

Research Article

Almost Sure Central Limit Theorem for Product of Partial Sums of Strongly Mixing Random Variables

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We give here an almost sure central limit theorem for product of sums of strongly mixing positive random variables.

1. Introduction and Results

In recent decades, there has been a lot of work on the almost sure central limit theorem (ASCLT), we can refer to Brosamler [1], Schatte [2], Lacey and Philipp [3], and Peligrad and Shao [4].

Khurelbaatar and Rempala [5] gave an ASCLT for product of partial sums of i.i.d. random variables as follows.

Theorem 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. positive random variables with $EX_1 = \mu > 0$ and $\text{Var}(X_1) = \sigma^2$. Denote $\gamma = \sigma/\mu$ the coefficient of variation. Then for any real x*

$$\lim_{n \rightarrow \infty} \frac{1}{\ln n} \sum_{k=1}^n \frac{1}{k} I \left(\left(\frac{\prod_{i=1}^k S_i}{k! \mu^k} \right)^{1/\gamma \sqrt{k}} \leq x \right) = F(x) \quad a.s., \quad (1.1)$$

where $S_n = \sum_{k=1}^n X_k$, $I(\cdot)$ is the indicator function, $F(\cdot)$ is the distribution function of the random variable $e^{-\mathcal{N}}$, and \mathcal{N} is a standard normal variable.

Recently, Jin [6] had proved that (1.1) holds under appropriate conditions for strongly mixing positive random variables and gave an ASCLT for product of partial sums of strongly mixing as follows.

Theorem 1.2. Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed positive strongly mixing random variable with $EX_1 = \mu > 0$ and $\text{Var}(X_1) = \sigma^2$, $d_k = 1/k$, $D_n = \sum_{k=1}^n d_k$. Denote by $\gamma = \sigma/\mu$ the coefficient of variation, $\sigma_n^2 = \text{Var}(\sum_{k=1}^n ((S_k - k\mu)/k\sigma))$ and $B_n^2 = \text{Var}(S_n)$. Assume

$$E|X_1|^{2+\delta} < \infty \quad \text{for some } \delta > 0, \quad \lim_{n \rightarrow \infty} \frac{B_n^2}{n} = \sigma_0^2 > 0, \quad (1.2)$$

$$\alpha(n) = O(n^{-r}) \quad \text{for some } r > 1 + \frac{2}{\delta}, \quad \inf_{n \in \mathbb{N}} \frac{\sigma_n^2}{n} > 0.$$

Then for any real x

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I \left(\left(\frac{\prod_{i=1}^k S_i}{k! \mu^k} \right)^{1/\gamma \sigma_k} \leq x \right) = F(x) \quad \text{a.s.} \quad (1.3)$$

The sequence $\{d_k, k \geq 1\}$ in (1.3) is called weight. Under the conditions of Theorem 1.2, it is easy to see that (1.3) holds for every sequence d_k^* with $0 \leq d_k^* \leq d_k$ and $D_n^* = \sum_{k \leq n} d_k^* \rightarrow \infty$ [7]. Clearly, the larger the weight sequence (d_k) is, the stronger is the result (1.3).

In the following sections, let $d_k = e^{(\ln k)^\alpha}/k$, $0 \leq \alpha < 1/2$, $D_n = \sum_{k=1}^n d_k$, " \ll " denote the inequality " \leq " up to some universal constant.

We first give an ASCLT for strongly mixing positive random variables.

Theorem 1.3. Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed positive strongly mixing random variable with $EX_1 = \mu > 0$ and $\text{Var}(X_1) = \sigma^2$, d_k and D_n as mentioned above. Denote by $\gamma = \sigma/\mu$ the coefficient of variation, $\sigma_n^2 = \text{Var}(\sum_{k=1}^n ((S_k - k\mu)/k\sigma))$ and $B_n^2 = \text{Var}(S_n)$. Assume that

$$E|X_1|^{2+\delta} < \infty \quad \text{for some } \delta > 0, \quad (1.4)$$

$$\alpha(n) = O(n^{-r}) \quad \text{for some } r > 1 + \frac{2}{\delta}, \quad (1.5)$$

$$\lim_{n \rightarrow \infty} \frac{B_n^2}{n} = \sigma_0^2 > 0, \quad (1.6)$$

$$\inf_{n \in \mathbb{N}} \frac{\sigma_n^2}{n} > 0. \quad (1.7)$$

Then for any real x

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I \left(\left(\frac{\prod_{i=1}^k S_i}{k! \mu^k} \right)^{1/\gamma \sigma_k} \leq x \right) = F(x) \quad \text{a.s.} \quad (1.8)$$

In order to prove Theorem 1.3 we first establish ASCLT for certain triangular arrays of random variables. In the sequel we shall use the following notation. Let $b_{k,n} = \sum_{i=k}^n (1/i)$ and $s_{k,n}^2 = \sum_{i=1}^k b_{i,n}^2$ for $k \leq n$ with $b_{k,n} = 0$ if $k > n$. $Y_k = (X_k - \mu)/\sigma$, $k \leq 1$, $\tilde{S}_n = \sum_{k=1}^n Y_k$ and $S_{n,n} = \sum_{k=1}^n b_{k,n} Y_k$.

In this setting we establish an ASCLT for the triangular array $(b_{k,n}Y_k)$.

Theorem 1.4. *Under the conditions of Theorem 1.3, for any real x*

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I \left\{ \frac{S_{k,k}}{\sigma_k} \leq x \right\} = \Phi(x) \quad \text{a.s.}, \quad (1.9)$$

where $\Phi(x)$ is the standard normal distribution function.

2. The Proofs

2.1. Lemmas

To prove theorems, we need the following lemmas.

Lemma 2.1 (see [8]). *Let $\{X_n, n \geq 1\}$ be a sequence of strongly mixing random variables with zero mean, and let $\{a_{k,n}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of real numbers. Assume that*

$$\sup_n \sum_{k=1}^n a_{k,n}^2 < \infty, \quad \max_{1 \leq k \leq n} |a_{k,n}| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.1)$$

If for a certain $\delta > 0$, $\{|X_k|^{2+\delta}\}$ is uniformly integrable, $\inf_k \text{Var}(X_k) > 0$,

$$\sum_{n=1}^{\infty} n^{2/\delta} \alpha(n) < \infty, \quad \text{Var} \left(\sum_{n=1}^n a_{k,n} X_k \right) = 1, \quad (2.2)$$

then

$$\sum_{k=1}^n a_{k,n} X_k \xrightarrow{d} \mathcal{N}(0, 1). \quad (2.3)$$

Lemma 2.2 (see [9]). *Let $d_k = e^{(\ln k)^\alpha}/k$, $0 \leq \alpha < 1/2$, $D_n = \sum_{k=1}^n d_k$; then*

$$D_n \sim C(\ln n)^{1-\alpha} \exp\{(\ln n)^\alpha\}, \quad (2.4)$$

where $C = 1/\alpha$ as $0 < \alpha < 1/2$, $C = 1$ as $\alpha = 0$.

Lemma 2.3 (see [8]). *Let $\{X_n, n \geq 1\}$ be a strongly mixing sequence of random variables such that $\sup_n E|X_n|^{2+\delta} < \infty$ for a certain $\delta > 0$ and every $n \geq 1$. Then there is a numerical constant $c(\delta)$ depending only on δ such that for every $n > 1$ one has*

$$\sup_j \sum_{i=j+1}^{n+j} |\text{Cov}(X_i, X_j)| \leq c(\delta) \left(\sum_{i=1}^n i^{2/\delta} \alpha(i) \right)^{\delta/(2+\delta)} \sup_k \|X_k\|_{2+\delta}^2, \quad (2.5)$$

where $\|X_k\|_p = E(|X_k|^p)^{1/p}$, $p > 1$.

Lemma 2.4 (see [9]). Let $\{\xi_k, k \geq 1\}$ be a sequence of random variables, uniformly bounded below and with finite variances, and let $\{d_k, k \geq 1\}$ be a sequence of positive number. Let for $n \geq 1$, $D_n = \sum_{k=1}^n d_k$ and $T_n = (1/D_n) \sum_{k=1}^n d_k \xi_k$. Assume that

$$D_n \rightarrow \infty \quad \frac{D_{n+1}}{D_n} \rightarrow 1, \quad (2.6)$$

as $n \rightarrow \infty$. If for some $\varepsilon > 0$, C and all n

$$ET_n^2 \leq C(\ln^{-1-\varepsilon} D_n), \quad (2.7)$$

then

$$T_n \xrightarrow{a.s.} 0 \quad \text{as } n \rightarrow \infty. \quad (2.8)$$

Lemma 2.5 (see [10]). Let $\{X_n, n \geq 1\}$ be a strongly mixing sequence of random variables with zero mean and $\sup_n E|X_n|^{2+\delta} < \infty$ for a certain $\delta > 0$. Assume that (1.5) and (1.6) hold. Then

$$\limsup_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{2\sigma_0^2 n \ln \ln n}} = 1 \quad a.s. \quad (2.9)$$

2.2. Proof of Theorem 1.4

From the definition of strongly mixing we know that $\{Y_k, k \geq 1\}$ remain to be a sequence of identically distributed strongly mixing random variable with zero mean and unit variance. Let $a_{k,n} = b_{k,n}/\sigma_n$; note that

$$\sum_{k=1}^n b_{k,n}^2 = b_{1,n} + 2 \sum_{k=2}^n \sum_{i=1}^{k-1} \frac{1}{k} = b_{1,n} + 2 \sum_{k=2}^n \frac{k-1}{k} = 2n - b_{1,n}, \quad n \geq 1, \quad (2.10)$$

and via (1.7) we have

$$\sup_n \sum_{k=1}^n a_{k,n}^2 = \sup_n \sum_{k=1}^n \frac{b_{k,n}^2}{\sigma_n^2} \ll \sup_n \frac{2n - b_{1,n}}{n} < \infty, \quad (2.11)$$

$$\max_{1 \leq k \leq n} |a_{k,n}| = \max_{1 \leq k \leq n} \frac{b_{k,n}}{\sigma_n} \ll \frac{\ln n}{\sqrt{n}} \rightarrow 0, \quad n \rightarrow \infty.$$

From the definition of Y_k and (1.4) we have that $\{|Y_k|^{2+\delta}\}$ is uniformly integrable; note that

$$\inf_k \text{Var}(Y_k) = EY_1^2 = 1 > 0, \quad \text{Var}\left(\sum_{k=1}^n a_{k,n} Y_k\right) = \frac{\text{Var}(\sum_{k=1}^n b_{k,n} Y_k)}{\sigma_n^2} = 1, \quad (2.12)$$

and applying (1.5)

$$\sum_{n=1}^{\infty} n^{2/\delta} \alpha(n) \ll \sum_{n=1}^{\infty} n^{-r+2/\delta} < \infty. \tag{2.13}$$

Consequently using Lemma 2.1, we can obtain

$$\frac{S_{n,n}}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } n \rightarrow \infty, \tag{2.14}$$

which is equivalent to

$$Ef\left(\frac{S_{n,n}}{\sigma_n}\right) \rightarrow Ef(\mathcal{N}) \quad \text{as } n \rightarrow \infty \tag{2.15}$$

for any bounded Lipschitz-continuous function f ; applying *Toeplitz Lemma*

$$\frac{1}{D_n} \sum_{k=1}^n d_k Ef\left(\frac{S_{k,k}}{\sigma_k}\right) \rightarrow Ef(\mathcal{N}) \quad \text{as } n \rightarrow \infty. \tag{2.16}$$

We notice that (1.9) is equivalent to

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k f\left(\frac{S_{k,k}}{\sigma_k}\right) = \Phi(x) \quad \text{a.s.} \tag{2.17}$$

for all bounded Lipschitz continuous f ; it therefore remains to prove that

$$T_n \triangleq \frac{1}{D_n} \sum_{k=1}^n d_k \left(f\left(\frac{S_{k,k}}{\sigma_k}\right) - Ef\left(\frac{S_{k,k}}{\sigma_k}\right) \right) \xrightarrow{\text{a.s.}} 0, \quad n \rightarrow \infty. \tag{2.18}$$

Let $\xi_k = f(S_{k,k}/\sigma_k) - Ef(S_{k,k}/\sigma_k)$,

$$\begin{aligned} E\left(\sum_{k=1}^n d_k \xi_k\right)^2 &\leq E\left(2 \sum_{1 \leq k < l \leq n} d_k d_l \xi_k \xi_l\right) \ll \sum_{1 \leq k < l \leq n} d_k d_l |E(\xi_k \xi_l)| \\ &= \sum_{\substack{1 \leq k < l \leq n \\ l \leq 2k}} d_k d_l |E(\xi_k \xi_l)| + \sum_{\substack{1 \leq k < l \leq n \\ l > 2k}} d_k d_l |E(\xi_k \xi_l)| \\ &\triangleq T_{1,n} + T_{2,n}. \end{aligned} \tag{2.19}$$

From Lemma 2.2, we obtain for some constant C_1

$$e^{(\ln n)^\alpha} \sim C_1 D_n (\ln D_n)^{1-1/\alpha}. \tag{2.20}$$

Using (2.20) and property of f , we have

$$T_{1,n} \ll e^{(\ln n)^\alpha} \sum_{k=1}^n d_k \sum_{l=k}^{2k} \frac{1}{l} \ll D_n e^{(\ln n)^\alpha} \ll D_n^2 (\ln D_n)^{1-1/\alpha}. \quad (2.21)$$

We estimate now $T_{2,n}$. For $l > 2k$,

$$\begin{aligned} S_{l,l} - S_{2k,2k} &= (b_{1,l}Y_1 + b_{2,l}Y_2 + \cdots + b_{l,l}Y_l) - (b_{1,2k}Y_1 + b_{2,2k}Y_2 + \cdots + b_{2k,2k}Y_{2k}) \\ &= b_{2k+1,l}\tilde{S}_{2k} + (b_{2k+1,l}Y_{2k+1} + \cdots + b_{l,l}Y_l). \end{aligned} \quad (2.22)$$

Notice that

$$\begin{aligned} |E\tilde{\xi}_k\tilde{\xi}_l| &= \left| \text{Cov} \left(f \left(\frac{S_{k,k}}{\sigma_k} \right), f \left(\frac{S_{l,l}}{\sigma_l} \right) \right) \right| \\ &\leq \left| \text{Cov} \left(f \left(\frac{S_{k,k}}{\sigma_k} \right), f \left(\frac{S_{l,l}}{\sigma_l} \right) - f \left(\frac{S_{l,l} - S_{2k,2k} - b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l} \right) \right) \right| \\ &\quad + \left| \text{Cov} \left(f \left(\frac{S_{k,k}}{\sigma_k} \right), f \left(\frac{S_{l,l} - S_{2k,2k} - b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l} \right) \right) \right|, \end{aligned} \quad (2.23)$$

and the properties of strongly mixing sequence imply

$$\left| \text{Cov} \left(f \left(\frac{S_{k,k}}{\sigma_k} \right), f \left(\frac{S_{l,l} - S_{2k,2k} - b_{2k+1,l}\tilde{S}_{2k}}{\sigma_l} \right) \right) \right| \ll \alpha(k). \quad (2.24)$$

Applying Lemma 2.3 and (2.10),

$$\begin{aligned} \text{Var}(S_{2k,2k}) &= \sum_{i=1}^{2k} b_{i,2k}^2 EY_i^2 + 2 \sum_{j=1}^{2k-1} \sum_{i=j+1}^{2k} b_{i,2k} b_{j,2k} \text{Cov}(Y_i, Y_j) \\ &\leq \sum_{i=1}^{2k} b_{i,2k}^2 + 2 \sum_{j=1}^{2k-1} b_{j,2k}^2 \sum_{i=j+1}^{2k} |\text{Cov}(Y_i, Y_j)| \ll k, \\ \text{Var}(\tilde{S}_{2k}) &= E \left(\sum_{i=1}^{2k} Y_i \right)^2 = \sum_{i=1}^{2k} EY_i^2 + 2 \sum_{i=1}^{2k-1} \sum_{j=i+1}^{2k} \text{Cov}(Y_i, Y_j) \ll k. \end{aligned} \quad (2.25)$$

Consequently, via the properties of f , the *Jensen* inequality, and (1.7),

$$\begin{aligned} & \left| \text{Cov} \left(f \left(\frac{S_{k,k}}{\sigma_k} \right), f \left(\frac{S_{l,l}}{\sigma_l} \right) - f \left(\frac{S_{l,l} - S_{2k,2k} - b_{2k+1,l} \tilde{S}_{2k}}{\sigma_l} \right) \right) \right| \\ & \ll \frac{E |S_{2k,2k} + b_{2k+1,l} \tilde{S}_{2k}|}{\sigma_l} \leq \frac{\sqrt{ES_{2k,2k}^2}}{\sigma_l} + \frac{\sqrt{E(b_{2k+1,l} \tilde{S}_{2k})^2}}{\sigma_l} \\ & = \frac{\sqrt{\text{Var}(S_{2k,2k})}}{\sigma_l} + b_{2k+1,l} \frac{\sqrt{\text{Var}(\tilde{S}_{2k})}}{\sigma_l} \ll \left(\frac{k}{l} \right)^\beta, \end{aligned} \tag{2.26}$$

where $0 < \beta < 1/2$. Hence for $l > 2k$ we have

$$|E\xi_k \xi_l| \ll \alpha(k) + \left(\frac{k}{l} \right)^\beta. \tag{2.27}$$

Consequently, we conclude from the above inequalities that

$$\begin{aligned} T_{2,n} & \ll \sum_{\substack{1 \leq k \leq l \leq n \\ l > 2k}} d_k d_l \left(\alpha(k) + \left(\frac{k}{l} \right)^\beta \right) \\ & = \sum_{\substack{1 \leq k \leq l \leq n \\ l > 2k}} d_k d_l \alpha(k) + \sum_{\substack{1 \leq k \leq l \leq n \\ l > 2k}} d_k d_l \left(\frac{k}{l} \right)^\beta \triangleq T_{2,n,1} + T_{2,n,2}. \end{aligned} \tag{2.28}$$

Applying (1.5) and Lemma 2.2 we can obtain for any $\eta > 0$

$$T_{2,n,1} \leq \sum_{k=1}^n \sum_{l=1}^n d_k d_l \alpha(k) \ll (\ln D_n)^{-1-\eta} \sum_{k=1}^n d_k \sum_{l=1}^n d_l = D_n^2 (\ln D_n)^{-1-\eta}. \tag{2.29}$$

Notice that

$$T_{2,n,2} = \sum_{\substack{1 \leq k \leq l \leq n \\ l > 2k \\ (l/k) \geq (\ln D_n)^{2/\beta}}} d_k d_l \left(\frac{k}{l} \right)^\beta + \sum_{\substack{1 \leq k \leq l \leq n \\ l > 2k \\ (l/k) < (\ln D_n)^{2/\beta}}} d_k d_l \left(\frac{k}{l} \right)^\beta \triangleq T_{2,n,2,1} + T_{2,n,2,2}, \tag{2.30}$$

$$T_{2,n,2,1} \leq \sum_{\substack{1 \leq k \leq l \leq n \\ l > 2k}} d_k d_l (\ln D_n)^{-2} \leq (\ln D_n)^{-2} \sum_{k=1}^n d_k \sum_{l=1}^n d_l = D_n^2 (\ln D_n)^{-2}. \tag{2.31}$$

Let $n_0 = \max\{l : k \leq l \leq n, (l/k) < (\ln D_n)^{2/\beta}\}$, then

$$\begin{aligned} T_{2,n,2,2} &\leq \sum_{k=1}^n \sum_{l=2k}^{n_0} d_k d_l \leq e^{(\ln n)^\alpha} \sum_{k=1}^n d_k \sum_{l=2k}^{n_0} \frac{1}{l} \ll e^{(\ln n)^\alpha} \sum_{k=1}^n d_k (\ln n_0 - \ln 2k) \\ &\ll e^{(\ln n)^\alpha} D_n \ln \ln D_n \ll D_n^2 \ln^{1-1/\alpha} D_n \ln \ln D_n. \end{aligned} \quad (2.32)$$

By (2.21), (2.29), (2.31), and (2.32), for some $\varepsilon > 0$ such that

$$ET_n^2 = \frac{1}{D_n^2} E \left(\sum_{k=1}^n d_k \xi_k \right)^2 \ll (\ln D_n)^{-1-\varepsilon}, \quad (2.33)$$

applying Lemma 2.4, we have

$$T_n \xrightarrow{\text{a.s.}} 0. \quad (2.34)$$

2.3. Proof of Theorem 1.3

Let $C_k = S_k / \mu k$; we have

$$\frac{1}{\gamma \sigma_n} \sum_{k=1}^n (C_k - 1) = \frac{1}{\gamma \sigma_n} \sum_{k=1}^n \left(\frac{S_k}{\mu k} - 1 \right) = \frac{1}{\sigma_n} \sum_{k=1}^n b_{k,n} Y_k = \frac{S_{n,n}}{\sigma_n}. \quad (2.35)$$

We see that (1.9) is equivalent to

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I \left(\frac{1}{\gamma \sigma_k} \sum_{i=1}^k (C_i - 1) \leq x \right) = \Phi(x), \quad \text{a.s. } \forall x. \quad (2.36)$$

Note that in order to prove (1.8) it is sufficient to show that

$$\lim_{n \rightarrow \infty} \frac{1}{D_n} \sum_{k=1}^n d_k I \left(\frac{1}{\gamma \sigma_k} \sum_{i=1}^k \ln C_i \leq x \right) = \Phi(x), \quad \text{a.s. } \forall x. \quad (2.37)$$

From Lemma 2.5, for sufficiently large k , we have

$$|C_k - 1| = O \left(\left(\frac{\ln(\ln k)}{k} \right)^{1/2} \right). \quad (2.38)$$

Since $\ln(1+x) = x + O(x^2)$ for $|x| < 1/2$, thus

$$\left| \sum_{k=1}^n \ln(C_k) - \sum_{k=1}^n (C_k - 1) \right| \ll \sum_{k=1}^n (C_k - 1)^2 \ll \sum_{k=1}^n \frac{\ln(\ln k)}{k} \ll \ln n \ln(\ln n) \quad \text{a.s.} \quad (2.39)$$

Hence for any $\varepsilon > 0$ and for sufficiently large n , we have

$$I\left(\frac{1}{\gamma^{\sigma_n}} \sum_{k=1}^n (C_k - 1) \leq x - \varepsilon\right) \leq I\left(\frac{1}{\gamma^{\sigma_n}} \sum_{k=1}^n \ln C_k \leq x\right) \leq I\left(\frac{1}{\gamma^{\sigma_n}} \sum_{k=1}^n (C_k - 1) \leq x + \varepsilon\right) \quad (2.40)$$

and thus (2.36) implies (2.37).

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