Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 895907, 23 pages
doi:10.1155/2010/895907

Research Article

An Algorithm for Finding a Common Solution for a System of Mixed Equilibrium Problem, Quasivariational Inclusion Problem, and Fixed Point Problem of Nonexpansive Semigroup

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Received 31 March 2010; Accepted 8 June 2010

Academic Editor: Vy Khoi Le

We introduce a hybrid iterative scheme for finding a common element of the set of solutions for a system of mixed equilibrium problems, the set of common fixed point for nonexpansive semigroup, and the set of solutions of the quasi-variational inclusion problem with multivalued maximal monotone mappings and inverse-strongly monotone mappings in Hilbert space. Under suitable conditions, some strong convergence theorems are proved. Our results extend some recent results announced by some authors.

1. Introduction

Throughout this paper we assume that H is a real Hilbert space, and C is a nonempty closed convex subset of H.

In the sequel, we denote the set of fixed points of S by $F(S)$.

A bounded linear operator $A : H \to H$ is said to be *strongly positive*, if there exists a constant $\overline{\gamma}$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.1}$$

Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ a multivalued mapping. The "so-called" *quasi-variational inclusion problem* (see, Chang [1, 2]) is to find an $u \in H$ such that

$$\theta \in B(u) + M(u). \tag{1.2}$$

A number of problems arising in structural analysis, mechanics, and economics can be studied in the framework of this kind of variational inclusions (see, e.g., [3]).

The set of solutions of variational inclusion (1.2) is denoted by $\mathrm{VI}(H, B, M)$.

Special Case

If $M = \partial\delta_C$, where C is a nonempty closed convex subset of H, and $\delta_C : H \to [0, \infty)$ is the indicator function of C, that is,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{1.3}$$

then the variational inclusion problem (1.2) is equivalent to find $u \in C$ such that

$$\langle B(u), v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.4}$$

This problem is called *Hartman-Stampacchia variational inequality problem* (see, e.g., [4]). The set of solutions of (1.4) is denoted by $\mathrm{VI}(C, B)$.

Recall that a mapping $B : H \to H$ is called *α-inverse strongly monotone* (see [5]), if there exists an $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \alpha \|Bx - By\|^2, \quad \forall x, y \in H. \tag{1.5}$$

A multivalued mapping $M : H \to 2^H$ is called *monotone*, if for all $x, y \in H$, $u \in Mx$, and $v \in My$, then it implies that $\langle u - v, x - y \rangle \geq 0$. A multivalued mapping $M : H \to 2^H$ is called *maximal monotone*, if it is monotone and if for any $(x, u) \in H \times H$

$$\langle u - v, x - y \rangle \geq 0, \quad \forall (y, v) \in \text{Graph } (M) \tag{1.6}$$

(the graph of mapping M) implies that $u \in Mx$.

Proposition 1.1 (see [5]). *Let $B : H \to H$ be an α-inverse strongly monotone mapping, then*

(a) *B is a $1/\alpha$-Lipschitz continuous and monotone mapping;*

(b) *if λ is any constant in $(0, 2\alpha]$, then the mapping $I - \lambda B$ is nonexpansive, where I is the identity mapping on H.*

Let $\Theta : C \times C \to R$ be an equilibrium bifunction (i.e., $\Theta(x, x) = 0$, for all $x \in C$), and let $\varphi : C \to R$ be a real-valued function.

Recently, Ceng and Yao [6] introduced the following *mixed equilibrium problem* (MEP), that is, to find $z \in C$ such that

$$\text{MEP} : \Theta(z, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C. \tag{1.7}$$

The set of solutions of (1.7) is denoted by $\mathrm{MEP}(\Theta, \varphi)$, that is,

$$\mathrm{MEP}(\Theta) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) \geq 0,\ \forall y \in C\}. \tag{1.8}$$

In particular, if $\varphi = 0$, this problem reduces to the *equilibrium problem*, that is, to find $z \in C$ such that

$$\mathrm{EP} : \Theta(z, y) \geq 0, \quad \forall y \in C. \tag{1.9}$$

Denote the set of solution of EP by $\mathrm{EP}(\Theta)$.

On the other hand, Li et al. [7] introduced two steps of iterative procedures for the approximation of common fixed point of a nonexpansive semigroup $\{T(s) : 0 \leq s < \infty\}$ on a nonempty closed convex subset C in a Hilbert space.

Very recently, Saeidi [8] introduced a more general iterative algorithm for finding a common element of the set of solutions for a system of equilibrium problems and of the set of common fixed points for a finite family of nonexpansive mappings and a nonexpansive semigroup.

Recall that a family of mappings $\mathcal{T} = \{T(s) : 0 \leq s < \infty\} : C \to C$ is called *a nonexpansive semigroup*, if it satisfies the following conditions:

(a) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$ and $T(0) = I$;

(b) $\|T(s)x - T(s)y\| \leq \|x - y\|$, for all $x, y \in C$.

(c) the mapping $T(\cdot)x$ is continuous, for each $x \in C$.

Motivated and inspired by Ceng and Yao [6], Li et al. [7], Saeidi [8], and [9–13], the purpose of this paper is to introduce a hybrid iterative scheme for finding a common element of the set of solutions for a system of mixed equilibrium problems, the set of common fixed point for a nonexpansive semigroup, and the set of solutions of the quasi-variational inclusion problem with multivalued maximal monotone mappings and inverse-strongly monotone mappings in Hilbert space. Under suitable conditions, some strong convergence theorems are proved. Our results extend the recent results in Zhang et al. [5], S. Takahashi and W. Takahashi [14], Chang et al. [15], Ceng and Yao [6], Li et al. [7] and, Saeidi [8].

2. Preliminaries

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and strong convergence of the sequence $\{x_n\}$ in H, respectively.

Definition 2.1. Let $M : H \to 2^H$ be a multivalued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad \forall u \in H \tag{2.1}$$

is called the *resolvent operator associated with* M, where λ is any positive number, and I is the identity mapping.

Proposition 2.2 (see [5]). (a) *The resolvent operator* $J_{M,\lambda}$ *associated with* M *is single-valued and nonexpansive for all* $\lambda > 0$*, that is,*

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \le \|x - y\|, \quad \forall x, y \in H, \ \forall \lambda > 0. \tag{2.2}$$

(b) *The resolvent operator* $J_{M,\lambda}$ *is 1-inverse-strongly monotone, that is,*

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \le \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y)\rangle, \quad \forall x, y \in H. \tag{2.3}$$

Definition 2.3. A single-valued mapping $P : H \to H$ is said to be *hemicontinuous*, if for any $x, y \in H$, the mapping $t \mapsto P(x + ty)$ converges weakly to Px (as $t \to 0+$).

It is well known that every continuous mapping must be hemicontinuous.

Lemma 2.4 (see [16]). *Let* E *be a real Banach space,* E^* *the dual space of* $E, T : E \to 2^{E^*}$ *a maximal monotone mapping, and* $P : E \to E^*$ *a hemicontinuous bounded monotone mapping with* $D(P) = E$*, then the mapping* $S = T + P : E \to 2^{E^*}$ *is a maximal monotone mapping.*

For solving the equilibrium problem for bifunction $\Theta : C \times C \to R$, let us assume that Θ satisfies the following conditions:

(H_1) $\Theta(x, x) = 0$ for all $x \in C$;

(H_2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \le 0$ for all $x, y \in C$;

(H_3) for each $y \in C$, $x \mapsto \Theta(x, y)$ is concave and upper semicontinuous.

(H_4) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex.

A map $\eta : C \times C \to H$ is called Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|\eta(x, y)\| \le L\|x - y\|, \quad \forall x, y \in C. \tag{2.4}$$

A differentiable function $K : C \to R$ on a convex set C is called

(i) η-convex [6] if

$$K(y) - K(x) \ge \langle K'(x), \eta(y, x)\rangle, \quad \forall x, y \in C, \tag{2.5}$$

where $K'(x))$ is the *Fréchet* derivative of K at x;

(ii) η-strongly convex [6] if there exists a constant $\mu > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y, x)\rangle \ge \left(\frac{\mu}{2}\right)\|x - y\|^2, \quad \forall x, y \in C. \tag{2.6}$$

Let $\Theta : C \times C \to R$ be an equilibrium bifunction satisfying the conditions (H_1)–(H_4). Let r be any given positive number. For a given point $x \in C$, consider the following *auxiliary problem for* MEP (for short, MEP(x, r)) to find $y \in C$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \ge 0, \quad \forall z \in C, \tag{2.7}$$

where $\eta : C \times C \to H$ is a mapping, and $K'(x)$ is the *Fréchet* derivative of a functional $K : C \to R$ at x. Let $V_r^{\Theta} : C \to C$ be the mapping such that for each $x \in C$, $V_r^{\Theta}(x)$ is the set of solutions of MEP(x, r), that is,

$$V_r^{\Theta}(x) = \Big\{ y \in C : \Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \geq 0,\ \forall z \in C \Big\}, \quad \forall x \in C. \tag{2.8}$$

Then the following conclusion holds.

Proposition 2.5 (see [6]). *Let C be a nonempty closed convex subset of $H, \varphi : C \to R$ a lower semicontinuous and convex functional. Let $\Theta : C \times C \to R$ be an equilibrium bifunction satisfying conditions (H_1)–(H_4). Assume that*

(i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $L > 0$ such that*

  (a) *$\eta(x, y) + \eta(y, x) = 0$, for all $x, y \in C$,*

  (b) *$\eta(\cdot,\cdot)$ is affine in the first variable,*

  (c) *for each fixed $y \in C$, $x \mapsto \eta(y, x)$ is continuous from the weak topology to the weak topology;*

(ii) *$K : C \to R$ is η-strongly convex with constant $\mu > 0$, and its derivative K' is continuous from the weak topology to the strong topology;*

(iii) *for each $x \in C$, there exists a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$, one has*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0. \tag{2.9}$$

Then the following hold:

(i) *V_r^{Θ} is single-valued;*

(ii) *V_r^{Θ} is nonexpansive if K' is Lipschitz continuous with constant $\nu > 0$ such that $\mu \geq L\nu$;*

(iii) *$F(V_r^{\Theta}) = MEP(\Theta)$;*

(iv) *$MEP(\Theta)$ is closed and convex.*

Lemma 2.6 (see [17]). *Let C be a nonempty bounded closed convex subset of H, and let $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, then for any $h \geq 0$*

$$\lim_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)x\,ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x\,ds\right) \right\| = 0. \tag{2.10}$$

Lemma 2.7 (see [7]). *Let C be a nonempty bounded closed convex subset of H, and let $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and* $\limsup_{s\to\infty}\limsup_{n\to\infty}\|T(s)x_n - x_n\| = 0$, *then $z \in F(\Im)$.*

3. The Main Results

In order to prove the main result, we first give the following lemma.

Lemma 3.1 (see [5]). (a) *$u \in H$ is a solution of variational inclusion* (1.2) *if and only if $u = J_{M,\lambda}(u - \lambda Bu)$, for all $\lambda > 0$, that is,*

$$VI(H,B,M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall\lambda > 0. \tag{3.1}$$

(b) *If $\lambda \in (0, 2\alpha]$, then $VI(H,B,M)$ is a closed convex subset in H.*

In the sequel, we assume that $H, C, M, A, B, f, T, F, \varphi_i, \eta_i, K_i$ $(i = 1,2,\ldots,N)$ satisfy the following conditions:

(1) H is a real Hilbert space, $C \subset H$ is a nonempty closed convex subset;

(2) $A : H \to H$ is a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$, $f : H \to H$ is a contraction mapping with a contraction constant h $(0 < h < 1)$, $0 < \gamma < \overline{\gamma}/h$, $B : C \to H$ is an α-inverse-strongly monotone mapping, and $M : H \to 2^H$ is a multivalued maximal monotone mapping;

(3) $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\} : C \to C$ is a nonexpansive semigroup;

(4) $\mathfrak{F} = \{\Theta_i : i = 1,2,\ldots,N\} : C \times C \to R$ is a finite family of bifunctions satisfying conditions (H_1)–(H_4), and $\varphi_i : C \to R$ $(i = 1,2,\ldots,N)$ is a finite family of lower semicontinuous and convex functional;

(5) $\eta_i : C \times C \to H$ is a finite family of Lipschitz continuous mappings with constant $L_i > 0$ $(i = 1,2,\ldots,N)$ such that

  (a) $\eta_i(x,y) + \eta_i(y,x) = 0$, for all $x, y \in C$,

  (b) $\eta_i(\cdot,\cdot)$ is affine in the first variable,

  (c) for each fixed $y \in C$, $x \mapsto \eta_i(y,x)$ is sequentially continuous from the weak topology to the weak topology;

(6) $K_i : C \to R$ is a finite family of η_i-strongly convex with constant $\mu_i > 0$, and its derivative K_i' is not only continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu_i > 0, \mu_i \geq L_i\nu_i$.

In the sequel we always denote by $F(\mathcal{T})$ the set of fixed points of the nonexpansive semi-group $\mathcal{T}$, $\mathrm{VI}(H,B,M)$ the set of solutions to the variational inequality (1.2), and $\mathrm{MEP}(\mathcal{F})$ the set of solutions to the following *auxiliary problem for a system of mixed equilibrium problems*:

$$\begin{aligned}
&\Theta_1\left(y_n^{(1)},x\right)+\phi_1(x)-\phi_1\left(y_n^{(1)}\right)+\frac{1}{r_1}\left\langle K'\left(y_n^{(1)}\right)-K'(x_n),\eta_1\left(x,y_n^{(1)}\right)\right\rangle\geq 0,\quad \forall x\in C,\\
&\Theta_2\left(y_n^{(2)},x\right)+\phi_2(x)-\phi_2\left(y_n^{(2)}\right)+\frac{1}{r_2}\left\langle K'\left(y_n^{(2)}\right)-K'\left(y_n^{(1)}\right),\eta_2\left(x,y_n^{(2)}\right)\right\rangle\geq 0,\quad \forall x\in C,\\
&\qquad\vdots\\
&\Theta_{N-1}\left(y_n^{(N-1)},x\right)+\phi_{N-1}(x)-\phi_{N-1}\left(y_n^{(N-1)}\right)\\
&\qquad+\frac{1}{r_{N-1}}\left\langle K'\left(y_n^{(N-1)}\right)-K'\left(y_n^{(N-2)}\right),\eta_{N-1}\left(x,y_n^{(N-1)}\right)\right\rangle\geq 0,\quad \forall x\in C,\\
&\Theta_N(y_n,x)+\phi_N(x)-\phi_N(y_n)+\frac{1}{r_N}\left\langle K'(y_n)-K'\left(y_n^{(N-1)}\right),\eta_N(x,y_n)\right\rangle\geq 0,\quad \forall x\in C,
\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}
y_n^{(1)}&=V_{r_1}^{\Theta_1}x_n,\\
y_n^{(i)}&=V_{r_i}^{\Theta_i}y_n^{(i-1)}=V_{r_i}^{\Theta_i}V_{r_{(i-1)}}^{\Theta_{i-1}}y_n^{(i-2)}=V_{r_i}^{\Theta_i}\cdots V_{r_2}^{\Theta_2}y_n^{(1)}\\
&=V_{r_i}^{\Theta_i}\cdots V_{r_2}^{\Theta_2}V_{r_1}^{\Theta_1}x_n,\quad i=2,3,\ldots,N-1,\\
y_n&=V_{r_N}^{\Theta_N}\cdots V_{r_2}^{\Theta_2}V_{r_1}^{\Theta_1}x_n,
\end{aligned}\tag{3.3}$$

and $V_{r_i}^{\Theta_i}: C\to C$, $i=1,2,\ldots,N$ is the mapping defined by (2.8).

In the sequel we denote by $\mathcal{V}^l=V_{r_l}^{\Theta_l}\cdots V_{r_2}^{\Theta_2}V_{r_1}^{\Theta_1}$ for $l\in\{1,2,\ldots,N\}$ and $\mathcal{V}^0=I$.

Theorem 3.2. *Let $H,C,A,B,M,f,T,F,\varphi_i,\eta_i,K_i$ $(i=1,2,\ldots,N)$ be the same as above. Let r_i $(i=1,2,\ldots,N)$ be a finite family of positive numbers, $\lambda\in(0,2\alpha]$, $\{\alpha_n\},\{\beta_n\}\subset[0,1]$, and $\{t_n\}\subset(0,\infty)$. If $\mathcal{G}:=F(\mathcal{T})\bigcap MEP(\mathcal{F})\bigcap VI(H,B,M)\neq\emptyset$ and the following conditions are satisfied:*

(i) *for each $x\in C$, there exists a bounded subset $D_x\subseteq C$ and $z_x\in C$ such that for any $y\in C\setminus D_x$*

$$\Theta_i(y,z_x)+\varphi_i(z_x)-\varphi_i(y)+\frac{1}{r_i}\langle K_i'(y)-K_i'(x),\eta_i(z_x,y)\rangle<0,\tag{3.4}$$

(ii) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$, $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$, *and* $\lim_{n\to\infty}t_n=\infty$, *then*

(1) *for each* $n \geq 1$, *there is a unique* $x_n \in C$ *such that*

$$x_n = \alpha_n \gamma f\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right)\frac{1}{t_n}\int_0^{t_n} T(s)(J_{M,\lambda}(I-\lambda B))^2 \mho^N x_n ds, \tag{3.5}$$

(2) *the sequence* $\{x_n\}$ *converges strongly to some point* $x^* \in \mathcal{G}$, *provided that* $V_{r_i}^{\Theta_i}$ *is firmly nonexpansive;*

(3) x^* *is the unique solution of the following variational inequality*

$$\left\langle (A - \gamma f)x^*, x^* - z\right\rangle \leq 0, \quad \forall z \in \mathcal{G}. \tag{3.6}$$

Proof. We observe that from condition (ii), we can assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$.

Since A is a linear bounded self-adjoint operator on H, then

$$\|A\| = \sup\{|\langle Au, u\rangle| : u \in H, \|u\| = 1\}. \tag{3.7}$$

Since

$$\begin{aligned} \left\langle ((1-\beta_n)I - \alpha_n A)u, u\right\rangle &= 1 - \beta_n - \alpha_n\langle Au, u\rangle \\ &\geq 1 - \beta_n - \alpha_n\|A\| \geq 0, \end{aligned} \tag{3.8}$$

this implies that $(1-\beta_n)I - \alpha_n A$ is positive. Hence we have

$$\begin{aligned} \left\|(1-\beta_n)I - \alpha_n A\right\| &= \sup\left\{\left|\left\langle ((1-\beta_n)I - \alpha_n A)u, u\right\rangle\right| : u \in H, \|u\| = 1\right\} \\ &= \sup\left\{1 - \beta_n - \alpha_n\langle Au, u\rangle : u \in H, \|u\| = 1\right\} \\ &\leq 1 - \beta_n - \alpha_n\overline{\gamma} < 1. \end{aligned} \tag{3.9}$$

For each given $n \geq 1$, let us define the mapping

$$W_n := \alpha_n \gamma f \frac{1}{t_n}\int_0^{t_n} T(s)ds + \beta_n I + \left((1-\beta_n)I - \alpha_n A\right)\frac{1}{t_n}\int_0^{t_n} T(s)(J_{M,\lambda}(I-\lambda B))^2 \mho^N ds. \tag{3.10}$$

Firstly we show that the mapping $W_n : C \to C$ is a contraction. Indeed, for any $x, y \in C$, we have

$$
\begin{aligned}
&\|W_n x - W_n y\| \\
&\quad = \Bigg\| \alpha_n \gamma f\left(\frac{1}{t_n}\int_0^{t_n} T(s)x ds\right) + \beta_n x + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)(J_{M,\lambda}(I-\lambda B))^2 \mho^N x ds \\
&\qquad\quad -\alpha_n \gamma f \frac{1}{t_n}\int_0^{t_n} T(s)y ds - \beta_n y - ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)(J_{M,\lambda}(I-\lambda B))^2 \mho^N y ds \Bigg\| \\
&\quad \le \alpha_n \gamma \left\| f\left(\frac{1}{t_n}\int_0^{t_n} T(s)x ds\right) - f\left(\frac{1}{t_n}\int_0^{t_n} T(s)y ds\right)\right\| + \beta_n \|x-y\| \\
&\qquad + (1-\beta_n - \alpha_n \overline{\gamma})\frac{1}{t_n}\int_0^{t_n} \left\|T(s)(J_{M,\lambda}(I-\lambda B))^2 \mho^N x - T(s)(J_{M,\lambda}(I-\lambda B))^2 \mho^N y\right\| ds \\
&\quad \le \alpha_n \gamma h \|x-y\| + \beta_n \|x-y\| + \|(1-\beta_n - \alpha_n \overline{\gamma})\| x-y\| \\
&\quad = (1-\alpha_n(\overline{\gamma} - \gamma h))\|x-y\|.
\end{aligned} \tag{3.11}
$$

This implies that $W_n : C \to C$ is a contraction mapping. Let $x_n \in C$ be the unique fixed point of W_n. Thus,

$$
\begin{aligned}
x_n = \alpha_n \gamma f\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right) + \beta_n x_n \\
+ ((1-\beta_n)I - \alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)(J_{M,\lambda}(I-\lambda B))^2 \mho^N x_n ds\right)
\end{aligned} \tag{3.12}
$$

is well defined.

Letting $y_n = \mho^N x_n$, $\xi_n = J_{M,\lambda}(I-\lambda B)y_n$, and $\rho_n = J_{M,\lambda}(I-\lambda B)\xi_n$, then

$$
x_n = \alpha_n \gamma f\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds. \tag{3.13}
$$

We divide the proof of Theorem 3.2 into 8 steps.

Step 1. First prove that the sequences $\{x_n\}, \{\rho_n\}, \{\xi_n\}$, and $\{y_n\}$ are bounded.

(a) Pick $p \in G$, since $y_n = \mho^N x_n$ and $p = \mho^N p$, we have

$$
\|y_n - p\| = \left\|\mho^N x_n - p\right\| \le \|x_n - p\|. \tag{3.14}
$$

(b) Since $p \in \mathrm{VI}(H,B,M)$ and $\rho_n = J_{M,\lambda}(I-\lambda B)\xi_n$, we have $p = J_{M,\lambda}(I-\lambda B)p$, and so

$$\begin{aligned}\|\rho_n - p\| &= \|J_{M,\lambda}(I-\lambda B)\xi_n - J_{M,\lambda}(I-\lambda B)p\| \\ &\le \|(I-\lambda B)\xi_n - (I-\lambda B)p\| \le \|\xi_n - p\| \\ &= \|J_{M,\lambda}(I-\lambda B)y_n - J_{M,\lambda}(I-\lambda B)p\| \\ &\le \|y_n - p\| \le \|x_n - p\|.\end{aligned} \tag{3.15}$$

Letting $u_n = (1/t_n)\int_0^{t_n} T(s)x_n ds$, $q_n = (1/t_n)\int_0^{t_n} T(s)\rho_n ds$, we have

$$\begin{aligned}\|u_n - p\| &= \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p \right\| \\ &\le \frac{1}{t_n}\int_0^{t_n} \|T(s)x_n - T(s)p\| ds \\ &\le \|x_n - p\|.\end{aligned} \tag{3.16}$$

Similarly, we have

$$\|q_n - p\| \le \|\rho_n - p\|. \tag{3.17}$$

Form (3.5), (3.9), (3.14), (3.15), (3.16), and (3.17) we have

$$\begin{aligned}&\|x_n - p\| \\ &\quad = \|\alpha_n\gamma f(u_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)q_n - p\| \\ &\quad = \|\alpha_n\gamma(f(u_n) - f(p)) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(q_n - p) + \alpha_n(\gamma f(p) - Ap)\| \\ &\quad \le \alpha_n\gamma h\|u_n - p\| + \beta_n\|x_n - p\| + ((1-\beta_n) - \alpha_n\overline{\gamma})\|q_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\ &\quad \le \alpha_n\gamma h\|x_n - p\| + \beta_n\|x_n - p\| + ((1-\beta_n) - \alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\|.\end{aligned} \tag{3.18}$$

So, $\|x_n - p\| \le (1/(\overline{\gamma} - \gamma h))\|\gamma f(p) - Ap\|$. This implies that $\{x_n\}$ is a bounded sequence in H. Therefore $\{y_n\}, \{\rho_n\}, \{\xi_n\}, \{\gamma f(u_n)\}$, and $\{q_n\}$ are all bounded.

Step 2. Next we prove that

$$\|x_n - T(s)x_n\| \longrightarrow 0, \quad (n \longrightarrow \infty). \tag{3.19}$$

Since $x_n = \alpha_n\gamma f(u_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)q_n$, then

$$\|x_n - q_n\| \le \alpha_n\|\gamma f(u_n) - Aq_n\| + \beta_n\|x_n - q_n\|. \tag{3.20}$$

Hence

$$\|x_n - q_n\| \leq \frac{\alpha_n}{1-\beta_n}\|\gamma f(u_n) - Aq_n\|. \tag{3.21}$$

From condition (ii), we have

$$\|x_n - q_n\| \longrightarrow 0. \tag{3.22}$$

Let $K = \{w \in C : \|w-p\| \leq (1/(\overline{\gamma}-\gamma h))\|\gamma f(p) - Ap\|\}$, then K is a nonempty bounded closed convex subset of C and $T(s)$-invariant. Since $\{x_n\} \subset K$ and K is bounded, there exists $r > 0$ such that $K \subset B_r$; it follows from Lemma 2.6 that

$$\lim_{n\to\infty}\|q_n - T(s)q_n\| \longrightarrow 0. \tag{3.23}$$

From (3.22) and (3.23), we have

$$\begin{aligned}\|x_n - T(s)x_n\| &= \|x_n - q_n + q_n - T(s)q_n + T(s)q_n - T(s)x_n\| \\ &\leq \|x_n - q_n\| + \|q_n - T(s)q_n\| + \|T(s)q_n - T(s)x_n\| \\ &\leq \|x_n - q_n\| + \|q_n - T(s)q_n\| + \|q_n - x_n\| \longrightarrow 0.\end{aligned} \tag{3.24}$$

Step 3. Next we prove that

$$\begin{aligned}&\text{(i)}\ \lim_{n\to\infty}\left\|\mho^{l+1}x_n - \mho^{l}x_n\right\| = 0, \quad \forall l \in \{0,1,\dots,N-1\}; \\ &\text{(ii) especially,}\ \lim_{n\to\infty}\left\|\mho^{N}x_n - x_n\right\| = \lim_{n\to\infty}\|y_n - x_n\| = 0.\end{aligned} \tag{3.25}$$

In fact, for any given $p \in G$ and $l \in \{0,1,\dots,N-1\}$, since $V_{r_{l+1}}^{\Theta_{l+1}}$ is firmly nonexpansive, we have

$$\begin{aligned}\left\|\mho^{l+1}x_n - p\right\|^2 &= \left\|V_{r_{l+1}}^{\Theta_{l+1}}(\mho^{l}x_n) - V_{r_{l+1}}^{\Theta_{l+1}}p\right\|^2 \\ &\leq \left\langle V_{r_{l+1}}^{\Theta_{l+1}}\left(\mho^{l}x_n\right) - p, \mho^{l}x_n - p\right\rangle \\ &= \left\langle \mho^{l+1}x_n - p, \mho^{l}x_n - p\right\rangle \\ &= \frac{1}{2}\left(\left\|\mho^{l+1}x_n - p\right\|^2 + \left\|\mho^{l}x_n - p\right\|^2 - \left\|\mho^{l}x_n - \mho^{l+1}x_n'\right\|^2\right).\end{aligned} \tag{3.26}$$

It follows that

$$\left\|\mho^{l+1}x_n - p\right\|^2 \leq \|x_n - p\|^2 - \left\|\mho^{l}x_n - \mho^{l+1}x_n\right\|^2. \tag{3.27}$$

From (3.5), we have

$$\begin{aligned}\|x_n - p\|^2 &= \|\alpha_n\gamma f(u_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)q_n - p\|^2\\ &= \|\alpha_n(\gamma f(u_n) - Ap) + \beta_n(x_n - q_n) + (I - \alpha_n A)(q_n - p)\|^2\\ &\le \|(I-\alpha_n A)(q_n - p) + \beta_n(x_n - q_n)\|^2 + 2\alpha_n\langle \gamma f(u_n) - Ap, x_n - p\rangle\\ &\le [\|(I-\alpha_n A)(q_n-p)\| + \beta_n\|(x_n - q_n)\|]^2 + 2\alpha_n\langle \gamma f(u_n) - Ap, x_n - p\rangle\\ &\le [(1-\alpha_n\overline{\gamma})\|\rho_n - p\| + \beta_n\|x_n - q_n\|]^2 + 2\alpha_n\langle \gamma f(u_n) - Ap, x_n - p\rangle\\ &= (1-\alpha_n\overline{\gamma})^2\|\rho_n - p\|^2 + \beta_n^2\|x_n - q_n\|^2 + 2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n - p\|\cdot\|x_n - q_n\|\\ &\quad + 2\alpha_n\|\gamma f(u_n) - Ap\|\cdot\|x_n - p\|.\end{aligned} \tag{3.28}$$

Since

$$\|\rho_n - p\| \le \|\xi_n - p\| \le \left\|\mho^N x_n - p\right\| \le \left\|\mho^{l+1}x_n - p\right\|, \quad \forall l \in \{0,1,\dots,N-1\}, \tag{3.29}$$

and this together with (3.27) and (3.28), it yields

$$\begin{aligned}&\|x_n - p\|^2\\ &\quad\le (1-\alpha_n\overline{\gamma})^2\left\{\|x_n - p\|^2 - \left\|\mho^l x_n - \mho^{l+1}x_n\right\|^2\right\} + \beta_n^2\|x_n - q_n\|^2\\ &\qquad + 2(1-\alpha_n\overline{\gamma})\cdot\beta_n\|\rho_n - p\|\cdot\|x_n - q_n\| + 2\alpha_n\|\gamma f(u_n) - Ap\|\cdot\|x_n - p\|\\ &\quad= \left(1 - 2\alpha_n\overline{\gamma} + (\alpha_n\overline{\gamma})^2\right)\|x_n - p\|^2 - (1-\alpha_n\overline{\gamma})^2\left\|\mho^l x_n - \mho^{l+1}x_n\right\|^2 + \beta_n^2\|x_n - q_n\|^2\\ &\qquad + 2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n - p\|\cdot\|x_n - q_n\| + 2\alpha_n\|\gamma f(u_n) - Ap\|\cdot\|x_n - p\|.\end{aligned} \tag{3.30}$$

Simplifying it we have

$$\begin{aligned}(1-\alpha_n\overline{\gamma})^2\left\|\mho^l x_n - \mho^{l+1}x_n\right\|^2 &\le \left(1 + \alpha_n(\overline{\gamma})^2\right)\|x_n - p\|^2 - \|x_n - p\|^2\\ &\quad + \beta_n^2\|x_n - q_n\|^2 + 2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n - p\|\cdot\|x_n - q_n\|\\ &\quad + 2\alpha_n\|\gamma f(u_n) - Ap\|\cdot\|x_n - p\|.\end{aligned} \tag{3.31}$$

Since $\alpha_n \to 0$ and $\|x_n - q_n\| \to 0$, by condition (ii), it yields $\|\mho^{l+1}x_n - \mho^l x_n\| \to 0$.

Step 4. Now we prove that for any given $p \in G$

$$\lim_{n\to\infty}\|By_n - Bp\| = 0. \tag{3.32}$$

In fact, it follows from (3.15) that

$$\begin{aligned}\|\rho_n - p\|^2 &\le \|\xi_n - p\|^2 = \|J_{M,\lambda}(I-\lambda B)y_n - J_{M,\lambda}(I-\lambda B)p\|^2 \\ &\le \|(I-\lambda B)y_n - (I-\lambda B)p\|^2 \\ &= \|y_n - p\|^2 - 2\lambda\langle y_n - p, By_n - Bp\rangle + \lambda^2\|By_n - Bp\|^2 \\ &\le \|y_n - p\|^2 + \lambda(\lambda - 2\alpha)\|By_n - Bp\|^2 \\ &\le \|x_n - p\|^2 + \lambda(\lambda - 2\alpha)\|By_n - Bp\|^2.\end{aligned} \tag{3.33}$$

Substituting (3.33) into (3.28), we obtain

$$\begin{aligned}\|x_n - p\|^2 \le{}& (1-\alpha_n\overline{\gamma})^2\left\{\|x_n - p\|^2 + \lambda(\lambda - 2\alpha)\|By_n - Bp\|^2\right\} + \beta_n^2\|x_n - q_n\|^2 \\ &+ 2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n - p\|\cdot\|x_n - q_n\| + 2\alpha_n\|\gamma f(u_n) - Ap\|\cdot\|x_n - p\|.\end{aligned} \tag{3.34}$$

Simplifying it, we have

$$\begin{aligned}&(1-\alpha_n\overline{\gamma})^2\lambda(2\alpha - \lambda)\|By_n - Bp\|^2 \\ &\quad\le \left(1 + \alpha_n(\overline{\gamma})^2\right)\|x_n - p\|^2 - \|x_n - p\|^2 + \beta_n^2\|x_n - q_n\|^2 \\ &\qquad + 2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n - p\|\cdot\|x_n - q_n\| + 2\alpha_n\|\gamma f(u_n) - Ap\|\cdot\|x_n - p\| \\ &\quad= \alpha_n(\overline{\gamma})^2\|x_n - p\|^2 + \beta_n^2\|x_n - q_n\|^2 \\ &\qquad + 2(1-\alpha_n\overline{\gamma})\beta_n\|\rho_n - p\|\cdot\|x_n - q_n\| + 2\alpha_n\|\gamma f(u_n) - Ap\|\cdot\|x_n - p\|.\end{aligned} \tag{3.35}$$

Since $\alpha_n \to 0, 0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1, \|x_n - q_n\| \to 0$, and $\{\gamma f(u_n) - Ap\}, \{x_n\}$ are bounded, these imply that $\|By_n - Bp\| \to 0\ (n \to \infty)$.

Step 5. Next we prove that

$$\begin{aligned}&\lim_{n\to\infty}\|y_n - \rho_n\| = 0, \\ &\lim_{n\to\infty}\|x_n - \rho_n\| = 0.\end{aligned} \tag{3.36}$$

In fact, since

$$\|y_n - \rho_n\| \le \|y_n - \xi_n\| + \|\xi_n - \rho_n\|, \tag{3.37}$$

for the purpose, it is sufficient to prove

$$\|y_n - \xi_n\| \longrightarrow 0, \qquad \|\xi_n - \rho_n\| \longrightarrow 0. \tag{3.38}$$

(a) First we prove that $\|y_n - \xi_n\| \to 0$. In fact, since

$$\begin{aligned}
&\|\xi_n - p\|^2 \\
&\quad = \left\| J_{M,\lambda}(I - \lambda B)y_n - J_{M,\lambda}(I - \lambda B)p \right\|^2 \\
&\quad \le \langle y_n - \lambda B y_n - (p - \lambda B p), \xi_n - p \rangle \\
&\quad = \frac{1}{2}\left\{ \left\| y_n - \lambda B y_n - (p - \lambda B p) \right\|^2 + \|\xi_n - p\|^2 - \left\| y_n - \lambda B y_n - (p - \lambda B p) - (\xi_n - p) \right\|^2 \right\} \\
&\quad \le \frac{1}{2}\left\{ \|y_n - p\|^2 + \|\xi_n - p\|^2 - \|y_n - \xi_n - \lambda(B y_n - B p)\|^2 \right\} \\
&\quad \le \frac{1}{2}\left\{ \|y_n - p\|^2 + \|\xi_n - p\|^2 - \|y_n - \xi_n\|^2 + 2\lambda\langle y_n - \xi_n, B y_n - B p\rangle - \lambda^2 \|B y_n - B p\|^2 \right\},
\end{aligned} \tag{3.39}$$

we have

$$\|\xi_n - p\|^2 \le \|y_n - p\|^2 - \|y_n - \xi_n\|^2 + 2\lambda\langle y_n - \xi_n, B y_n - B p\rangle - \lambda^2 \|B y_n - B p\|^2. \tag{3.40}$$

Substituting (3.40) into (3.28), it yields that

$$\begin{aligned}
\|x_n - p\|^2 \le (1 - \alpha_n \overline{\gamma})^2 \Big\{ &\|y_n - p\|^2 - \|y_n - \xi_n\|^2 + 2\lambda\langle y_n - \xi_n, B y_n - B p\rangle \\
&- \lambda^2 \|B y_n - B p\|^2 \Big\} + \beta_n^2 \|x_n - q_n\|^2 \\
&+ 2(1 - \alpha_n \overline{\gamma})\beta_n \|\rho_n - p\| \cdot \|x_n - q_n\| + 2\alpha_n \|\gamma f(u_n) - A p\| \cdot \|x_n - p\|.
\end{aligned} \tag{3.41}$$

Simplifying it we have

$$\begin{aligned}
(1 - \alpha_n \overline{\gamma})^2 \|y_n - \xi_n\|^2 \le \alpha_n \overline{\gamma}^2 \|x_n - p\|^2 &+ 2\left(1 - \alpha_n \overline{\gamma}^2\right)\lambda\langle y_n - \xi_n, B y_n - B p\rangle \\
&- (1 - \alpha_n \overline{\gamma})^2 \lambda^2 \|B y_n - B p\|^2 + \beta_n^2 \|x_n - q_n\|^2 \\
&+ 2(1 - \alpha_n \overline{\gamma})\beta_n \|\rho_n - p\| \cdot \|x_n - q_n\| + 2\alpha_n \|\gamma f(u_n) - A p\| \cdot \|x_n - p\|.
\end{aligned} \tag{3.42}$$

Since $\alpha_n \to 0$, $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, $\|x_n - q_n\| \to 0$, $\|B y_n - B p\| \to 0$ $(n \to \infty)$, and $\{\gamma f(u_n) - A p\}$, $\{x_n\}$, $\{\rho_n\}$ are bounded, these imply that $\|y_n - \xi_n\| \to 0$ $(n \to \infty)$.

(b) Next we prove that

$$\lim_{n\to\infty} \|\xi_n - \rho_n\| = 0. \tag{3.43}$$

In fact, since $\|\xi_n - \rho_n\| = \|J_{M,\lambda}(I - \lambda B)y_n - J_{M,\lambda}(I - \lambda B)\xi_n\| \le \|y_n - \xi_n\| \to 0$, so $\|y_n - \rho_n\| = \|y_n - \xi_n + \xi_n - \rho_n\| \le \|y_n - \xi_n\| + \|\xi_n - \rho_n\| \to 0$. This together with (3.25) shows that $\|x_n - \rho_n\| \to 0$.

Step 6. Next we prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in G$, and x^* is the unique solution of the variational inequality (3.6).

(a) We first prove that $x^* \in F(\mathcal{T})$. In fact, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \rightharpoonup x^*$. From Lemma 2.7 and Step 2, we obtain $x^* \in F(\mathcal{T})$.

(b) Now we prove that $x^* \in \cap_{l=1}^{N} \mathrm{MEP}(\Theta_l, \varphi_l)$.

Since $x_{n_k} \rightharpoonup x^*$ and noting Step 3, without loss of generality, we may assume that $\mho^l x_{n_k} \rightharpoonup x^*$, for all $l \in \{0,1,2,\ldots,N-1\}$. Hence for any $x \in C$ and for any $l \in \{0,1,2,\ldots,N-1\}$, we have

$$\left\langle \frac{K'_{l+1}(\mho^{l+1}x_{n_k}) - K'_{l+1}(\mho^{l}x_{n_k})}{r_{l+1}}, \eta_{l+1}\left(x, \mho^{l+1}x_{n_k}\right)\right\rangle \geq -\Theta_{l+1}\left(\mho^{l+1}x_{n_k}, x\right) - \varphi_{l+1}(x) + \varphi_{l+1}\left(\mho^{l+1}x_{n_k}\right). \tag{3.44}$$

By the assumptions and by condition (H_2) we know that the function φ_i and the mapping $x \mapsto (-\Theta_{l+1}(x,y))$ both are convex and lower semicontinuous, hence they are weakly lower semicontinuous. These together with $(K'_{l+1}(\mho^{l+1}x_{n_k}) - K'_{l+1}(\mho^{l}x_{n_k}))/r_{l+1} \to 0$ and $\mho^{l+1}x_{n_k} \rightharpoonup x^*$, we have

$$\begin{aligned} 0 &= \liminf_{k\to\infty}\left\{\left\langle \frac{K'_{l+1}(\mho^{l+1}x_{n_k}) - K'_{l+1}(\mho^{l}x_{n_k})}{r_{l+1}}, \eta_{l+1}\left(x, \mho^{l+1}x_{n_k}\right)\right\rangle\right\} \\ &\geq \liminf_{k\to\infty}\left\{-\Theta_{l+1}\left(\mho^{l+1}x_{n_k}, x\right) - \varphi_{l+1}(x) + \varphi_{l+1}\left(\mho^{l+1}x_{n_k}\right)\right\}. \end{aligned} \tag{3.45}$$

That is,

$$\Theta_{l+1}(x^*, x) + \varphi_{l+1}(x) - \varphi_{l+1}(x^*) \geq 0 \tag{3.46}$$

for all $x \in C$ and $l \in \{0,1,\ldots,N-1\}$, hence $x^* \in \cap_{l=1}^{N} \mathrm{MEP}(\Theta_l, \varphi_l)$.

(c) Now we prove that $x^* \in \mathrm{VI}(H, B, M)$.

In fact, since B is α-inverse-strongly monotone, it follows from Proposition 1.1 that B is a $1/\alpha$-Lipschitz continuous monotone mapping and $D(B) = H$ (where $D(B)$ is the domain of B). It follows from Lemma 2.4 that $M + B$ is maximal monotone. Let $(\nu, g) \in \mathrm{Graph}\ (M + B)$, that is, $g - B\nu \in M(\nu)$. Since $x_{n_k} \rightharpoonup x^*$ and noting Step 3, without loss of generality, we may assume that $\mho^l x_{n_k} \rightharpoonup x^*$; in particular, we have $y_{n_k} = \mho^N x_{n_k} \rightharpoonup x^*$. From $\|y_n - \rho_n\| \to 0$, we can prove that $\rho_{n_k} \rightharpoonup x^*$. Again since $\rho_{n_k} = J_{M,\lambda}(I - \lambda B)\xi_{n_k}$, we have

$$\xi_{n_k} - \lambda B\xi_{n_k} \in (I + \lambda M)\rho_{n_k}, \text{ that is, } \frac{1}{\lambda}(\xi_{n_k} - \rho_{n_k} - \lambda B\xi_{n_k}) \in M(\rho_{n_k}). \tag{3.47}$$

By virtue of the maximal monotonicity of M, we have

$$\left\langle \nu - \rho_{n_k}, g - B\nu - \frac{1}{\lambda}(\xi_{n_k} - \rho_{n_k} - \lambda B\xi_{n_k})\right\rangle \geq 0. \tag{3.48}$$

So,

$$\begin{aligned}\langle v-\rho_{n_k}, g\rangle &\geq \left\langle v-\rho_{n_k}, Bv+\frac{1}{\lambda}(\xi_{n_k}-\rho_{n_k}-\lambda B\xi_{n_k})\right\rangle \\ &= \left\langle v-\rho_{n_k}, Bv-B\rho_{n_k}+B\rho_{n_k}-B\xi_{n_k}+\frac{1}{\lambda}(\xi_{n_k}-\rho_{n_k})\right\rangle \\ &\geq 0+\langle v-\rho_{n_k}, B\rho_{n_k}-B\xi_{n_k}\rangle+\left\langle v-\rho_{n_k}, \frac{1}{\lambda}(\xi_{n_k}-\rho_{n_k})\right\rangle.\end{aligned} \tag{3.49}$$

Since $\|\xi_n-\rho_n\| \to 0$, $\|B\xi_n-B\rho_n\| \to 0$, and $\rho_{n_k} \rightharpoonup x^*$, we have

$$\lim_{n_k\to\infty}\langle v-\rho_{n_k}, g\rangle = \langle v-x^*, g\rangle \geq 0. \tag{3.50}$$

Since $M+B$ is maximal monotone, this implies that $\theta \in (M+B)(x^*)$, that is, $x^* \in \mathrm{VI}(H,B,M)$, and so $x^* \in \mathcal{G}$.

(d) Now we prove that x^* is the unique solution of variational inequality (3.6).

(1^0) We first prove that $\{x_{n_k}\} \to x^*$.

Since for all $z \in G$,

$$\begin{aligned}\|x_n-z\|^2 &= \langle x_n-z, x_n-z\rangle \\ &= \langle \alpha_n\gamma f(u_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n A)q_n-z, x_n-z\rangle \\ &= \langle \alpha_n(\gamma f(u_n)-Az)+\beta_n(x_n-z)+((1-\beta_n)I-\alpha_n A)(q_n-z), x_n-z\rangle \\ &\leq \alpha_n\langle \gamma f(u_n)-Az, x_n-z\rangle+\beta_n\|x_n-z\|^2+(1-\beta_n-\alpha_n\overline{\gamma})\|q_n-z\|\cdot\|x_n-z\| \\ &\leq (1-\alpha_n\overline{\gamma})\|x_n-z\|^2+\alpha_n\langle \gamma f(u_n)-Az, x_n-z\rangle.\end{aligned} \tag{3.51}$$

It follows that

$$\begin{aligned}\|x_n-z\|^2 &\leq \frac{1}{\overline{\gamma}}\langle \gamma f(u_n)-Az, x_n-z\rangle \\ &= \frac{1}{\overline{\gamma}}\langle \gamma f(u_n)-\gamma f(z)+\gamma f(z)-Az, x_n-z\rangle \\ &\leq \frac{1}{\overline{\gamma}}\left\{\gamma h\|x_n-z\|^2+\langle \gamma f(z)-Az, x_n-z\rangle\right\}.\end{aligned} \tag{3.52}$$

Therefore,

$$\|x_n - z\|^2 \le \frac{1}{\overline{\gamma} - \gamma h}\langle \gamma f(z) - Az, x_n - z\rangle. \tag{3.53}$$

Now, replacing n in (3.53) with n_k and letting $k \to \infty$ and $x_{n_k} \rightharpoonup x^*$, we have $x_{n_k} \to x^*$.

(2^0) Next we prove that x^* is the unique solution of the variational inequality (3.6).

Since

$$x_n = \alpha_n \gamma f\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right)\frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds, \tag{3.54}$$

we have

$$\begin{aligned}
&\alpha_n (A - \gamma f)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right) \\
&\quad = -\left\{(1-\beta_n)\left(x_n - \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds\right)\right\} + \alpha_n A\frac{1}{t_n}\int_0^{t_n}(T(s)x_n - T(s)\rho_n)ds \\
&\quad = -(1-\beta_n)\left(I - \frac{1}{t_n}\int_0^{t_n} T(s)(J_{M,\lambda}(I-\lambda B))^2 \mho^N ds\right)x_n + \alpha_n A\frac{1}{t_n}\int_0^{t_n}(T(s)x_n - T(s)\rho_n)ds.
\end{aligned} \tag{3.55}$$

Hence for any $z \in G$ we have,

$$\begin{aligned}
&\alpha_n\left\langle (A-\gamma f)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right), x_n - z\right\rangle \\
&\quad = -(1-\beta_n)\left\langle \left(I - \frac{1}{t_n}\int_0^{t_n} T(s)(J_{M,\lambda}(I-\lambda B))^2 \mho^N ds\right)x_n \right. \\
&\qquad\qquad \left. - \left(I - \frac{1}{t_n}\int_0^{t_n} T(s)(J_{M,\lambda}(I-\lambda B))^2 \mho^N ds\right)z, x_n - z\right\rangle \\
&\qquad + \alpha_n\left\langle A\frac{1}{t_n}\int_0^{t_n}(T(s)x_n - T(s)\rho_n)ds, x_n - z\right\rangle,
\end{aligned} \tag{3.56}$$

then

$$\left\langle (A-\gamma f)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right), x_n - z\right\rangle$$
$$= -\frac{1-\beta_n}{\alpha_n}$$
$$\times\left\langle\left(I-\frac{1}{t_n}\int_0^{t_n} T(s)J_{M,\lambda}^2(I-\lambda B)\mho^N ds\right)x_n - \left(I-\frac{1}{t_n}\int_0^{t_n} T(s)J_{M,\lambda}^2(I-\lambda B)\mho^N ds\right)z, x_n - z\right\rangle \tag{3.57}$$
$$+\left\langle A\frac{1}{t_n}\int_0^{t_n}(T(s)x_n - T(s)\rho_n)ds, x_n - z\right\rangle.$$

It is easily seen that $I-(1/t_n)\int_0^{t_n} T(s)(J_{M,\lambda}(I-\lambda B))^2\mho^N ds$ is monotone. Thus from (3.57) we have that

$$\left\langle (A-\gamma \mathrm{f})\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right), x_n - z\right\rangle \le \left\langle A\frac{1}{t_n}\int_0^{t_n}(T(s)x_n - T(s)\rho_n)ds, x_n - z\right\rangle. \tag{3.58}$$

Now, in (3.58) replacing n by n_k and letting $k\to\infty$ and $x_{n_k}\to x^*$, from (3.36), we have

$$\frac{1}{t_{n_k}}\int_0^{t_{n_k}}(T(s)x_{n_k} - T(s)\rho_{n_k})ds \longrightarrow 0. \tag{3.59}$$

So, we have

$$\langle (A-\gamma f)x^*, x^* - z\rangle \le 0 \quad \forall z\in \mathcal{G}. \tag{3.60}$$

It follows from [18, Theorem 3.2] that the solution of the variational inequality (3.6) is unique, that is, x^* is a unique solution of (3.6).

Step 7. Next we prove that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle \le 0. \tag{3.61}$$

(a) First, we prove that

$$\limsup_{n\to\infty}\left\langle \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds - x^*, \gamma f(x^*) - Ax^*\right\rangle \le 0. \tag{3.62}$$

Indeed, there exists a subsequence $\{\rho_{n_i}\}$ of $\{\rho_n\}$ such that

$$\limsup_{n\to\infty}\left\langle \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds - x^*, \gamma f(x^*) - Ax^*\right\rangle = \lim_{i\to\infty}\left\langle \frac{1}{t_{n_i}}\int_0^{t_{n_i}} T(s)\rho_{n_i} ds - x^*, \gamma f(x^*) - Ax^*\right\rangle. \tag{3.63}$$

We may also assume that $\rho_{n_i} \rightharpoonup w$. This together with (3.22) and (3.36) shows that $q_{n_i} = (1/t_{n_i})\int_0^{t_{n_i}} T(s)\rho_{n_i} ds \rightharpoonup w$. Since $\|x_n - q_n\| \to 0$, we have $x_{n_i} \rightharpoonup w$. Again by the same method as given in Step 6 we can prove that $w \in \mathcal{G}$. So, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left\langle \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds - x^*, \gamma f(x^*) - Ax^*\right\rangle \\
&\quad = \lim_{i\to\infty}\left\langle \frac{1}{t_{n_i}}\int_0^{t_{n_i}} T(s)\rho_{n_i} ds - x^*, \gamma f(x^*) - Ax^{*\prime}\right\rangle \\
&\quad = \lim_{i\to\infty}\left\langle q_{n_i} - x^*, \gamma f(x^*) - Ax^*\right\rangle \\
&\quad = \left\langle w - x^*, \gamma f(x^*) - Ax^*\right\rangle \le 0.
\end{aligned} \tag{3.64}$$

(b) Now we prove that

$$\limsup_{n\to\infty}\left\langle \gamma f(x^*) - Ax^*, x_n - x^*\right\rangle \le 0. \tag{3.65}$$

From $\|x_n - q_n\| \to 0$ and (a), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left\langle \gamma f(x^*) - Ax^*, x_n - x^*\right\rangle \\
&\quad = \limsup_{n\to\infty}\left\langle \gamma f(x^*) - Ax^*, x_n - q_n + q_n - x^*\right\rangle \\
&\quad \le \limsup_{n\to\infty}\left\langle \gamma f(x^*) - Ax^*, x_n - q_n\right\rangle + \limsup_{n\to\infty}\left\langle \gamma f(x^*) - Ax^*, q_n - x^*\right\rangle \\
&\quad \le 0.
\end{aligned} \tag{3.66}$$

Step 8. Finally we prove that

$$x_n \longrightarrow x^*. \tag{3.67}$$

Indeed, from (3.5), (3.15), and (3.17), we have

$$\begin{aligned}
&\|x_n - x^*\|^2 \\
&\quad = \left\|\alpha_n(\gamma f(u_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(q_n - x^*)\right\|^2 \\
&\quad \le \left\|\beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(q_n - x^*)\right\|^2 + 2\alpha_n\langle \gamma f(u_n) - Ax^*, x_n - x^*\rangle \\
&\quad \le \left[\|((1-\beta_n)I - \alpha_n A)(q_n - x^*)\| + \beta_n\|x_n - x^*\|\right]^2 + 2\alpha_n\gamma\langle f(u_n) - f(x^*), x_n - x^*\rangle \\
&\qquad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle \\
&\quad \le \left[(1-\beta_n - \alpha_n\overline{\gamma})\|\rho_n - x^*\| + \beta_n\|x_n - x^*\|\right]^2 + 2\alpha_n\gamma h\|x_n - x^*\|^2 \\
&\qquad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle \\
&\quad = \left((1-\alpha_n\overline{\gamma})^2 + 2\alpha_n\gamma h\right)\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle.
\end{aligned} \tag{3.68}$$

This implies that

$$\|x_n - x^*\|^2 \le \frac{2}{2(\overline{\gamma} - \gamma h) - \overline{\gamma}^2}\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle. \tag{3.69}$$

Combining (3.61) and (3.69), we obtain that $x_n \to x^*$.

This completes the proof of Theorem 3.2. □

Corollary 3.3. *Let $H, C, f, T, F, A, B, \varphi_i, \eta_i, K_i$ $(i = 1, 2, \ldots, N)$ be the same as in Theorem 3.2. Let r_i $(i = 1, 2, \ldots, N)$ be a finite family of positive parameter, $\lambda \in (0, 2\alpha]$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{t_n\} \subset (0,\infty)$. If $\mathcal{G} := F(\mathcal{T}) \bigcap MEP(\mathcal{F}) \bigcap VI(H, B, M) \neq \emptyset$ and conditions (i) and (ii) in Theorem 3.2 are satisfied, then*

(1) *for each $n \ge 1$ there is a unique $x_n \in C$ such that*

$$\begin{aligned}
x_n &= \alpha_n\gamma f\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right) + \beta_n x_n \\
&\quad + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)(P_C(I - \lambda B))^2\mathcal{V}^N x_n d;
\end{aligned} \tag{3.70}$$

(2) *the sequence $\{x_n\}$ converges strongly to some point $x^* \in \mathcal{G}$, provided that $V_{r_i}^{\Theta_i}$ is firmly nonexpansive;*

(3) *x^* is the unique solution of variational inequality* (3.6).

Proof. Taking $M = \partial\delta_C : H \to 2^H$ in Theorem 3.2, where $\delta_C : H \to [0,\infty)$ is the indicator function of C, that is,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{3.71}$$

then the variational inclusion problem (1.2) is equivalent to variational inequality (1.4), that is, to find $u \in C$ such that

$$\langle B(u), v - u\rangle \geq 0, \quad \forall v \in C. \tag{3.72}$$

Again, since $M = \partial\delta_C$, then $J_{M,\lambda} = P_C$. Therefore we have

$$\rho_n = P_C(I - \lambda B)\xi_n, \qquad \xi_n = P_C(I - \lambda B)y_n. \tag{3.73}$$

The conclusion of Corollary 3.3 can be obtained from Theorem 3.2 immediately. □

4. Applications to Optimization Problem

Let H be a real Hilbert space, C a nonempty closed convex subset of H, $A : H \to H$ a strongly positive linear bounded operator with a constant $\overline{\gamma} > 0$, and $T : C \to C$ a nonexpansive mapping. In this section we will utilize the results presented in Section 3 to study the following *optimization problem*:

$$\min_{x\in F(T)} \frac{1}{2}(\langle Ax, x\rangle - h(x)), \tag{4.1}$$

where $F(T)$ is the set of fixed points of T in C and $h : C \to R$ is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, $x \in C$), where $f : C \to C$ is a contractive mapping with a contractive constant $h \in (0,1)$. We have the following theorem.

Theorem 4.1. *Let H, C, f, T, A be the same as above. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0,1]$ satisfying condition (ii) in Theorem 3.2. If $F(T)$ is a nonempty compact subset of C, then for each $n \geq 1$ there is a unique $x_n \in C$ such that*

$$x_n = \alpha_n\gamma f(T(x_n)) + \beta_n x_n + \big((1-\beta_n)I - \alpha_n A\big)Tx_n, \quad \forall n \geq 1, \tag{4.2}$$

and the sequence $\{x_n\}$ converges strongly to some point $x^* \in F(T)$ which is the unique minimal point of optimization problem (4.1).

Proof. Taking $\Theta_i = 0, \varphi_i = 0, K_i = 0, \eta_i = 0, r_i = 1$ $(i = 1,2,\ldots,N)$, $B = 0$, $\mathfrak{T} = T$ in Corollary 3.3, hence we have $\mathfrak{F} = 0$, $V_{r_i}^{\Theta_i} = I, i = 1,2,\ldots,N$, $y_n = \xi_n = \rho_n = x_n$, $(1/t_n)\int_0^{t_n} T(s)x_n ds = Tx_n$, for all $n \geq 1$, $F(\mathfrak{T}) = F(T)$, $\mathrm{MEP}(\mathfrak{F}) = \mathrm{VI}(H,B,M) = C$, $\mathcal{G} = F(T)$. Hence from Corollary 3.3

we know that the sequence $\{x_n\}$ defined by (4.2) converges strongly to some point $x^* \in F(T)$ which is the unique solution of the following variational inequality:

$$\langle (A-\gamma f)x^*, x-x^*\rangle \geq 0, \quad x \in F(T). \tag{4.3}$$

Since T is nonexpansive, then $F(T)$ is convex. Again by the assumption that $F(T)$ is compact, therefore it is a compact and convex subset of C, and $(1/2)(\langle Ax,x\rangle - h(x)) : C \to R$ is a continuous mapping. By virtue of the well-known Weierstrass theorem, there exists a point $y^* \in F(T)$ which is a minimal point of optimization problem (4.1). As is known to all, (4.3) is the optimality necessary condition [19] for the optimization problem (4.1). Therefore we also have

$$\langle (A-\gamma f)y^*, x-y^*\rangle \geq 0, \quad \forall x \in F(T). \tag{4.4}$$

Since x^* is the unique solution of (4.3), we have $x^* = y^*$.

This completes the proof of Theorem 4.1. □

Acknowledgment

The authors would like to express their thanks to the referees for their valuable suggestions and comments.

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