Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2011, Article ID 936428, 15 pages
doi:10.1155/2011/936428

Research Article

Boundedness and Nonemptiness of Solution Sets for Set-Valued Vector Equilibrium Problems with an Application

Ren-You Zhong,[1] Nan-Jing Huang,[1] and Yeol Je Cho[2]

[1] *Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China*
[2] *Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea*

Correspondence should be addressed to Yeol Je Cho, yjcho@gsnu.ac.kr

Received 25 October 2010; Accepted 19 January 2011

Academic Editor: K. Teo

This paper is devoted to the characterizations of the boundedness and nonemptiness of solution sets for set-valued vector equilibrium problems in reflexive Banach spaces, when both the mapping and the constraint set are perturbed by different parameters. By using the properties of recession cones, several equivalent characterizations are given for the set-valued vector equilibrium problems to have nonempty and bounded solution sets. As an application, the stability of solution set for the set-valued vector equilibrium problem in a reflexive Banach space is also given. The results presented in this paper generalize and extend some known results in Fan and Zhong (2008), He (2007), and Zhong and Huang (2010).

1. Introduction

Let X and Y be reflexive Banach spaces. Let K be a nonempty closed convex subset of X. Let $F : K \times K \to 2^Y$ be a set-valued mapping with nonempty values. Let P be a closed convex pointed cone in Y with $\operatorname{int} P \neq \emptyset$. The cone P induces a partial ordering in Y, which was defined by $y_1 \leq_P y_2$ if and only if $y_2 - y_1 \in P$. We consider the following set-valued vector equilibrium problem, denoted by SVEP(F, K), which consists in finding $x \in K$ such that

$$F(x, y) \cap (-\operatorname{int} P) = \emptyset, \quad \forall y \in K. \tag{1.1}$$

It is well known that (1.1) is closely related to the following dual set-valued vector equilibrium problem, denoted by DSVEP(F, K), which consists in finding $x \in$ K such that

$$F(y, x) \subset (-P), \quad \forall y \in K. \tag{1.2}$$

We denote the solution sets of SVEP(F, K) and DSVEP(F, K) by S and S^D, respectively.

Let (Z_1, d_1) and (Z_2, d_2) be two metric spaces. Suppose that a nonempty closed convex set $L \subset X$ is perturbed by a parameter u, which varies over (Z_1, d_1), that is, $L : Z_1 \to 2^X$ is a set-valued mapping with nonempty closed convex values. Assume that a set-valued mapping $F : X \times X \to 2^Y$ is perturbed by a parameter v, which varies over (Z_2, d_2), that is, $F : X \times X \times Z_2 \to 2^Y$. We consider a parametric set-valued vector equilibrium problem, denoted by SVEP$(F(\cdot,\cdot,v), L(u))$, which consists in finding $x \in L(u)$ such that

$$F(x, y, v) \cap (-\operatorname{int} P) = \emptyset, \quad \forall y \in L(u). \tag{1.3}$$

Similarly, we consider the parameterized dual set-valued vector equilibrium problem, denoted by DSVEP$(F(\cdot,\cdot,v), L(u))$, which consists in finding $x \in L(u)$ such that

$$F(y, x, v) \subset (-P), \quad \forall y \in L(u). \tag{1.4}$$

We denote the solution sets of SVEP$(F(\cdot,\cdot,v), L(u))$ and DSVEP$(F(\cdot,\cdot,v), L(u))$ by $S(u, v)$ and $S^D(u, v)$, respectively.

In 1980, Giannessi [1] extended classical variational inequalities to the case of vector-valued functions. Meanwhile, vector variational inequalities have been researched quite extensively (see, e.g., [2]). Inspired by the study of vector variational inequalities, more general equilibrium problems [3] have been extended to the case of vector-valued bifunctions, known as vector equilibrium problems. It is well known that the vector equilibrium problem provides a unified model of several problems, for example, vector optimization, vector variational inequality, vector complementarity problem, and vector saddle point problem (see [4–9]). In recent years, the vector equilibrium problem has been intensively studied by many authors (see, e.g., [1–3, 10–26] and the references therein).

Among many desirable properties of the solution sets for vector equilibrium problems, stability analysis of solution set is of considerable interest (see, e.g, [27–33] and the references therein). Assuming that the barrier cone of K has nonempty interior, McLinden [34] presented a comprehensive study of the stability of the solution set of the variational inequality, when the mapping is a maximal monotone set-valued mapping. Adly [35], Adly et al. [36], and Addi et al. [37] discussed the stability of the solution set of a so-called semicoercive variational inequality. He [38] studied the stability of variational inequality problem with either the mapping or the constraint set perturbed in reflexive Banach spaces. Recently, Fan and Zhong [39] extended the corresponding results of He [38] to the case that the perturbation was imposed on the mapping and the constraint set simultaneously. Very recently, Zhong and Huang [40] studied the stability analysis for a class of Minty mixed variational inequalities in reflexive Banach spaces, when both the mapping and the constraint set are perturbed. They got a stability result for the Minty mixed variational inequality with Φ-pseudomonotone mapping in a reflexive Banach space, when both the mapping and the constraint set are perturbed by different parameters, which generalized and extended some known results in [38, 39].

Inspired and motivated by the works mentioned above, in this paper, we further study the characterizations of the boundedness and nonemptiness of solution sets for set-valued vector equilibrium problems in reflexive Banach spaces, when both the mapping and the constraint set are perturbed. We present several equivalent characterizations for the vector equilibrium problem to have nonempty and bounded solution set by using the properties of recession cones. As an application, we show the stability of the solution set for the set-valued vector equilibrium problem in a reflexive Banach space, when both the mapping and the constraint set are perturbed by different parameters. The results presented in this paper extend some corresponding results of Fan and Zhong [39], He [38], Zhong and Huang [40] from the variational inequality to the vector equilibrium problem.

The rest of the paper is organized as follows. In Section 2, we recall some concepts in convex analysis and present some basic results. In Section 3, we present several equivalent characterizations for the set-valued vector equilibrium problems to have nonempty and bounded solution sets. In Section 4, we give an application to the stability of the solution sets for the set-valued vector equilibrium problem.

2. Preliminaries

In this section, we introduce some basic notations and preliminary results.

Let X be a reflexive Banach space and K be a nonempty closed convex subset of X. The symbols "$\rightarrow$" and "$\rightharpoonup$" are used to denote strong and weak convergence, respectively.

The barrier cone of K, denoted by $\operatorname{barr}(K)$, is defined by

$$\operatorname{barr}(K) := \left\{ x^* \in X^* : \sup_{x\in K} \langle x^*, x \rangle < \infty \right\}. \tag{2.1}$$

The recession cone of K, denoted by K_∞, is defined by

$$K_\infty := \{ d \in X : \exists t_n \longrightarrow 0, \exists x_n \in K, t_n x_n \rightharpoonup d \}. \tag{2.2}$$

It is known that for any given $x_0 \in K$,

$$K_\infty = \{ d \in X : x_0 + \lambda d \in K, \forall \lambda > 0 \}. \tag{2.3}$$

We give some basic properties of recession cones in the following result which will be used in the sequel. Let $\{K_i\}_{i\in I}$ be any family of nonempty sets in X. Then

$$\left(\bigcap_{i\in I} K_i \right)_\infty \subset \bigcap_{i\in I} (K_i)_\infty. \tag{2.4}$$

If, in addition, $\bigcap_{i\in I} K_i \neq \emptyset$ and each set K_i is closed and convex, then we obtain an equality in the previous inclusion, that is,

$$\left(\bigcap_{i\in I} K_i\right)_\infty = \bigcap_{i\in I} (K_i)_\infty. \tag{2.5}$$

Let $\Phi : K \to R \cup \{+\infty\}$ be a proper convex and lower semicontinuous function. The recession function Φ_∞ of Φ is defined by

$$\Phi_\infty(x) := \lim_{t\to+\infty} \frac{\Phi(x_0 + tx) - \Phi(x_0)}{t}, \tag{2.6}$$

where x_0 is any point in $\operatorname{Dom}\Phi$. Then it follows that

$$\Phi_\infty(x) := \lim_{t\to+\infty} \frac{\Phi(tx)}{t}. \tag{2.7}$$

The function $\Phi_\infty(\cdot)$ turns out to be proper convex, lower semicontinuous and so weakly lower semicontinuous with the property that

$$\Phi(u + v) \leq \Phi(u) + \Phi_\infty(v), \quad \forall u \in \operatorname{Dom}\Phi, v \in X. \tag{2.8}$$

Definition 2.1. A set-valued mapping $G : K \to 2^Y$ is said to be

(i) upper semicontinuous at $x_0 \in K$ if, for any neighborhood $\mathcal{N}(G(x_0))$ of $G(x_0)$, there exists a neighborhood $\mathcal{N}(x_0)$ of x_0 such that

$$G(x) \subset \mathcal{N}(G(x_0)), \quad \forall x \in \mathcal{N}(x_0); \tag{2.9}$$

(ii) lower semicontinuous at $x_0 \in K$ if, for any $y_0 \in G(x_0)$ and any neighborhood $\mathcal{N}(y_0)$ of y_0, there exists a neighborhood $\mathcal{N}(x_0)$ of x_0 such that

$$G(x) \bigcap \mathcal{N}(y_0) \neq \emptyset, \quad \forall x \in \mathcal{N}(x_0). \tag{2.10}$$

We say G is continuous at x_0 if it is both upper and lower semicontinuous at x_0, and we say G is continuous on K if it is both upper and lower semicontinuous at every point of K.

It is evident that G is lower semicontinuous at $x_0 \in K$ if and only if, for any sequence $\{x_n\}$ with $x_n \to x_0$ and $y_0 \in G(x_0)$, there exists a sequence $\{y_n\}$ with $y_n \in G(x_n)$ such that $y_n \to y_0$.

Definition 2.2. A set-valued mapping $G : K \to 2^Y$ is said to be weakly lower semicontinuous at $x_0 \in K$ if, for any $y_0 \in G(x_0)$ and for any sequence $\{x_n\} \in K$ with $x_n \rightharpoonup x_0$, there exists a sequence $y_n \in G(x_n)$ such that $y_n \to y_0$.

We say G is weakly lower semicontinuous on K if it is weakly lower semicontinuous at every point of K. By Definition 2.2, we know that a weakly lower semicontinuous mapping is lower semicontinuous.

Definition 2.3. A set-valued mapping $G : K \to 2^Y$ is said to be

(i) upper P-convex on K if for any x_1 and $x_2 \in K$, $t \in [0,1]$,

$$tG(x_1) + (1-t)G(x_2) \subset G(tx_1 + (1-t)x_2) + P; \tag{2.11}$$

(ii) lower P-convex on K if for any x_1 and $x_2 \in K$, $t \in [0,1]$,

$$G(tx_1 + (1-t)x_2) \subset tG(x_1) + (1-t)G(x_2) - P. \tag{2.12}$$

We say that G is P-convex if G is both upper P-convex and lower P-convex.

Definition 2.4. Let $\{A_n\}$ be a sequence of sets in X. We define

$$\omega\text{-}\limsup_{n\to\infty} A_n := \{x \in X : \exists \{n_k\}, \qquad x_{n_k} \in A_{n_k} \text{ such that } x_{n_k} \rightharpoonup x\}. \tag{2.13}$$

Lemma 2.5 (see [36]). *Let K be a nonempty closed convex subset of X with* $\mathrm{int}(\mathrm{barr}(K)) \neq \emptyset$. *Then there exists no sequence $\{x_n\} \subset K$ such that $\|x_n\| \to \infty$ and $x_n/\|x_n\| \rightharpoonup 0$.*

Lemma 2.6 (see [39]). *Let K be a nonempty closed convex subset of X with* $\mathrm{int}(\mathrm{barr}(K)) \neq \emptyset$. *Then there exists no sequence $\{d_n\} \subset K_\infty$ with each $\|d_n\| = 1$ such that $d_n \rightharpoonup 0$.*

Lemma 2.7 (see [39]). *Let (Z,d) be a metric space and $u_0 \in Z$ be a given point. Let $L : Z \mapsto 2^X$ be a set-valued mapping with nonempty values and let L be upper semicontinuous at u_0. Then there exists a neighborhood U of u_0 such that $(L(u))_\infty \subset (L(u_0))_\infty$ for all $u \in U$.*

Lemma 2.8 (see [41]). *Let K be a nonempty convex subset of a Hausdorff topological vector space E and $G : K \to 2^E$ be a set-valued mapping from K into E satisfying the following properties:*

(i) *G is a* KKM *mapping, that is, for every finite subset A of K,* $\mathrm{co}(A) \subset \bigcup_{x\in A} G(x)$;

(ii) *$G(x)$ is closed in E for every $x \in K$;*

(iii) *$G(x_0)$ is compact in E for some $x_0 \in K$.*

Then $\bigcap_{x\in K} G(x) \neq \emptyset$.

3. Boundedness and Nonemptiness of Solution Sets

In this section, we present several equivalent characterizations for the set-valued vector equilibrium problem to have nonempty and bounded solution set. First of all, we give some assumptions which will be used for next theorems.

Let K be a nonempty convex and closed subset of X. Assume that $F : K \times K \to 2^Y$ is a set-valued mapping satisfying the following conditions:

(f_0) for each $x \in K$, $F(x,x) = 0$;

(f_1) for each $x, y \in K$, $F(x,y) \cap (-\operatorname{int} P) = \emptyset$ implies that $F(y,x) \subset (-P)$;

(f_2) for each $x \in K$, $F(x,\cdot)$ is P-convex on K;

(f_3) for each $x \in K$, $F(x,\cdot)$ is weakly lower semicontinuous on K;

(f_4) for each $x, y \in K$, the set $\{\xi \in [x,y] : F(\xi,y) \bigcap (-\operatorname{int} P) = \emptyset\}$ is closed, here $[x,y]$ stands for the closed line segment joining x and y.

Remark 3.1. If

$$F(x,y) = \langle Ax, y - x\rangle + \Phi(y) - \Phi(x), \quad \forall x, y \in K, \tag{3.1}$$

where $A{:}K \to 2^{X^*}$ is a set-valued mapping, $\Phi : K \to \mathbb{R} \bigcup \{+\infty\}$ is a proper, convex, lower semicontinuous function and $P = \mathbb{R}^+$, then condition (f_1) reduces to the following Φ-pseudomonotonicity assumption which was used in [40]. (See [40, Definition 2.2(iii)] of [40]): for all $(x,x^*), (y,y^*)$ in the graph(A),

$$\langle x^*, y - x\rangle + \Phi(y) - \Phi(x) \geq 0 \Longrightarrow \langle y^*, y - x\rangle + \Phi(y) - \Phi(x) \geq 0. \tag{3.2}$$

Remark 3.2. If, for each $y \in K$, the mapping $F(\cdot,y)$ is lower semicontinuous in K, then condition (f_4) is fulfilled. Indeed, for each $x, y \in K$ and for any sequence $\{\xi_n\} \subset \{\xi \in [x,y] : F(\xi,y) \bigcap (-\operatorname{int} P) = \emptyset\}$ with $\xi_n \to \xi_0$, we have $\xi_0 \in [x,y]$ and $F(\xi_0,y) \bigcap (-\operatorname{int} P) = \emptyset$. By the lower semicontinuity of $F(\cdot,y)$, for any $z \in F(\xi_0,y)$, there exists $z_n \in F(\xi_n,y)$ such that $z_n \to z$. Since $F(\xi_n,y) \bigcap (-\operatorname{int} P) = \emptyset$, we have $z_n \in Y \setminus (-\operatorname{int} P)$ and so $z \in Y \setminus (-\operatorname{int} P)$ by the closedness of $Y \setminus (-\operatorname{int} P)$. This implies that $F(\xi_0,y) \bigcap (-\operatorname{int} P) = \emptyset$ and the set $\{\xi \in [x,y] : F(\xi,y) \bigcap (-\operatorname{int} P) = \emptyset\}$ is closed.

The following example shows that conditions (f_0)–(f_4) can be satisfied.

Example 3.3. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $P = \mathbb{R}^2_+$ and $K = [1,2]$. Let

$$F(x,y) = \left(y - x, [1, 1 + x](y - x)\right)^{\top}, \quad \forall x, y \in K. \tag{3.3}$$

It is obvious that (f_0) holds. Since for each $x, y \in K$, $F(x,\cdot)$ and $F(\cdot,y)$ are lower semicontinuous on K, by Remark 3.2, we known that conditions (f_3) and (f_4) hold. For each $x, y \in K$, if $F(x,y) \cap (-R^2_+) = \emptyset$, then we have $y - \mathrm{x} \geq 0$. This implies that

$$F(y,x) = \left(x - y, [1, 1 + y](x - y)\right)^{\top} \subset \left(-R^2_+\right) \tag{3.4}$$

and so (f_1) holds. Moreover, for each $x \in K$, $y_1, y_2 \in K$ and $t_1, t_2 \in [0,1]$ with $t_1 + t_2 = 1$, it is easy to verify that

$$F(x, t_1 y_1 + t_2 y_2) = t_1 F(x, y_1) + t_2 F(x, y_2) \tag{3.5}$$

which shows that $F(x,\cdot)$ is R^2_+-convex on K and so (f_2) holds. Thus, F satisfies all conditions (f_0)–(f_4).

Theorem 3.4. *Let K be a nonempty closed convex subset of X and $F : K\times K \to 2^Y$ be a set-valued mapping satisfying assumptions (f_0)-(f_4). Then $S = S^D$.*

Proof. From the assumption (f_1), it is easy to see that $S \subset S^D$. We now prove that $S^D \subset S$. Let $x \in S^D$. Then for all $y \in K$, $F(y,x) \subset (-P)$. Set $x_t = x + t(y - x)$, where $t \in [0,1]$. Clearly, $x_t \in K$. From the upper P-convexity of $F(x,\cdot)$, we have

$$(1-t)F(x_t,x) + tF(x_t,y) \subset F(x_t,x_t) + P. \tag{3.6}$$

Since $F(x_t,x) \subset (-P)$, we obtain

$$tF(x_t,y) \subset -(1-t)F(x_t,x) + 0 + P \subset P + P \subset P. \tag{3.7}$$

This implies that $F(x_t,y) \subset P$ and so $F(x_t,y) \cap (-\operatorname{int} P) = \emptyset$. Letting $t \to 0^+$, by assumption (f_4), we have $F(x,y) \cap (-\operatorname{int} P) = \emptyset$. Thus, $x \in S$ and $S^D \subset S$. This completes the proof. □

Theorem 3.5. *Let K be a nonempty closed convex subset of X and $F : K\times K \to 2^Y$ be a set-valued mapping satisfying assumptions (f_0)–(f_4). If the solution set S is nonempty, then*

$$S_\infty = S^D_\infty = R_1 := \bigcap_{y\in K}\{d \in K_\infty : F(y, y+\lambda d) \subset (-P),\ \forall\lambda > 0\}. \tag{3.8}$$

Proof. From the proof of Theorem 3.4, we know that

$$S = S^D = \{x \in K : F(y,x) \subset (-P), \forall y \in K\} = \bigcap_{y\in K}\{x \in K : F(y,x) \subset (-P)\}. \tag{3.9}$$

Let $S_y = \{x \in X : F(y,x) \subset (-P)\}$. Then $S = S^D = \bigcap_{y\in K}(K \cap S_y)$. By the assumptions (f_2) and (f_3), we know that the set S_y is nonempty closed and convex. It follows from (2.5) and Theorem 3.4 that

$$\begin{aligned} S_\infty = S^D_\infty &= \left(\bigcap_{y\in K} K\cap S_y\right)_\infty = \bigcap_{y\in K}(K\cap S(y))_\infty \\ &= \bigcap_{y\in K} K_\infty \cap (S(y))_\infty \\ &= \bigcap_{y\in K}\{d \in K_\infty : d \in (S(y)_\infty)\} \\ &= \bigcap_{y\in K}\{d \in K_\infty : y+\lambda d \in S(y),\ \forall\lambda > 0\} \\ &= \bigcap_{y\in K}\{d \in K_\infty : F(y, y+\lambda d) \subset -P,\ \forall\lambda > 0\}. \end{aligned} \tag{3.10}$$

Then this completes the proof. □

Remark 3.6. If

$$F(y,x) = \langle Ay, x-y\rangle + \Phi(x) - \Phi(y), \quad \forall x, y \in K, \tag{3.11}$$

where $A : K \to 2^{X^*}$ is a set-valued mapping, $\Phi : K \to \mathbb{R}\bigcup\{+\infty\}$ is a proper, convex, lower semicontinuous function and $P = \mathbb{R}^+$, then it follows from (3.8) and (2.8) that

$$\begin{aligned} S_\infty^D &= \bigcap_{y\in K}\{d \in K_\infty : F(y, y+\lambda d) \subset (-P),\ \forall \lambda > 0\} \\ &= K_\infty \cap \{d \in X : \langle y^*, y+\lambda d - y\rangle + \Phi(y+\lambda d) - \Phi(y) \le 0, \forall y \in K, y^* \in A(y),\ \forall \lambda > 0\} \\ &= K_\infty \cap \{d \in X : \langle y^*, d\rangle + \Phi_\infty(d) \le 0, \forall y^* \in A(K)\}. \end{aligned} \tag{3.12}$$

Thus, we know that Theorem 3.5 is a generalization of [40, Theorem 3.1]. Moreover, by [40, Remark 3.1], Theorem 3.5 is also a generalization of [38, Lemma 3.1].

Theorem 3.7. *Let K be a nonempty closed convex subset of X and $F : K \times K \to 2^Y$ be a set-valued mapping satisfying assumptions (f_0)–(f_4). Suppose that* $\text{int}(\text{barr}(K) \neq \emptyset$. *Then the following statements are equivalent:*

(i) *the solution set of* SVEP(F,K) *is nonempty and bounded;*

(ii) *the solution set of* DSVEP(F,K) *is nonempty and bounded;*

(iii) $R_1 = \bigcap_{y\in K}\{d \in K_\infty : F(y, y+\lambda d) \subset (-P),\ \forall \lambda > 0\} = \{0\}$;

(iv) *there exists a bounded set $C \subset K$ such that for every $x \in K \setminus C$, there exists some $y \in C$ such that $F(y,x) \not\subset (-P)$.*

Proof. The implications (i)⇔(ii) and (ii)⇒(iii) follow immediately from Theorems 3.4 and 3.5 and the definition of recession cone.

Now we prove that (iii) implies (iv). If (iv) does not hold, then there exists a sequence $\{x_n\} \subset K$ such that for each n, $\|x_n\| \ge n$ and $F(y, x_n) \subset (-P)$ for every $y \in K$ with $\|y\| \le n$. Without loss of generality, we may assume that $d_n = x_n/\|x_n\|$ weakly converges to d. Then $d \in K_\infty$ by the definition of the recession cone. Since $\text{int}(\text{barr}K) \neq \emptyset$, by Lemma 2.5, we know that $d \neq 0$. Let $y \in K$ and $\lambda > 0$ be any fixed points. For n sufficiently large, by the lower P-convexity of $F(y,\cdot)$,

$$F\left(y, \left(1-\frac{\lambda}{\|x_n\|}\right)y + \frac{\lambda}{\|x_n\|}x_n\right) \subset \left(1-\frac{\lambda}{\|x_n\|}\right)F(y,y) + \frac{\lambda}{\|x_n\|}F(y,x_n) - P \subset 0 - P - P \subset -P. \tag{3.13}$$

Since

$$\left(1-\frac{\lambda}{\|x_n\|}\right)y + \frac{\lambda}{\|x_n\|}x_n \rightharpoonup y + \lambda d \tag{3.14}$$

and $F(y,\cdot)$ is weakly lower semicontinuous, we know that $F(y, y+\lambda d) \subset -P$ and so $d \in R_1$. However, it contradicts the assumption that $R_1 = \{0\}$. Thus (iv) holds.

Since (i) and (ii) are equivalent, it remains to prove that (iv) implies (ii). Let $G : K \to 2^K$ be a set-valued mapping defined by

$$G(y) := \{x \in K : F(y,x) \subset (-P)\}, \quad \forall y \in K. \tag{3.15}$$

We first prove that $G(y)$ is a closed subset of K. Indeed, for any $x_n \in G(y)$ with $x_n \to x_0$, we have $F(y,x_n) \subset (-P)$. It follows from the weakly lower semicontinuity of $F(y,\cdot)$ that $F(y,x_0) \subset (-P)$. This shows that $x_0 \in G(y)$ and so $G(y)$ is closed.

We next prove that G is a KKM mapping from K to K. Suppose to the contrary that there exist $t_1,t_2,\dots,t_n \in [0,1]$ with $t_1+t_2+\cdots+t_n = 1$, $y_1,y_2,\dots,y_n \in K$ and $\overline{y} = t_1y_1+t_2y_2+\cdots+t_ny_n \in \operatorname{co}\{y_1,y_2,\dots,y_n\}$ such that $\overline{y} \notin \cup_{i\in\{1,2,\dots,n\}} G(y_i)$. Then

$$F(y_i,\overline{y}) \not\subset (-P), \quad i = 1,2,\dots,n. \tag{3.16}$$

By assumption (f_1), we have

$$F(\overline{y},y_i) \cap (-\operatorname{int} P) \neq \emptyset, \quad i = 1,2,\dots,n. \tag{3.17}$$

It follows from the upper P-convexity of $F(\overline{y},\cdot)$ that

$$t_1F(\overline{y},y_1) + t_2F(\overline{y},y_2) + \cdots + t_nF(\overline{y},y_n) \subset F(\overline{y},\overline{y}) + P \subset P, \tag{3.18}$$

which is a contradiction with (3.17). Thus we know that G is a KKM mapping.

We may assume that C is a bounded closed convex set (otherwise, consider the closed convex hull of C instead of C). Let $\{y_1,\dots,y_m\}$ be finite number of points in K and let $M := \operatorname{co}(C \cup \{y_1,\dots,y_m\})$. Then the reflexivity of the space X yields that M is weakly compact convex. Consider the set-valued mapping G' defined by $G'(y) := G(y) \cap M$ for all $y \in M$. Then each $G'(y)$ is a weakly compact convex subset of M and G' is a KKM mapping. We claim that

$$\emptyset \neq \bigcap_{y\in M} G'(y) \subset C. \tag{3.19}$$

Indeed, by Lemma 2.8, intersection in (3.19) is nonempty. Moreover, if there exists some $x_0 \in \bigcap_{y\in M} G'(y)$ but $x_0 \notin C$, then by (iv), we have $F(y,x_0) \not\subset (-P)$ for some $y \in C$. Thus, $x_0 \notin G(y)$ and so $x_0 \notin G'(y)$, which is a contradiction to the choice of x_0.

Let $z \in \bigcap_{y\in M} G'(y)$. Then $z \in C$ by (3.19) and so $z \in \bigcap_{i=1}^m (G(y_i) \cap C)$. This shows that the collection $\{G(y) \cap C : y \in K\}$ has finite intersection property. For each $y \in K$, it follows from the weak compactness of $G(y) \cap C$ that $\bigcap_{y\in K}(G(y) \cap C)$ is nonempty, which coincides with the solution set of DSVEP(F,K). □

Remark 3.8. Theorem 3.7 establishes the necessary and sufficient conditions for the vector equilibrium problem to have nonempty and bounded solution sets. If

$$F(y,x) = \langle Ay, x-y\rangle + \Phi(x) - \Phi(y), \quad \forall x,y \in K, \tag{3.20}$$

where $A : K \to 2^{X^*}$ is a set-valued mapping, $\Phi : K \to \mathbb{R}\bigcup\{+\infty\}$ is a proper, convex, lower semicontinuous function and $P = \mathbb{R}^+$, then problem (1.2) reduces to the following Minty mixed variational inequality: finding $x \in K$ such that

$$\langle y^*, y - x\rangle + \Phi(y) - \Phi(x) \geq 0, \quad \forall y \in K, y^* \in A(y), \tag{3.21}$$

which was considered by Zhong and Huang [40]. Therefore, Theorem 3.7 is a generalization of [40, Theorem 3.2]. Moreover, by [40, Remark 3.2], Theorem 3.7 is also a generalization of Theorem 3.4 due to He [38].

Remark 3.9. By using a asymptotic analysis methods, many authors studied the necessary and sufficient conditions for the nonemptiness and boundedness of the solution sets to variational inequalities, optimization problems, and equilibrium problems, we refer the reader to references [42–49] for more details.

4. An Application

As an application, in this section, we will establish the stability of solution set for the set-valued vector equilibrium problem when the mapping and the constraint set are perturbed by different parameters.

Let (Z_1, d_1) and (Z_2, d_2) be two metric spaces. $F : X \times X \times Z_2 \to 2^Y$ is a set-valued mapping satisfying the following assumptions:

(f_0') for each $u \in Z_1$, $v \in Z_2$, $x \in L(u)$, $F(x, x, v) = 0$;

(f_1') for each $u \in Z_1$, $v \in Z_2$, $x, y \in L(u)$, $F(x, y, v) \cap (-\operatorname{int} P) = \emptyset$ implies that $F(y, x, v) \subset (-P)$;

(f_2') for each $u \in Z_1$, $v \in Z_2$, $x \in L(u)$, $F(x, \cdot, v)$ is P-convex on $L(u)$;

(f_3') for each $u \in Z_1, v \in Z_2$, $x, y \in L(u)$ and $z \in F(x, y, v)$, for any sequences $\{x_n\}$, $\{y_n\}$ and $\{v_n\}$ with $x_n \to x$, $y_n \rightharpoonup y$ and $v_n \to v$, there exists a sequence $\{z_n\}$ with $z_n \in F(x_n, y_n, v_n)$ such that $z_n \to z$.

The following Theorem 4.1 plays an important role in proving our results.

Theorem 4.1. *Let (Z_1, d_1) and (Z_2, d_2) be two metric spaces, $u_0 \in Z_1$ and $v_0 \in Z_2$ be given points. Let $L : Z_1 \to 2^X$ be a continuous set-valued mapping with nonempty closed convex values and* $\operatorname{int}(\operatorname{barr}(L(u_0))) \neq \emptyset$. *Suppose that $F : X \times X \times Z_2 \to 2^Y$ is a set-valued mapping satisfying the assumptions f_0'–f_3'. If*

$$R_1(u_0, v_0) = \bigcap_{y \in L(u_0)} \left\{d \in L(u_0)_\infty : F(y, y + \lambda d, v_0) \subset (-P), \forall \lambda > 0\right\} = \{0\}, \tag{4.1}$$

then there exists a neighborhood $U \times V$ of (u_0, v_0) such that

$$R_1(u, v) = \bigcap_{y \in L(u)} \left\{d \in L(u)_\infty : F(y, y + \lambda d, v) \subset (-P),\ \forall \lambda > 0\right\} = \{0\}, \quad \forall (u, v) \in U \times V. \tag{4.2}$$

Proof. Assume that the conclusion does not hold, then there exist a sequence $\{(u_n, v_n)\}$ in $Z_1 \times Z_2$ with $(u_n, v_n) \to (u_0, v_0)$ such that $R_1(u_n, v_n) \neq \{0\}$.

Since $R_1(u_n, v_n)$ is cone, we can select a sequence $\{d_n\}$ with $d_n \in R_1(u_n, v_n)$ such that $\|d_n\| = 1$ for every $n = 1, 2, \ldots$. As X is reflexive, without loss of generality, we can assume that $d_n \rightharpoonup d_0$, as $n \to +\infty$. Since L is a continuous set-valued mapping, hence, L is upper semicontinuous and lower semicontinuous at u_0. From the upper semicontinuity of L, by Lemma 2.7, we have $(L(u_n))_\infty \subset (L(u_0))_\infty$ as n large enough and hence $d_n \in (L(u_0))_\infty$ as n large enough. Since $(L(u_0))_\infty$ is a closed convex cone and hence weakly closed. This implies that $d_0 \in (L(u_0))_\infty$. Moreover, it follows from Lemma 2.6 that $d_0 \neq 0$.

For any $\lambda > 0$, $y \in L(u_0)$ and $y^* \in F(y, y + \lambda d_0, v_0)$, from the lower semicontinuity of L, there exists $y_n \in L(u_n)$ such that $y_n \to y$. Since $d_n \rightharpoonup d_0$, it follows that $y_n + \lambda d_n \rightharpoonup y + d_0$. Together with $v_n \to v_0$, from assumption (f_3'), there exists $y_n^* \in F(y_n, y_n + \lambda d_n, v_n)$ such that $y_n^* \to y^*$. Since $d_n \in R_1(u_n, v_n)$, we have $F(y_n, y_n + \lambda d_n, v_n) \subset (-P)$ and $y_n^* \in -P$. Letting $n \to \infty$, we obtain that $y^* \in (-P)$. Since $y \in L(u_0)$ and $y^* \in F(y, y+\lambda d_0, v_0)$ are arbitrary, from the above discussion, we obtain $d_0 \in R_1(u_0, v_0)$ with $d_0 \neq 0$. This contradicts our assumption that $R_1(u_0, v_0) = \{0\}$. This completes the proof. □

Remark 4.2. If

$$F(y, x, v) = \langle A(y, v), x - y \rangle + \Phi(x) - \Phi(y), \quad \forall x, y \in L(u), \tag{4.3}$$

where $A : X \times Z_2 \to 2^{X^*}$ is a set-valued mapping, $\Phi : X \to \mathbb{R} \bigcup \{+\infty\}$ is a proper, convex, lower semicontinuous function and $P = \mathbb{R}^+$, from Remark 3.6, we know that (4.1) and (4.2) in Theorem 4.1 reduce to (4.1) and (4.2) in [40, Theorem 4.1], respectively. Therefore, Theorem 4.1 is a generalization of [40, Theorem 4.1]. Moreover, by [40, Remark 4.1], Theorem 4.1 is also a generalization of [39, Theorem 3.1].

From Theorem 4.1, we derive the following stability result of the solution set for the vector equilibrium problem.

Theorem 4.3. *Let (Z_1, d_1) and (Z_2, d_2) be two metric spaces, $u_0 \in Z_1$ and $v_0 \in Z_2$ be given points. Let $L : Z_1 \mapsto 2^X$ be a continuous set-valued mapping with nonempty closed convex values and* $\mathrm{int}(\mathrm{barr}(L(u_0))) \neq \emptyset$. *Suppose that $F : X \times X \times Z_2 \to 2^Y$ is a set-valued mapping satisfying the assumptions $({f'}_0)$-$({f'}_3)$. If $S(u_0, v_0)$ is nonempty and bounded, then*

- (i) *there exists a neighborhood $U \times V$ of (u_0, v_0) such that for every $(u, v) \in U \times V$, $S(u, v)$ is nonempty and bounded;*
- (ii) $\omega\text{-}\limsup_{(u,v)\to(u_0,v_0)} S(u, v) \subset S(u_0, v_0)$.

Proof. If $S(u_0, v_0)$ is nonempty and bounded, then by Theorem 3.7 we have $R_1(u_0, v_0) = \{0\}$. It follows from Theorem 4.1 that there exists a neighborhood $U \times V$ of (u_0, v_0), such that $R_1(u, v) = \{0\}$ for every $(u, v) \in U \times V$. By using Theorem 3.7 again, we have $S(u, v)$ is nonempty and bounded for every $(u, v) \in U \times V$. This verifies the first assertion.

Next, we prove the second assertion $\omega\text{-}\limsup_{(u,v)\to(u_0,v_0)} S(u, v) \subset S(u_0, v_0)$. For any given sequence $\{(u_n, v_n)\} \in U \times V$ with $(u_n, v_n) \to (u_0, v_0)$, we need to prove that $\omega\text{-}\limsup_{n\to\infty} S(u_n, v_n) \subset S(u_0, v_0)$. Let $x \in \omega\text{-}\limsup_{n\to\infty} S(u_n, v_n)$. Then there exists a sequence $\{x_{n_j}\}$ with each $x_{n_j} \in S(u_{n_j}, v_{n_j})$ such that x_{n_j} weakly converges to x. We claim that there exists $z_{n_j} \in L(u_0)$ such that $\lim_{j\to\infty} \|x_{n_j} - z_{n_j}\| = 0$. Indeed, if the claim does hold, then there exist that a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ and some $\varepsilon_0 > 0$, such that $d(x_{n_{j_k}}, L(u_0)) \geq \varepsilon_0$, for all $k = 1, 2, \ldots$. This implies that $x_{n_{j_k}} \notin L(u_0) + \varepsilon_0 B(0,1)$ and so $L(u_{n_{j_k}}) \not\subset L(u_0) + \varepsilon_0 B(0,1)$, which contradicts with the upper semicontinuity of $L(\cdot)$. Thus, we have the claim. Moreover, we obtain $x \in L(u_0)$ as $L(u_0)$ is a closed convex subset of X and hence weakly closed.

Now we prove $F(y, x, v_0) \subset (-P)$ for all $y \in L(u_0)$ and hence $x \in S^D(u_0, v_0) = \mathrm{S}(u_0, v_0)$. For any $y \in L(u_0)$ and $y^* \in F(y, x, v_0)$, from the lower semicontinuity of L, there exist $y_{n_j} \in L(u_{n_j})$ such that $\lim_{j\to\infty} y_{n_j} = y$. Moreover, from assumption (f_3'), there exists a sequence of elements $y_{n_j}^* \in F(y_{n_j}, x_{n_j}, v_{n_j})$ such that $y_{n_j}^* \to y^*$. Since $x_{n_j} \in S(u_{n_j}, v_{n_j})$, we have $F(y_{n_j}, x_{n_j}, v_{n_j}) \subset (-P)$ and so $y_{n_j}^* \in -P$. Letting $j \to \infty$, we obtain that $y^* \in (-P)$. Since $y^* \in F(y, x, v_0)$ is arbitrary, we have $F(y, x, v_0) \subset (-P)$. This yields that $x \in S^D(u_0, v_0) = S(u_0, v_0)$. Thus, have the second assertion. This completes the proof. □

Remark 4.4. If

$$F(y,x,v) = \langle A(y,v), x-y\rangle + \Phi(x) - \Phi(y), \quad \forall x, y \in L(u), \tag{4.4}$$

where $A : X \times Z_2 \to 2^{X^*}$ is a set-valued mapping, $\Phi : X \to \mathbb{R}\bigcup\{+\infty\}$ is a proper, convex, lower semicontinuous function and $P = \mathbb{R}^+$, then problem (1.4) reduces to the following parametric Minty mixed variational inequality: finding $x \in L(u)$ such that

$$\langle y^*, y-x\rangle + \Phi(y) - \Phi(x) \geq 0, \quad \forall y \in L(u), y^* \in A(y,v), \tag{4.5}$$

which was considered by Zhong and Huang [40]. Therefore, Theorem 4.3 is a generalization of [40, Theorem 4.2]. Moreover, by [40, Remark 4.2], Theorem 4.3 ia also a generalizationof Theorems 4.1 and 4.4 due to He [38] and Theorem 3.5 due to Fan and Zhong [39].

The following examples show the necessity of the conditions of Theorem 4.3.

Example 4.5. Let $X = Y = \mathbb{R}$, $P = \mathbb{R}^+$, $Z_1 = Z_2 = [-1, 1]$ and $u_0 = v_0 = 0$,

$$L(u) \equiv [0,1], \qquad F(x,y,v) = \begin{cases} \{0\}, & v \neq 0, \\ y^2 - x^2, & v = 0. \end{cases} \tag{4.6}$$

Note that $L(\cdot)$ is continuous on Z_1. However, $F(\cdot,\cdot,\cdot)$ is not lower semicontinuous at $(1/2, 1/4, 0) \in X \times X \times Z_2$. Clearly, we have $S(0,0) = \{0\}$ and $S(0,v) = [0,1]$ for any $v \neq 0$. Thus,

$$\limsup_{v\to 0} S(0,v) = [0,1] \not\subset S(0,0). \tag{4.7}$$

Example 4.6. Let $X = Y = \mathbb{R}$, $P = \mathbb{R}^+$, $Z_1 = Z_2 = [-1, 1]$ and $u_0 = v_0 = 0$,

$$L(u) = \begin{cases} [2,3], & u = 0, \\ [1,3], & u \neq 0, \end{cases} \qquad F(x,y,v) = y^2 - x^2, \quad \text{for any } x, y \in L(u), v \in Z_2. \tag{4.8}$$

Note that F satisfies the assumptions (f_0')–(f_3'), and $L(u)$ is upper semicontinuous. However, $L(u)$ is not lower semicontinuous at $u = 0$. Clearly, we have $S(0,0) = \{1\}$ and $S(u,0) = \{2\}$ for any $u \neq 0$. Thus,

$$\limsup_{u\to 0} S(u,0) = \{2\} \not\subset S(0,0). \tag{4.9}$$

Example 4.7. Let $X = Y = \mathbb{R}$, $P = \mathbb{R}^+$, $Z_1 = Z_2 = [-1,1]$, $u_0 = v_0 = 0$,

$$L(u) = \begin{cases} [2,3], & u = 0, \\ [1,3], & u \neq 0, \end{cases} \qquad F(x,y,v) = y^2 - x^2, \quad \text{for any } x,y \in L(u), v \in Z_2. \tag{4.10}$$

Note that F satisfies the assumptions (f_0')–(f_3') and $L(u)$ is lower semicontinuous. However, $L(u)$ is not upper semicontinuous at $u = 0$. Clearly, we have $S(0,0) = \{2\}$ and $S(u,0) = \{1\}$ for any $u \neq 0$. Thus,

$$\limsup_{u\to 0} S(u,0) = \{1\} \not\subset S(0,0). \tag{4.11}$$

Acknowledgments

The authors are grateful to the editor and reviewers for their valuable comments and suggestions. This work was supported by the Key Program of NSFC (Grant no. 70831005), the National Natural Science Foundation of China (10671135) and the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00050).

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