB^{-r/2})^{1/p}\geq m_1^{s/p}m_2^{-r/p}.$$

Replacing A by $(B^{-r/2}A^sB^{-r/2})^{1/p}$ and also x by $B^{r/2}x$ in (1.13), we have

$$\begin{aligned}(B^r\sharp_{1/p}A^sx,x)&\geq (A^sx,x)^{1/p}(B^rx,x)^{1/q}\\ &\geq K_-\left(\frac{m_1^{s/p}}{m_2^{r/p}},\frac{M_1^{s/p}}{M_2^{r/p}}\right)^{1/p}(B^r\sharp_{1/p}A^sx,x).\end{aligned}$$

Put $s=p$ and $r=q$ in (1.8) and (1.9) respectively, we have (1.10) and (1.11) respectively. Whence the proof of Theorem 1.7 is complete.

Remark 1.8 We remark that (1.10) in Theorem 1.7 just equals to Theorem A and (1.10) is equivalent to (1.8) and also (1.11) is equivalent to (1.9).

2 APPLICATIONS OF THEOREM 1.5 TO ORDER PRESERVING POWER INEQUALITIES

$0<A\leq B$ ensures $A^p\leq B^p$ for any $p\in[0,1]$ by well-known Löwner–Heinz theorem. However it is well known that $0<A\leq B$ does not always ensure $A^p\leq B^p$ for any $p>1$. Related to this result, a simple proof of the following interesting result is given in [2].

THEOREM B [2] *Let $0<A\leq B$ and $0<m\leq A\leq M$. Then*

$$A^p\leq\left(\frac{M}{m}\right)^p B^p \quad \textit{for } p\geq 1.$$

Here we give more precise estimation than Theorem B as follows.

THEOREM 2.1 *Let A and B be positive operators on a Hilbert space H such that $M_1\geq A\geq m_1>0$, $M_2\geq B\geq m_2>0$ and $0<A\leq B$. Then*

$$\text{(1-A)} \qquad A^p\leq K_{1,p}B^p\leq\left(\frac{M_1}{m_1}\right)^{p-1}B^p,$$

and

$$\text{(2-B)} \qquad A^p\leq K_{2,p}B^p\leq\left(\frac{M_2}{m_2}\right)^{p-1}B^p$$

hold for any $p\geq 1$, where $K_{1,p}$ and $K_{2,p}$ are defined by the following:

$$K_{1,p}=\frac{(p-1)^{p-1}}{p^p(M_1-m_1)}\frac{(M_1^p-m_1^p)^p}{(m_1M_1^p-M_1m_1^p)^{p-1}} \tag{2.1}$$

and

$$K_{2,p}=\frac{(p-1)^{p-1}}{p^p(M_2-m_2)}\frac{(M_2^p-m_2^p)^p}{(m_2M_2^p-M_2m_2^p)^{p-1}}. \tag{2.2}$$

We prepare the following Proposition 2.2 to give a proof of Theorem 2.1.

PROPOSITION 2.2 *If* $x \geq 1$, *then*

$$\frac{(p-1)^{p-1}(x^p-1)^p}{p^p(x-1)(x^p-x)^{p-1}} \leq x^{p-1} \quad \textit{for } 1<p<\infty \tag{2.3}$$

and the equality holds iff $x \downarrow 1$.

We need the following three lemmas to give a proof of Proposition 2.2.

LEMMA 2.3 *Let* $1<p<\infty$, $1/p+1/q=1$. *If* $t \geq 1$, *then*

$$0 \leq (p-1)t - pt^{1/q} + 1 \tag{2.4}$$

and the equality holds iff $t=1$.

Proof Put $f(t)=(p-1)t-pt^{1/q}+1$. Then $f(1)=0$ and

$$f'(t) = (p-1)(1-t^{-1/p}) \geq 0$$

for $t \geq 1$ and $1<p<\infty$, so we have (2.4).

LEMMA 2.4 *Let* $1<p<\infty$. *If* $t \geq 1$, *then*

$$\frac{t^{1/p}}{t}\frac{(t-1)}{(t^{1/p}-1)} \leq p \tag{2.5}$$

holds and the equality holds iff $t \downarrow 1$.

Proof Multiplying (2.4) by $t^{1/p}$, then

$$0 \leq (p-1)tt^{1/p} - pt + t^{1/p},$$

that is,

$$(t-1)t^{1/p} \leq pt(t^{1/p}-1),$$

so we have (2.5).

LEMMA 2.5 *Let* $1<p<\infty$, $1/p+1/q=1$. *If* $t\geq 1$, *then*

$$\frac{t-1}{(t^{1/p}-1)^{1/p}(t^{1/q}-1)^{1/q}t^{2/pq}}\leq p^{1/p}q^{1/q} \tag{2.6}$$

holds and the equality holds iff $t\downarrow 1$.

Proof Taking $1/p$ exponent of (2.5) and also taking $1/q$ exponent of (2.5), we have

$$\left(\frac{t^{1/p}(t-1)}{t(t^{1/p}-1)}\right)^{1/p}\leq p^{1/p} \tag{2.7}$$

and

$$\left(\frac{t^{1/q}(t-1)}{t(t^{1/q}-1)}\right)^{1/q}\leq p^{1/q}, \tag{2.8}$$

multiplying (2.7) by (2.8), so we have (2.6).

Proof of Proposition 2.2 Modifying the right-hand side of (2.6), we have

$$\frac{t-1}{(t^{1/p}-1)^{1/p}(t^{1/q}-1)^{1/q}t^{2/pq}}\leq \frac{p}{(p-1)^{(p-1)/p}} \quad \text{for } t\geq 1$$

taking p exponent of the inequality stated above,

$$\frac{(t-1)^p}{(t^{1/p}-1)(t^{1/q}-1)^{p/q}t^{2/q}}\leq \frac{p^p}{(p-1)^{p-1}},$$

and putting $t=x^p$, so we have (2.9) for $1<p<\infty$ and $x\geq 1$

$$\frac{(x^p-1)^p}{(x^{p-1}-1)^{p-1}(x-1)x^{2p-2}}\leq \frac{p^p}{(p-1)^{p-1}} \tag{2.9}$$

holds and the equality holds iff $t\downarrow 1$, so the proof of Proposition 2.2 is complete since $p/q=p-1$ holds.

Proof of Theorem 2.1 We have only to consider $p>1$ since the result is trivial in case $p=1$. First of all, whenever $M\geq m>0$ we recall the following inequality by putting $x = M/m \geq 1$ in Proposition 2.2,

$$\frac{(p-1)^{p-1}}{p^p}\frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}} \leq \left(\frac{M}{m}\right)^{p-1} \quad \text{for } p > 1. \tag{2.10}$$

For $p > 1$, we have

$$\begin{aligned}(A^p x, x) &\leq K_{1,p}(Ax, x)^p \quad \text{by (i) of Theorem 1.5,}\\ &\leq K_{1,p}(Bx, x)^p \quad \text{by } 0 < A \leq B,\\ &\leq K_{1,p}(B^p x, x)\\ &\leq \left(\frac{M_1}{m_1}\right)^{p-1}(B^p x, x)\end{aligned}$$

and the third inequality follows by Hölder–McCarthy inequality and the last one follows by (2.10), so that we obtain (1-A).

As $0 < B^{-1} \leq A^{-1}$ and $M_2^{-1} \leq B^{-1} \leq m_2^{-1}$, then by applying (1-A) we have

$$\begin{aligned}B^{-p} &\leq \frac{(p-1)^{p-1}}{p^p(m_2^{-1} - M_2^{-1})}\frac{(m_2^{-p} - M_2^{-p})^p}{(M_2^{-1}m_2^{-p} - m_2^{-1}M_2^{-p})^{p-1}}A^{-p}\\ &= \frac{(p-1)^{p-1}}{p^p(M_2 - m_2)}\frac{(M_2^p - m_2^p)^p}{(m_2 M_2^p - M_2 m_2^p)^{p-1}}A^{-p}\\ &= K_{2,p}A^{-p}\\ &\leq \left(\frac{M_2}{m_2}\right)^{p-1}A^{-p},\end{aligned}$$

taking inverses of both sides of the above inequality, we obtain (2-B), so the proof of Theorem 2.1 is complete.

Remark 2.6 (1-A) and (2-B) of Theorem 2.1 are more precise estimation than Theorem B since $K_{j,p} \leq (M_j/m_j)^{p-1} \leq (M_j/m_j)^p$ holds for $j = 1, 2$ and $p \geq 1$.

Results different from Theorem 1.1 and related results are given in [3,4].

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References

[1] Ky Fan, Some matrix inequalities, *Abh. Math. Sem. Univ. Hamburg*, **29** (1966), 185–196.
[2] M. Fujii, S. Izumino, R. Nakamoto and Y. Seo, *Operator inequalities related to Cauchy–Schwarz and Hölder–McCarthy inequalities*, preprint.
[3] T. Furuta, *Axtensions of Mond–Pecaric Generalization of Kantorovich inequality*, preprint.
[4] T. Furuta, *Two extensions of Ky Fan generalization and Mond–Pecaric matrix version generalization of Kantorovich inequality*, preprint.
[5] C.A. McCarthy, c_p, *Israel J. Math.*, **5** (1967), 249–271.
[6] B. Mond and J.E. Pecaric, Convex inequalities in Hilbert spaces, *Houston Journal of Mathematics*, **19** (1993), 405–420.
[7] B. Mond and J.E. Pecaric, A matrix version of the Ky Fan Generalization of the Kantorovich inequality, *Linear and Multilinear Algebra*, **36** (1994), 217–221.