Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 816363, 29 pages
doi:10.1155/2010/816363

Research Article

Derivatives of Orthonormal Polynomials and Coefficients of Hermite-Fejér Interpolation Polynomials with Exponential-Type Weights

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Received 10 November 2009; Accepted 14 January 2010

Academic Editor: Vijay Gupta

Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^2 : \mathbb{R} \to [0, \infty)$ be an even function. In this paper, we consider the exponential-type weights $w_\rho(x) = |x|^\rho \exp(-Q(x))$, $\rho > -1/2$, $x \in \mathbb{R}$, and the orthonormal polynomials $p_n(w_\rho^2; x)$ of degree n with respect to $w_\rho(x)$. So, we obtain a certain differential equation of higher order with respect to $p_n(w_\rho^2; x)$ and we estimate the higher-order derivatives of $p_n(w_\rho^2; x)$ and the coefficients of the higher-order Hermite-Fejér interpolation polynomial based at the zeros of $p_n(w_\rho^2; x)$.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$. Let $Q \in C^2 : \mathbb{R} \to \mathbb{R}^+$ be an even function and let $w(x) = \exp(-Q(x))$ be such that $\int_0^\infty x^n w^2(x)dx < \infty$ for all $n = 0, 1, 2, \ldots$. For $\rho > -1/2$, we set

$$w_\rho(x) := |x|^\rho w(x), \quad x \in \mathbb{R}. \tag{1.1}$$

Then we can construct the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ of degree n with respect to $w_\rho^2(x)$. That is,

$$\int_{-\infty}^{\infty} p_{n,\rho}(x) p_{m,\rho}(x) w_\rho^2(x)dx = \delta_{mn}(\text{Kronecker's delta}),$$
$$p_{n,\rho}(x) = \gamma_n x^n + \cdots, \quad \gamma_n = \gamma_{n,\rho} > 0. \tag{1.2}$$

We denote the zeros of $p_{n,\rho}(x)$ by

$$-\infty < x_{n,n,\rho} < x_{n-1,n,\rho} < \cdots < x_{2,n,\rho} < x_{1,n,\rho} < \infty. \tag{1.3}$$

A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be quasi-increasing if there exists $C > 0$ such that $f(x) \le Cf(y)$ for $0 < x < y$. For any two sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ of nonzero real numbers (or functions), we write $b_n \lesssim c_n$ if there exists a constant $C > 0$ independent of n (or x) such that $b_n \le Cc_n$ for n being large enough. We write $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. We denote the class of polynomials of degree at most n by $\mathcal{P}_n$.

Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t, and polynomials of degree at most n. The same symbol does not necessarily denote the same constant in different occurrences.

We shall be interested in the following subclass of weights from [1].

Definition 1.1. Let $Q : \mathbb{R} \to \mathbb{R}^+$ be even and satisfy the following properties.

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.

(b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.

(c) One has

$$\lim_{x\to\infty} Q(x) = \infty. \tag{1.4}$$

(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \ne 0 \tag{1.5}$$

is quasi-increasing in $(0,\infty)$ with

$$T(x) \ge \Lambda > 1, \quad x \in \mathbb{R}^+ \setminus \{0\}. \tag{1.6}$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \le C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus \{0\}. \tag{1.7}$$

Then we write $w \in \mathcal{F}(C^2)$. If there also exist a compact subinterval $J(\ni 0)$ of $\mathbb{R}$ and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \ge C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J, \tag{1.8}$$

then we write $w \in \mathcal{F}(C^2+)$.

In the following we introduce useful notations.

(a) Mhaskar-Rahmanov-Saff (MRS) numbers a_x is defined as the positive roots of the following equations:

$$x = \frac{2}{\pi}\int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{1/2}} du, \quad x > 0. \tag{1.9}$$

(b) Let

$$\eta_x = (xT(a_x))^{-2/3}, \quad x > 0. \tag{1.10}$$

(c) The function $\varphi_u(x)$ is defined as the following:

$$\varphi_u(x) = \begin{cases} \dfrac{a_{2u}^2 - x^2}{u\left[(a_u + x + a_u\eta_u)(a_u - x + a_u\eta_u)\right]^{1/2}}, & |x| \le a_u, \\ \varphi_u(a_u), & a_u < |x|. \end{cases} \tag{1.11}$$

In [2, 3] we estimated the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ associated with the weight $w_\rho^2 = |x|^{2\rho}\exp(-2Q(x)),\ \rho > -1/2$ and obtained some results with respect to the derivatives of orthonormal polynomials $p_{n,\rho}(x)$. In this paper, we will obtain the higher derivatives of $p_{n,\rho}(x)$. To estimate of the higher derivatives of the orthonormal polynomials sequence, we need further assumptions for $Q(x)$ as follows.

Definition 1.2. Let $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$ and let ν be a positive integer. Assume that $Q(x)$ is ν-times continuously differentiable on $\mathbb{R}$ and satisfies the followings.

(a) $Q^{(\nu+1)}(x)$ exists and $Q^{(i)}(x)$, $0 \le i \le \nu+1$ are positive for $x > 0$.

(b) There exist positive constants $C_i > 0$ such that for $x \in \mathbb{R} \setminus \{0\}$

$$\left|Q^{(i+1)}(x)\right| \le C_i \left|Q^{(i)}(x)\right| \frac{|Q'(x)|}{Q(x)}, \quad i = 1, \ldots, \nu. \tag{1.12}$$

(c) There exist constants $0 \le \delta < 1$ and $c_1 > 0$ such that on $(0, c_1]$

$$Q^{(\nu+1)}(x) \le C\left(\frac{1}{x}\right)^{\delta}. \tag{1.13}$$

Then we write $w(x) \in \mathcal{F}_\nu(C^2+)$. Furthermore, $w(x) \in \mathcal{F}_\nu(C^2+)$ and $Q(x)$ satisfies one of the following.

(a) $Q'(x)/Q(x)$ is quasi-increasing on a certain positive interval $[c_2, \infty)$.

(b) $Q^{(\nu+1)}(x)$ is nondecreasing on a certain positive interval $[c_2, \infty)$.

(c) There exists a constant $0 \le \delta < 1$ such that $Q^{(\nu+1)}(x) \le C(1/x)^{\delta}$ on $[c_2, \infty)$.

Then we write $w(x) \in \widetilde{\mathcal{F}}_\nu(C^2+)$.

Now, consider some typical examples of $\mathcal{F}(C^2+)$. Define for $\alpha > 1$ and $l \geq 1$,

$$Q_{l,\alpha}(x) := \exp_l(|x|^\alpha) - \exp_l(0). \tag{1.14}$$

More precisely, define for $\alpha + m > 1$, $m \geq 0$, $l \geq 1$ and $\alpha \geq 0$,

$$Q_{l,\alpha,m}(x) := |x|^m (\exp_l(|x|^\alpha) - \alpha^* \exp_l(0)) \tag{1.15}$$

where $\alpha^* = 0$ if $\alpha = 0$, otherwise $\alpha^* = 1$, and define

$$Q_\alpha(x) := (1 + |x|)^{|x|^\alpha} - 1, \quad \alpha > 1. \tag{1.16}$$

In the following, we consider the exponential weights with the exponents $Q_{l,\alpha,m}(x)$. Then we have the following examples (see [4]).

Example 1.3. Let ν be a positive integer. Let $m + \alpha - \nu > 0$. Then one has the following.

(a) $w(x) = \exp(-Q_{l,\alpha,m}(x))$ belongs to $\mathcal{F}_\nu(C^2+)$.

(b) If $l \geq 2$ and $\alpha > 0$, then there exists a constant $c_1 > 0$ such that $Q'_{l,\alpha,m}(x)/Q_{l,\alpha,m}(x)$ is quasi-increasing on (c_1, ∞).

(c) When $l = 1$, if $\alpha \geq 1$, then there exists a constant $c_2 > 0$ such that $Q'_{l,\alpha,m}(x)/Q_{l,\alpha,m}(x)$ is quasi-increasing on (c_2, ∞), and if $0 < \alpha < 1$, then $Q'_{l,\alpha,m}(x)/Q_{l,\alpha,m}(x)$ is quasidecreasing on (c_2, ∞).

(d) When $l = 1$ and $0 < \alpha < 1$, $Q^{(\nu+1)}_{l,\alpha,m}(x)$ is nondecreasing on a certain positive interval (c_2, ∞).

In this paper, we will consider the orthonormal polynomials $p_{n,\rho}(x)$ with respect to the weight class $\widetilde{\mathcal{F}}_\nu(C^2+)$. Our main themes in this paper are to obtain a certain differential equation for $p_{n,\rho}(x)$ of higher-order and to estimate the higher-order derivatives of $p_{n,\rho}(x)$ at the zeros of $p_{n,\rho}(x)$ and the coefficients of the higher-order Hermite-Fejér interpolation polynomials based at the zeros of $p_{n,\rho}(x)$. More precisely, we will estimate the higher-order derivatives of $p_{n,\rho}(x)$ at the zeros of $p_{n,\rho}(x)$ for two cases of an odd order and of an even order. These estimations will play an important role in investigating convergence or divergence of higher-order Hermite-Fejér interpolation polynomials (see [5–16]).

This paper is organized as follows. In Section 2, we will obtain the differential equations for $p_{n,\rho}(x)$ of higher-order. In Section 3, we will give estimations of higher-order derivatives of $p_{n,\rho}(x)$ at the zeros of $p_{n,\rho}(x)$ in a certain finite interval for two cases of an odd order and of an even order. In addition, we estimate the higher-order derivatives of $p_{n,\rho}(x)$ at all zeros of $p_{n,\rho}(x)$ for two cases of an odd order and of an even order. Furthermore, we will estimate the coefficients of higher-order Hermite-Fejér interpolation polynomials based at the zeros of $p_{n,\rho}(x)$, in Section 4.

2. Higher-Order Differential Equation for Orthonormal Polynomials

In the rest of this paper we often denote $p_{n,\rho}(x)$ and $x_{k,n,\rho}$ simply by $p_n(x)$ and x_{kn}, respectively. Let $\rho_n = \rho$ if n is odd, $\rho_n = 0$ otherwise, and define the integrating functions $A_n(x)$ and $B_n(x)$ with respect to $p_n(x)$ as follows:

$$A_n(x) := 2b_n \int_{-\infty}^{\infty} p_n^2(u)\overline{Q(x,u)}w_\rho^2(u)du,$$
$$B_n(x) := 2b_n \int_{-\infty}^{\infty} p_n(u)p_{n-1}(u)\overline{Q(x,u)}w_\rho^2(u)du, \tag{2.1}$$

where $\overline{Q(x,u)} = (Q'(x)-Q'(u))/(x-u)$ and $b_n = (\gamma_{n-1})/\gamma_n$. Then in [3, Theorem 4.1] we have a relation of the orthonormal polynomial $p_n(x)$ with respect to the weight $w_\rho^2(x)$:

$$p_n'(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x) - 2\rho_n \frac{p_n(x)}{x}. \tag{2.2}$$

Theorem 2.1 (cf. [6, Theorem 3.3]). *Let $\rho > -1/2$ and $w(x) \in \mathcal{F}(C^2)$. Then for $|x| > 0$ one has the second-order differential relation as follows:*

$$a(x)p_n''(x) + b(x)p_n'(x) + c(x)p_n(x) + D(x) + E(x) = 0. \tag{2.3}$$

Here, one knows that for any integer $n \geqslant 1$,

$$a(x) = A_n(x), \qquad b(x) = -2Q'(x)A_n(x) - A_n'(x),$$
$$c(x) = \frac{b_n A_n^2(x)A_{n-1}(x)}{b_{n-1}} + A_n(x)B_n(x)B_{n-1}(x) - \frac{xA_n(x)A_{n-1}(x)B_n(x)}{b_{n-1}}$$
$$+ A_n(x)B_n'(x) - A_n'(x)B_n(x) - 2\rho_n\frac{A_n(x)A_{n-1}(x)}{b_{n-1}} \tag{2.4}$$
$$=: c_1(x) + c_2(x) + c_3(x) + c_4(x) + c_5(x) + c_6(x),$$
$$D(x) = d(x)\frac{p_n(x)}{x}, \qquad E(x) = e_1(x)\frac{p_n'(x)}{x} + e_2(x)\frac{p_n(x)}{x^2},$$

where

$$d(x) = 2\rho_n\big(A_n(x)B_n(x) - A_n'(x)\big) + 2\rho_{n-1}A_n(x)B_n(x),$$
$$e_1(x) = 2\big(\rho_n + \rho_{n-1}\big)A_n(x), \qquad e_2(x) = -2\rho_n A_n(x). \tag{2.5}$$

Especially, when n is odd, one has

$$a(x)p_n''(x) + b(x)p_n'(x) + c(x)p_n(x) + d(x)q_{n-1}(x) + 2\rho A_n(x)q_{n-1}'(x) = 0, \tag{2.6}$$

where $q_{n-1}(x)$ is the polynomial of degree $n-1$ with $p_n(x) = xq_{n-1}(x)$.

Proof. We may similarly repeat the calculation [6, Proof of Theorem 3.3], and then we obtain the results. We stand for $A_n := A_n(x), B_n := B_n(x)$ simply. Applying (2.2) to $p'_{n-1}(x)$ we also see

$$p'_{n-1}(x) = A_{n-1}p_{n-2}(x) - B_{n-1}p_{n-1}(x) - 2\rho_{n-1}\frac{p_{n-1}(x)}{x}, \tag{2.7}$$

and so if we use the recurrence formula

$$xp_{n-1}(x) = b_n p_n(x) + b_{n-1}p_{n-2}(x) \tag{2.8}$$

and use (2.2) too, then we obtain the following:

$$\begin{aligned} p'_{n-1}(x) = \frac{1}{b_{n-1}A_n}\Big\{&(xA_{n-1} - b_{n-1}B_{n-1})p'_n(x) \\ &+ (xA_{n-1}B_n - b_{n-1}B_nB_{n-1} - b_nA_nA_{n-1})p_n(x) \\ &+ \frac{2\rho_n}{x}(xA_{n-1} - b_{n-1}B_{n-1})p_n(x) - \frac{2\rho_{n-1}b_{n-1}}{x}\left(p'_n(x) + B_np_n(x)\right)\Big\}. \end{aligned} \tag{2.9}$$

We differentiate the left and right sides of (2.2) and substitute (2.2) and (2.9). Then consequently, we have, for $n \geq 1$,

$$\begin{aligned} p''_n(x) = &-\left\{B_{n-1} + B_n - \frac{xA_{n-1}}{b_{n-1}} - \frac{A'_n}{A_n}\right\}p'_n(x) \\ &-\left\{\frac{b_nA_{n-1}A_n}{b_{n-1}} + B_{n-1}B_n - \frac{xA_{n-1}B_n}{b_{n-1}} + B'_n - \frac{A'_nB_n}{A_n} - 2\rho\frac{A_{n-1}}{b_{n-1}}\right\}p_n(x) \\ &- 2\rho_n\left(B_n - \frac{A'_n}{A_n}\right)\frac{p_n(x)}{x} - 2\rho_n\frac{xp'_n(x) - p_n(x)}{x^2} - 2\rho_{n-1}\frac{p'_n(x) + B_np_n(x)}{x}. \end{aligned} \tag{2.10}$$

Using the recurrence formula (2.8) and $u/(u-x) = 1 + x/(u-x)$, we have

$$\begin{aligned} B_n + B_{n-1} &= 2\int_{-\infty}^{\infty} p_{n-1}(u)\{b_np_n(u) + b_{n-1}p_{n-2}(u)\}\overline{Q(x,u)}w_\rho^2(u)du \\ &= 2\int_{-\infty}^{\infty} p_{n-1}^2(u)Q'(u)w_\rho^2(u)du - 2Q'(x) + 2x\int_{-\infty}^{\infty} p_{n-1}^2(u)\overline{Q(x,u)}w_\rho^2(u)du \\ &= -2Q'(x) + \frac{xA_{n-1}}{b_{n-1}}, \end{aligned} \tag{2.11}$$

because $Q'(u)$ is an odd function. Therefore, we have

$$b(x) = -2Q'(x)A_n - A'_n. \tag{2.12}$$

When n is odd, since $xp'_n(x) - p_n(x) = x^2q'_{n-1}(x)$, (2.6) is proved. □

For the higher-order differential equation for orthonormal polynomials, we see that for $j = 0, 1, 2, \dots, \nu - 2$ and $|x| > 0$

$$D^{(j)}(x) = \sum_{t=0}^{j}\left(\sum_{i=t}^{j}\frac{(-1)^{i-t}j!}{(j-i)!t!}d^{(j-i)}(x)x^{-(i-t+1)}\right)p_n^{(t)}(x),$$

$$\begin{aligned} E^{(j)}(x) = &\sum_{t=0}^{j}\left(\sum_{i=t}^{j}\frac{(-1)^{i-t}j!}{(j-i)!t!}e_1^{(j-i)}(x)x^{-(i-t+1)}\right)p_n^{(t+1)}(x) \\ &+\sum_{t=0}^{j}\left(\sum_{i=t}^{j}\frac{(-1)^{i-t}j!(i-t+1)}{(j-i)!t!}e_2^{(j-i)}(x)x^{-(i-t+2)}\right)p_n^{(t)}(x). \end{aligned} \tag{2.13}$$

Let $\binom{j}{-1} = 0$ for nonnegative integer j. In the following theorem, we show the higher-order differential equation for orthonormal polynomials.

Theorem 2.2. *Let $\rho > -1/2$ and $w(x) \in \mathcal{F}(C^2)$. Let $\nu \geqslant 2$ and $j = 0, 1, \dots, \nu - 2$. Then one has the following equation for $|x| > 0$:*

$$B_{j+2}^{[j]}(x)p_n^{(j+2)}(x) + B_{j+1}^{[j]}(x)p_n^{(j+1)}(x) + \sum_{s=0}^{j}B_s^{[j]}(x)p_n^{(s)}(x) = 0, \tag{2.14}$$

where

$$B_{j+2}^{[j]}(x) = a(x), \qquad B_{j+1}^{[j]}(x) = ja'(x) + b(x) + \frac{e_1(x)}{x}, \tag{2.15}$$

and for $j \geq 1$ and $1 \leq s \leq j$

$$\begin{aligned} B_s^{[j]}(x) = &\binom{j}{s-2}a^{(j-s+2)}(x) + \binom{j}{s-1}b^{(j-s+1)}(x) + \binom{j}{s}c^{(j-s)}(x) \\ &+\sum_{i=s}^{j}\frac{(-1)^{i-s}j!}{(j-i)!s!}d^{(j-i)}(x)x^{-(i-s+1)} + \sum_{i=s-1}^{j}\frac{(-1)^{i-s+1}j!}{(j-i)!(s-1)!}e_1^{(j-i)}(x)x^{-(i-s+2)} \\ &+\sum_{i=s}^{j}\frac{(-1)^{i-s}j!(i-s+1)}{(j-i)!s!}e_2^{(j-i)}(x)x^{-(i-s+2)}, \end{aligned} \tag{2.16}$$

and for $j \geq 0$

$$B_0^{[j]}(x) = c^{(j)}(x) + \sum_{i=0}^{j}\frac{(-1)^{i}j!}{(j-i)!}d^{(j-i)}(x)x^{-(i+1)} + \sum_{i=0}^{j}\frac{(-1)^{i}j!(i+1)}{(j-i)!}e_2^{(j-i)}(x)x^{-(i+2)}. \tag{2.17}$$

Proof. It comes from Theorem 2.1 and (2.13). □

Corollary 2.3. *Under the same assumptions as Theorem 2.1, if n is odd, then*

$$C_{j+2}^{[j]}(0)p_n^{(j+2)}(0)+C_{j+1}^{[j]}(0)p_n^{(j+1)}(0)+\sum_{s=1}^{j}C_s^{[j]}(0)p_n^{(s)}(0)=0,\quad j\geq 1,$$
$$C_2^{[0]}(0)p_n''(0)+C_1^{[0]}(0)p_n'(0)=0,\quad j=0, \tag{2.18}$$

where $C_{j+2}^{[j]}(x)=A_n(0)+(2\rho/(j+2))A_n(0)$ *and for* $1\leq s\leq j+1$

$$C_s^{[j]}(0)=\binom{j}{s-2}a^{(j-s+2)}(0)+\binom{j}{s-1}b^{(j-s+1)}(0)+\binom{j}{s}c^{(j-s)}(0)$$
$$+\frac{1}{s}\left(\binom{j}{s-1}d^{(j-s+1)}(0)+\binom{j}{s-2}2\rho A_n^{(j-s+2)}(0)\right). \tag{2.19}$$

Proof. Let n be odd. Then we will consider (2.6). Since $q_{n-1}^{(j)}(0)=p_n^{(j+1)}(0)/(j+1)$, we have

$$\left(d(x)q_{n-1}(x)+2\rho A_n(x)q_{n-1}'(x)\right)^{(j)}\Big|_{x=0}$$
$$=2\rho A_n(0)\frac{p_n^{(j+2)}(0)}{j+2}+\left(d(0)+2j\rho A_n'(0)\right)\frac{p_n^{(j+1)}(0)}{j+1}$$
$$+\sum_{s=2}^{j}\left(\binom{j}{s-1}d^{(j-s+1)}(0)+\binom{j}{s-2}2\rho A^{(j-s+2)}(0)\right)\frac{p_n^{(s)}(0)}{s}+d^{(j)}(0)p_n'(0), \tag{2.20}$$

and we have

$$\left(a(x)p_n''(x)+b(x)p_n'(x)+c(x)p_n(x)\right)^{(j)}\Big|_{x=0}$$
$$=a(0)p_n^{(j+2)}(0)+\left(ja'(0)+b(0)\right)p_n^{(j+1)}(0)$$
$$+\sum_{s=0}^{j}\left(\binom{j}{s-2}a^{(j-s+2)}(0)+\binom{j}{s-1}b^{(j-s+1)}(0)+\binom{j}{s}c^{(j-s)}(0)\right)p_n^{(s)}(0). \tag{2.21}$$

Therefore, we have the result from (2.6). □

In the rest of this paper, we let $\rho>-1/2$ and $w(x)=\exp(-Q(x))\in\tilde{\mathcal{F}}_\nu(C^2+)$ for positive integer $\nu\geq 1$ and assume that $1+2\rho-\delta\geq 0$ for $\rho<0$ and

$$a_n\lesssim n^{1/(1+\nu-\delta)}, \tag{2.22}$$

where $0\leq\delta<1$ is defined in (1.13).

In Section 3, we will estimate the higher-order derivatives of orthonormal polynomials at the zeros of orthonormal polynomials with respect to exponential-type weights.

3. Estimation of Higher-Order Derivatives of Orthonormal Polynomials

From [3, Theorem 4.2] we know that there exist C and $n_0 > 0$ such that for $n \ge n_0$ and $|x| \le a_n(1+\eta_n)$,

$$\frac{A_n(x)}{2b_n} \sim \varphi_n(x)^{-1}\left(a_n^2(1+2\eta_n)^2 - x^2\right)^{-1/2}, \qquad |B_n(x)| \lesssim A_n(x). \tag{3.1}$$

If $T(x)$ is unbounded, then (2.22) is trivially satisfied. Additionally we have, from [17, Theorem 1.3], that if we assume that $Q''(x)$ is nondecreasing, then for $|x| \le \varepsilon a_n$ with $0 < \varepsilon < 1/2$

$$|B_n(x)| < \lambda(\varepsilon, n) A_n(x), \tag{3.2}$$

where there exists a constant $C > 0$ such that

$$\lambda(\varepsilon, n) = C \cdot \max\left\{\left(\frac{1}{n\theta}+1\right)\theta^{\Lambda-1},\ \varepsilon^{(1-1/\Lambda)(\Lambda-1)},\ \varepsilon^{1/\Lambda},\ \lambda(n)\right\}, \tag{3.3}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\lambda(\varepsilon, n) = 0. \tag{3.4}$$

Here, $\theta = \varepsilon^{(\Lambda-1)/2\Lambda}$ and $\lambda(n) = O(e^{-n^C})$ for some $C > 0$.

For the higher derivatives of $A_n(x)$ and $B_n(x)$, we have the following results in [17, Theorem 1.8].

Theorem 3.1 (see[17, Theorem 1.4]). *For $|x| \le a_n(1+\eta_n)$ and $j = 0, \ldots, \nu - 1$*

$$\left|A_n^{(j)}(x)\right| \lesssim A_n(x)\left(\frac{T(a_n)}{a_n}\right)^j, \qquad \left|B_n^{(j)}(x)\right| \lesssim A_n(x)\left(\frac{T(a_n)}{a_n}\right)^j. \tag{3.5}$$

Moreover, there exists $\varepsilon(n) > 0$ such that for $|x| \le a_n/2$ and $j = 1, \ldots, \nu - 1$,

$$\left|A_n^{(j)}(x)\right| \le \varepsilon(n) A_n(x)\left(\frac{n}{a_n}\right)^j, \qquad \left|B_n^{(j)}(x)\right| \le \varepsilon(n) A_n(x)\left(\frac{n}{a_n}\right)^j, \tag{3.6}$$

with $\varepsilon(n) \to 0$ as $n \to \infty$.

Corollary 3.2. *Let $0 < \beta_1 < 1/2$. Then there exists a positive constant $C \ne C(n)$ such that one has for $|x| \le \beta_1 a_n$ and $j = 1, \ldots, \nu - 1$,*

$$\left|A_n^{(j)}(x)\right| \le C A_n(x)\left(\frac{n}{a_n}\right)^j, \qquad \left|B_n^{(j)}(x)\right| \le C A_n(x)\left(\frac{n}{a_n}\right)^j. \tag{3.7}$$

In the following, we have the estimation of the higher-order derivatives of orthonormal polynomials.

Theorem 3.3. *Let* $1 \le 2s+1 \le \nu$ *and* $0 < \alpha < 1/2$. *Then for* $a_n/\alpha n \le |x_{kn}| \le \alpha a_n$ *the following equality holds for* n *large enough:*

$$p_n^{(2s+1)}(x_{kn}) = (-1)^s \beta^s(x_{kn}, n)\left(\frac{n}{a_n}\right)^{2s}\left(1 + \tilde{\rho}_{2s+1}(\alpha, x_{kn}, n)\right)p_n'(x_{kn}), \tag{3.8}$$

where

$$\beta(x, n) := \frac{b_n}{b_{n-1}}\left(\frac{a_n}{n}\right)^2 A_n(x)A_{n-1}(x), \tag{3.9}$$

and $|\tilde{\rho}_{2s+1}(\alpha, x_{kn}, n)| \le C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n))$. *Moreover, for* $1 \le 2s \le \nu$

$$\left|p_n^{(2s)}(x_{kn})\right| \lesssim C\mu_1(\alpha, n)\left(\frac{n}{a_n}\right)^{2s-1}\left|p_n'(x_{kn})\right|. \tag{3.10}$$

Here,

$$\begin{gathered}\mu_1(\alpha, n) := \left(\varepsilon(n) + \alpha^{\Lambda-1} + \alpha\right), \qquad \mu_2(\alpha, n) := \frac{\log n}{n} + \varepsilon(n) + \alpha\lambda(\alpha, n) + \alpha^2, \\ \mu_3(\alpha, n) := \lambda(\alpha, n)\lambda(\alpha, n-1) + \alpha\lambda(\alpha, n) + \varepsilon(n) + \varepsilon(n)\lambda(\alpha, n) + \frac{1}{n}.\end{gathered} \tag{3.11}$$

Corollary 3.4. *Suppose the same assumptions as Theorem 3.3. Given any* $\delta > 0$, *there exists a small fixed positive constant* $0 < \alpha_0(\delta) < 1/2$ *such that* (3.8) *holds satisfying* $|\tilde{\rho}_{2s+1}(\alpha_0, x_{kn}, n)| \le \delta$ *and*

$$\left|p_n^{(2s)}(x_{kn})\right| \le \delta\left(\frac{n}{a_n}\right)^{2s-1}\left|p_n'(x_{kn})\right| \tag{3.12}$$

for $a_n/\alpha_0 n \le |x_{kn}| \le \alpha_0 a_n$.

Corollary 3.5. *For* $|x_{kn}| \le a_n/2$ *and* $1 \le j \le \nu$

$$\left|p_n^{(j)}(x_{kn})\right| \lesssim \left(\frac{n}{a_n}\right)^{j-1}\left|p_n'(x_{kn})\right|. \tag{3.13}$$

Theorem 3.6. *Let* $0 < |x_{kn}| \le a_n(1+\eta_n)$ *and let* $\nu = 2, 3, \ldots$, $j = 1, 2, \ldots, \nu - 2$. *Then*

$$\left|p_n^{(j+2)}(x_{kn})\right| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{j+1}\left|p_n'(x_{kn})\right|, \tag{3.14}$$

and especially if j is even, then

$$\left|p_n^{(j+2)}(x_{kn})\right| \lesssim \left(\frac{T(a_n)}{a_n} + \left|Q'(x_{kn})\right| + \frac{1}{|x_{kn}|}\right)\left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^j \left|p_n'(x_{kn})\right|. \tag{3.15}$$

We note that for n large enough,

$$|x_{kn}| < a_n(1+\eta_n), \quad k = 1,2,\ldots,n, \tag{3.16}$$

because we know that $x_{1n} < a_{n+\rho/2}$ from [3, Theorem 2.2] and

$$\begin{aligned} a_{n+\rho/2} - a_n &= a_{n+\rho/2}\left(1 - \frac{a_n}{a_{n+\rho/2}}\right) \le C_1 \frac{a_{n+\rho/2}}{T(a_n)} \log\left(1 + \frac{\rho/2}{n}\right) \\ &\le C_2 \frac{a_n}{nT(a_n)} \le a_n o(\eta_n). \end{aligned} \tag{3.17}$$

To prove these results we need some lemmas.

Lemma 3.7. (a) *For $s \ge r > 0$*

$$T(a_r)\left(1 - \frac{a_r}{a_s}\right) \le C \log\frac{s}{r}. \tag{3.18}$$

(b) *For $|x| \le (1/2)a_n$*

$$\left|Q'(x)\right| \le C\left(\frac{x}{a_n}\right)^{\Lambda-1} \frac{n}{a_n}. \tag{3.19}$$

(c) *For $|x| \le a_n(1+\eta_n)$*

$$|A_n(x)| \sim \frac{n}{a_{2n} - |x|}. \tag{3.20}$$

(d) *Let $0 \le j \le \nu - 1$. Then for $|x| \le a_n/2$*

$$\left|Q^{(j+1)}(x)\right| \lesssim \left|Q'(a_n/2)\right| \left(\frac{T(a_n/2)}{a_n}\right)^j, \tag{3.21}$$

and for $a_n/2 \le |x| \le a_n(1+\eta_n)$

$$\left|Q^{(j+1)}(x)\right| \lesssim \left|Q'(x)\right| \left(\frac{T(a_n)}{a_n}\right)^j. \tag{3.22}$$

Proof. (a) It is [1, Lemma 3.11(c)]. (b) It is [1, Lemma 3.8(c)]. (c) It comes from (3.1). (d) Since $j+1 \le \nu$, $Q^{(j+1)}(x)$ is increasing. So, we obtain (d) by (1.12). □

Lemma 3.8. *Let $a(x), b(x), c(x), d(x)$, and $e_i(x)$, $i = 1, 2$, be defined in Theorem 2.1.*

(a) *For $|x| \le a_n/2$ and $1 \le k \le \nu - 1$, there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \to 0$ as $n \to 0$ such that*

$$\left|a^{(k)}(x)\right| \lesssim \varepsilon(n)\left(\frac{n}{a_n}\right)^{k+1}. \tag{3.23}$$

Moreover, for $|x| \le a_n(1+\eta_n)$ and $1 \le k \le \nu - 1$,

$$\left|a^{(k)}(x)\right| \lesssim \left(\frac{T(a_n)}{a_n}\right)^{k} A_n(x). \tag{3.24}$$

(b) *For $|x| \le a_n/2$ and $1 \le k \le \nu - 2$, there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \to 0$ as $n \to 0$ such that*

$$\left|b^{(k)}(x)\right| \lesssim \varepsilon(n)\left(\frac{n}{a_n}\right)^{k+2}. \tag{3.25}$$

Moreover, for $|x| \le a_n(1+\eta_n)$ and $1 \le k \le \nu - 1$,

$$\left|b^{(k)}(x)\right| \lesssim \left(Q'(x) + \frac{n}{a_n}\right)\left(\frac{T(a_n)}{a_n}\right)^{k} A_n(x). \tag{3.26}$$

(c) *For $|x| \le a_n/2$ and $1 \le k \le \nu - 3$, there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \to 0$ as $n \to 0$ such that*

$$\left|c_i^{(k)}(x)\right| \lesssim \varepsilon(n)\left(\frac{n}{a_n}\right)^{k+3}, \quad i = 1,2,3,4,5,6. \tag{3.27}$$

Moreover, for $|x| \le a_n(1+\eta_n)$ and $1 \le k \le \nu - 3$,

$$\left|c_i^{(k)}(x)\right| \lesssim \left(\frac{T(a_n)}{a_n}\right)^{k} A_n^3(x), \quad i = 1,2,3,4,5,6. \tag{3.28}$$

(d) *For $|x| \le a_n/2$ and $1 \le k \le \nu - 3$, there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \to 0$ as $n \to 0$ such that*

$$\left|d^{(k)}(x)\right| \lesssim \varepsilon(n)\left(\frac{n}{a_n}\right)^{k+2}, \qquad \left|e_i^{(k)}(x)\right| \lesssim \varepsilon(n)\left(\frac{n}{a_n}\right)^{k+1}, \quad i = 1,2. \tag{3.29}$$

Moreover, for $|x| \le a_n(1+\eta_n)$ *and* $0 \le k \le \nu - 3$,

$$\left|d^{(k)}(x)\right| \lesssim \left(A_n(x) + \frac{T(a_n)}{a_n}\right)\left(\frac{T(a_n)}{a_n}\right)^k A_n(x),$$
$$\left|e_i^{(k)}(x)\right| \lesssim \left(\frac{T(a_n)}{a_n}\right)^k A_n(x), \quad i = 1, 2. \tag{3.30}$$

Proof. (a) Since $a(x) = A_n(x)$, we prove it by Theorem 3.1.

(b) For $1 \le k \le \nu - 2$, we see

$$b^{(k)}(x) = -\left(A_n^{(k+1)}(x) + 2\sum_{p=0}^{k}\binom{k}{p}Q^{(p+1)}(x)A_n^{(k-p)}(x)\right). \tag{3.31}$$

From (3.18), we know that $T(a_n/2) \lesssim \log n$. Therefore by (3.19), (3.21), and (3.6) we have for $0 \le x \le a_n/2$

$$\left|Q^{(p+1)}(x)A_n^{(k-p)}(x)\right| \lesssim \left|Q'\left(\frac{a_n}{2}\right)\right|\left(\frac{T(a_n/2)}{a_n}\right)^p \left|A_n^{(k-p)}(x)\right| \lesssim \varepsilon(n)\left(\frac{n}{a_n}\right)^{k+2}, \tag{3.32}$$

and for $|x| \le a_n(1+\eta_n)$ we have by (3.21) and (3.22)

$$\left|Q^{(p+1)}(x)A_n^{(k-p)}(x)\right| \lesssim \left(Q'(x) + \frac{n}{a_n}\right)\left(\frac{T(a_n)}{a_n}\right)^k A_n(x). \tag{3.33}$$

Consequently we have (b).

(c) Next we estimate $c^{(k)}(x)$. Suppose $|x| \le a_n/2$. Let us set $c(x) = \sum_{i=1}^{6} c_i(x)$. By (3.6) and (3.20) we have

$$\left|c_1^{(k)}(x)\right| \lesssim \sum_{t,u,v,t+u+v=k} A_n^{(t)}(x)A_n^{(u)}(x)A_{n-1}^{(v)}(x)$$
$$\lesssim \varepsilon(n)\sum_{t,u,v,t+u+v=k}\left(\frac{n}{a_n}\right)^k A_n^3(x) \lesssim \varepsilon(n)\left(\frac{n}{a_n}\right)^{k+3}. \tag{3.34}$$

For $c_i^{(k)}(x)$ $(i = 2,3,4,5,6)$, we obtain the same estimate as $c_1^{(k)}$:

$$\left|c_i^{(k)}(x)\right| \lesssim \varepsilon(n)\left(\frac{n}{a_n}\right)^{k+3}, \quad i = 2,3,4,5,6. \tag{3.35}$$

For $|x| \le a_n(1+\eta_n)$, we have similarly to the case of $|x| \le a_n/2$

$$\left|c_i^{(k)}(x)\right| \lesssim \left(\frac{T(a_n)}{a_n}\right)^k A_n^3(x), \quad i = 1,2,3,4,5,6. \tag{3.36}$$

(d) It is similar to (c). Consequently we have the following lemma. □

Lemma 3.9. *Let $0 < \alpha < 1/2$, $0 \leq j \leq \nu - 2$, and $L_1 > 0$. Let $a_n/\alpha n \leq |x| \leq \alpha a_n$. Then*

$$\left| \frac{B_{j+1}^{[j]}(x)}{B_{j+2}^{[j]}(x)} \right| \leq C\mu_1(\alpha, n) \frac{n}{a_n}, \tag{3.37}$$

where $\mu_1(\alpha, n)$ is defined in Theorem 3.3 and for $L_1(a_n/n) \leq |x| \leq a_n/2$

$$\left| \frac{B_{j+1}^{[j]}(x)}{B_{j+2}^{[j]}(x)} \right| \leq C \frac{n}{a_n}. \tag{3.38}$$

Moreover, for $|x| \leq a_n(1 + \eta_n)$,

$$\left| \frac{B_{j+1}^{[j]}(x)}{B_{j+2}^{[j]}(x)} \right| \lesssim \frac{T(a_n)}{a_n} + \left| Q'(x) \right| + \frac{1}{|x|}. \tag{3.39}$$

Proof. Since

$$\begin{aligned} \left| B_{j+1}^{[j]}(x) \right| &= \left| j a'(x) + b(x) + \frac{e_1(x)}{x} \right| \\ &\lesssim \left| (j-1) A_n'(x) - 2Q'(x) A_n(x) \right| + \left| \frac{A_n(x)}{x} \right|, \end{aligned} \tag{3.40}$$

we have (3.39) for $|x| \leq a_n(1 + \eta_n)$ by (3.5). For $a_n/\alpha n \leq |x| \leq \alpha a_n$ we have from (3.6) and (3.19) that

$$\left| B_{j+1}^{[j]}(x) \right| \leq \left(\varepsilon(n) + C_1 \alpha^{\Lambda - 1} + C_2 \alpha \right) \frac{n}{a_n} A_n(x) \leq C\mu_1(\alpha, n) \frac{n}{a_n} A_n(x). \tag{3.41}$$

Moreover, we can obtain (3.38) for $L_1(a_n/n) \leq |x| \leq a_n/2$ from the above easily. □

Lemma 3.10. *Let $0 < \alpha < 1/2$ and $0 \leq j \leq \nu - 2$. Let $a_n/\alpha n \leq |x| \leq \alpha a_n$. Then for $a_n/\alpha n \leq |x| \leq \alpha a_n$*

$$-\frac{B_j^{[j]}(x)}{B_{j+2}^{[j]}(x)} = (-1)\beta(x, n)\left(1 + f_j(\alpha, x_{kn}, n)\right) \left(\frac{n}{a_n} \right)^2 \tag{3.42}$$

with $|f_j(\alpha, x_{kn}, n)| \leq C(\mu_2(\alpha, n) + \mu_3(\alpha, n))$, where $\mu_2(\alpha, n)$, $\mu_3(\alpha, n)$, and $\beta(x, n)$ are defined in Theorem 3.3. For $L_1(a_n/n) \leq |x| \leq (1/2)a_n$ one has

$$\left| \frac{B_j^{[j]}(x)}{B_{j+2}^{[j]}(x)} \right| \leq C \left(\frac{n}{a_n} \right)^2. \tag{3.43}$$

On the other hand, one has for $L_1(a_n/n) < |x| \le a_n(1+\eta_n)$,

$$\left|\frac{B_j^{[j]}(x)}{B_{j+2}^{[j]}(x)}\right| \lesssim \left(A_n(x) + \frac{T(a_n)}{a_n}\right)^2. \tag{3.44}$$

Proof. First, we know that

$$\begin{aligned} B_j^{[j]}(x) &= \frac{j(j-1)}{2}a''(x) + jb'(x) + c(x) \\ &\quad + d(x)x^{-1} + je_1'(x)x^{-1} - je_1(x)x^{-2} + e_2(x)x^{-2}. \end{aligned} \tag{3.45}$$

Suppose $a_n/\alpha n \le |x| \le \alpha a_n$. Since from (3.18) and (3.19)

$$\left|Q''\left(\frac{a_n}{2}\right)\right| \lesssim \frac{\log n}{n}\left(\frac{n}{a_n}\right)^2, \qquad \left|Q'\left(\frac{a_n}{2}\right)\right| \lesssim \frac{n}{a_n}, \tag{3.46}$$

we have from (3.6)

$$\left|\frac{j(j-1)}{2}a''(x) + jb'(x)\right| \le C_1\left(\frac{\log n}{n} + \varepsilon(n)\right)\left(\frac{n}{a_n}\right)^2 A_n(x). \tag{3.47}$$

Since

$$|d(x)| \le C_1(\lambda(\alpha,n) + \varepsilon(n))\frac{n}{a_n}A_n(x), \tag{3.48}$$

we know from (3.6) that

$$\left|d(x)x^{-1} + je_1'(x)x^{-1} - je_1(x)x^{-2} + e_2(x)x^{-2}\right| \le C\alpha(\lambda(\alpha,n) + \varepsilon(n) + \alpha)\left(\frac{n}{a_n}\right)^2 A_n(x). \tag{3.49}$$

Therefore we have for $a_n/\alpha n \le |x| \le \alpha a_n$

$$\left|B_j^{[j]}(x) - c(x)\right| \le C\mu_2(\alpha,n)\left(\frac{n}{a_n}\right)^2 A_n(x). \tag{3.50}$$

Since from (3.3)

$$\begin{aligned} |c_2(x) + c_3(x)| &= \left|A_n(x)B_n(x)B_{n-1}(x) + \frac{x}{b_{n-1}}A_n(x)A_{n-1}(x)B_n(x)\right| \\ &\le C(\lambda(\alpha,n)\lambda(\alpha,n-1) + \alpha\lambda(\alpha,n))\left(\frac{n}{a_n}\right)^2 A_n(x) \end{aligned} \tag{3.51}$$

and similarly

$$|c_4(x) + c_5(x) + c_6(x)| \leq C\left(\varepsilon(n) + \varepsilon(n)\lambda(\alpha, n) + \frac{1}{n}\right)\left(\frac{n}{a_n}\right)^2 A_n(x), \tag{3.52}$$

we have

$$|c_2(x) + c_3(x) + c_4(x) + c_5(x) + c_6(x)| \leq C\mu_3(\alpha, n)\left(\frac{n}{a_n}\right)^2 A_n(x). \tag{3.53}$$

Then we have

$$\left|\frac{B_j^{[j]}(x)}{B_{j+2}^{[j]}(x)} - \frac{c_1(x)}{B_{j+2}^{[j]}(x)}\right| \leq C(\mu_2(\alpha, n) + \mu_3(\alpha, n))\left(\frac{n}{a_n}\right)^2. \tag{3.54}$$

Therefore, since

$$\frac{c_1(x)}{B_{j+2}^{[j]}(x)} = \beta(x, n)\left(\frac{n}{a_n}\right)^2, \tag{3.55}$$

there exist constants $f_j(\alpha, x_{kn}, n)$ with $|f_j(\alpha, x_{kn}, n)| \leq C(\mu_2(\alpha, n) + \mu_3(\alpha, n))$ such that we have for $a_n/\alpha n \leq |x| \leq \alpha a_n$

$$-\frac{B_j^{[j]}(x)}{B_{j+2}^{[j]}(x)} = (-1)\beta(x, n)(1 + f_j(\alpha, x_{kn}, n))\left(\frac{n}{a_n}\right)^2. \tag{3.56}$$

Especially, from the above estimates we can see (3.43) for $L_1(a_n/n) \leq |x| \leq a_n/2$. On the other hand, suppose $L_1(a_n/n) \leq |x| \leq a_n(1 + \eta_n)$. Then since from Theorem 2.1 and (3.5)

$$|c(x)| \lesssim A_n^3(x) + \frac{T(a_n)}{a_n} A_n^2(x) \lesssim \left(A_n(x) + \frac{T(a_n)}{a_n}\right)^2 A_n(x) \tag{3.57}$$

and $|Q'(x)| + n/a_n \lesssim A_n(x)$, we have from Lemma 3.8

$$\left|B_j^{[j]}(x)\right| \lesssim \left(A_n(x) + \frac{T(a_n)}{a_n}\right)^2 A_n(x). \tag{3.58}$$

Therefore, we have (3.44) for $L_1(a_n/n) < |x| \leq a_n(1 + \eta_n)$. □

Lemma 3.11. *Let $0 < \alpha < 1/2$ and $1 \le j \le \nu - 2$. Let $L_1(a_n/n) \le |x| \le a_n/2$. Then for $\ell = 1, 2, \ldots, j-1$, there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \to 0$ as $n \to 0$ such that*

$$\left|\frac{B_\ell^{[j]}(x)}{B_{j+2}^{[j]}(x)}\right| \le \varepsilon(n)\left(\frac{n}{a_n}\right)^{j-\ell+2}. \tag{3.59}$$

Moreover, one has for $L_1(a_n/n) \le |x| \le a_n(1+\eta_n)$,

$$\left|\frac{B_\ell^{[j]}(x)}{B_{j+2}^{[j]}(x)}\right| \lesssim \frac{T(a_n)}{a_n}\left(A_n(x) + \frac{T(a_n)}{a_n}\right)^{j-\ell+1}. \tag{3.60}$$

Proof. For $\ell = 1, 2, \ldots, j-1$ we have from Lemma 3.8 that there exists $\varepsilon(n)$ satisfying $\varepsilon(n) \to 0$ as $n \to 0$ such that

$$\begin{aligned}
\left|B_\ell^{[j]}(x)\right| &= \left|a^{(j-\ell+2)}(x)\right| + \left|b^{(j-\ell+1)}(x)\right| + \left|c^{(j-\ell)}(x)\right| \\
&\quad + |x|^{-1}\sum_{i=\ell}^{j}\left|d^{(j-i)}(x)\right| + |x|^{-1}\sum_{i=\ell-1}^{j}\left|e_1^{(j-i)}(x)\right| + |x|^{-2}\sum_{i=\ell}^{j}\left|e_2^{(j-i)}(x)\right| \\
&\le \varepsilon(n)\left(\frac{n}{a_n}\right)^{j-\ell+3} + \varepsilon(n)\frac{\alpha n}{a_n}\left(\frac{n}{a_n}\right)^{j-\ell+2} + \varepsilon(n)\left(\frac{\alpha n}{a_n}\right)^{2}\left(\frac{n}{a_n}\right)^{j-\ell+1} \\
&\le \varepsilon(n)\left(\frac{n}{a_n}\right)^{j-\ell+3}.
\end{aligned} \tag{3.61}$$

Similarly, for $\ell = 1, 2, \ldots, j-1$ and $L_1(a_n/n) < |x| \le a_n(1+\eta_n)$,

$$\left|B_\ell^{[j]}(x)\right| \lesssim \frac{T(a_n)}{a_n}\left(A_n(x) + \frac{T(a_n)}{a_n}\right)^{j-\ell+1} A_n(x). \tag{3.62}$$

Therefore, we have the results. □

Proof of Theorem 3.3. First we know that the following differential equation is satisfied:

$$\begin{aligned}
p_n^{(j+2)}(x_{kn}) &= -\frac{B_{j+1}^{[j]}(x_{kn})}{B_{j+2}^{[j]}(x_{kn})}p_n^{(j+1)}(x_{kn}) - \frac{B_{j}^{[j]}(x_{kn})}{B_{j+2}^{[j]}(x_{kn})}p_n^{(j)}(x_{kn}) \\
&\quad - \frac{B_{j-1}^{[j]}(x_{kn})}{B_{j+2}^{[j]}(x_{kn})}p_n^{(j-1)}(x_{kn}) - \cdots - \frac{B_{1}^{[j]}(x_{kn})}{B_{j+2}^{[j]}(x_{kn})}p_n'(x_{kn}).
\end{aligned} \tag{3.63}$$

Suppose $L_1(a_n/n) \le |x_{kn}| \le (1/2)a_n$. Then since we see from (3.63) and (3.38) that

$$\left|p_n''(x_{kn})\right| \le C\frac{n}{a_n}\left|p_n'(x_{kn})\right|, \tag{3.64}$$

we have by (3.63) and mathematical induction

$$\left|p_n^{(j+1)}(x_{kn})\right| \lesssim \left(\frac{n}{a_n}\right)^j \left|p_n'(x_{kn})\right|. \tag{3.65}$$

Next, suppose $a_n/\alpha n \leq |x_{kn}| \leq \alpha a_n$. More precisely, from Lemma 3.9 we have

$$\left|p_n''(x_{kn})\right| \leq C\mu_1(\alpha, n)\frac{n}{a_n}\left|p_n'(x_{kn})\right|. \tag{3.66}$$

Then by (3.63), (3.42), and (3.66) there exists a constant $\tilde{\rho}_1(\alpha, x_{kn}, n)$ with

$$\left|\tilde{\rho}_1(\alpha, x_{kn}, n)\right| \leq \left|f_1(\alpha, x_{kn}, n) + C\mu_1(\alpha, x_{kn})\right| \leq C\sum_{i=1}^{3}\mu_i(\alpha, n), \tag{3.67}$$

such that we have that

$$p_n^{(3)}(x_{kn}) = (-1)\beta(x_{kn}, n)\left(\frac{n}{a_n}\right)^2 (1 + \tilde{\rho}_1(\alpha, x_{kn}, n))p_n'(x_{kn}). \tag{3.68}$$

Suppose that there exist constants $\tilde{\rho}_{2s-1}(\alpha, x_{kn}, n)$ with $|\tilde{\rho}_{2s-1}(\alpha, x_{kn}, n)| \leq C(\mu_1(\alpha, n)+\mu_2(\alpha, n)+\mu_3(\alpha, n))$ such that

$$p_n^{(2s-1)}(x_{kn}) = (-1)^{s-1}\beta^{s-1}(x_{kn}, n)\left(\frac{n}{a_n}\right)^{2s-2} (1 + \tilde{\rho}_{2s-1}(\alpha, x_{kn}, n))p_n'(x_{kn}), \tag{3.69}$$

$$\left|p_n^{(2s)}(x_{kn})\right| \leq C\mu_1(\alpha, n)\left(\frac{n}{a_n}\right)^{2s-1} \left|p_n'(x_{kn})\right|. \tag{3.70}$$

Then we have by (3.38) and (3.70)

$$\left|\frac{B_{2s}^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})}p_n^{(2s)}(x_{kn})\right| \lesssim C\mu_1(\alpha, n)\left(\frac{n}{a_n}\right)^{2s+1} \left|p_n'(x_{kn})\right|, \tag{3.71}$$

and we have by (3.42) and (3.69)

$$-\frac{B_{2s-1}^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})}p_n^{(2s-1)}(x_{kn}) = (-1)^s\beta^s(x_{kn}, n)\left(\frac{n}{a_n}\right)^{2s} \left(1 + \tilde{\tilde{\rho}}_{2s-1}(\alpha, x_{kn}, n)\right)p_n'(x_{kn}), \tag{3.72}$$

where $\tilde{\tilde{\rho}}_{2s-1}(\alpha, x_{kn}, n) = f_{2s-1}(\alpha, x_{kn}, n)\tilde{\rho}_{2s-1}(\alpha, x_{kn}, n) + f_{2s-1}(\alpha, x_{kn}, n) + \tilde{\rho}_{2s-1}(\alpha, x_{kn}, n)$ and $|\tilde{\tilde{\rho}}_{2s-1}(\alpha, x_{kn}, n)| \le C(\mu_1(\alpha, n)+\mu_2(\alpha, n)+\mu_3(\alpha, n))$. Also, we have by (3.59) that for $1 \le \ell \le 2s-2$

$$\left| \frac{B_{\ell}^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})} p_n^{(\ell)}(x_{kn}) \right| \lesssim \varepsilon(n) \left(\frac{n}{a_n} \right)^{2s} |p_n'(x_{kn})|. \tag{3.73}$$

Therefore, there exists $\tilde{\rho}_{2s+1}(\alpha, x_{kn}, n)$ satisfying $|\tilde{\rho}_{2s+1}(\alpha, x_{kn}, n)| \le C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n))$ such that

$$p_n^{(2s+1)}(x_{kn}) = (-1)^s \beta^s(x_{kn}, n) \left(\frac{n}{a_n} \right)^{2s} (1 + \tilde{\rho}_{2s+1}(x_{kn}, n)) p_n'(x_{kn}). \tag{3.74}$$

Moreover, we have by (3.37) and (3.65)

$$\left| \frac{B_{2s+1}^{[2s]}(x_{kn})}{B_{2s+2}^{[2s]}(x_{kn})} p_n^{(2s+1)}(x_{kn}) \right| \lesssim C\mu_1(\alpha, n) \left(\frac{n}{a_n} \right)^{2s+1} |p_n'(x_{kn})|, \tag{3.75}$$

and by (3.43) and (3.70)

$$\left| \frac{B_{2s}^{[2s]}(x_{kn})}{B_{2s+2}^{[2s]}(x_{kn})} p_n^{(2s)}(x_{kn}) \right| \le C\mu_1(\alpha, n) \left(\frac{n}{a_n} \right)^{2s} |p_n'(x_{kn})|. \tag{3.76}$$

Also we obtain by (3.59) and (3.65) that for $1 \le \ell \le 2s - 1$

$$\left| \frac{B_{\ell}^{[2s]}(x_{kn})}{B_{2s+2}^{[2s]}(x_{kn})} p_n^{(\ell)}(x_{kn}) \right| \le \varepsilon(n) \left(\frac{n}{a_n} \right)^{2s+1} |p_n'(x_{kn})|. \tag{3.77}$$

Therefore, since we have by (3.63) that

$$\left| p_n^{(2s+2)}(x_{kn}) \right| \le C\mu_1(\alpha, n) \left(\frac{n}{a_n} \right)^{2s+1} |p_n'(x_{kn})|, \tag{3.78}$$

we proved the results. □

Proof of Theorem 3.4 . From (3.3), Theorem 3.1, and the definitions of $\mu_i(\alpha, n)$ ($i = 1, 2, 3$) in Theorem 3.3, if for any $\delta > 0$ we choose a fixed constant $\alpha_0(\delta) > 0$ small enough, then there exists an integer $N = N(\alpha_0)$ such that we can make $\mu_1(\alpha_0, n)$, $\mu_2(\alpha_0, n)$, and $\mu_3(\alpha_0, n)$ small enough for $a_n/\alpha_0 n \le |x| \le \alpha_0 a_n$ with $n > N$. □

Proof of Corollary 3.5. Since we have from Lemma 3.8 that $|C_{j+2}^{[j]}(0)| \sim n/a_n$, $|C_{j+1}^{[j]}(0)| \lesssim (n/a_n)^2$ for $j \geq 0$ and $|C_s^{[j]}(0)| \lesssim (n/a_n)^{j+3-s}$ for $1 \leq s \leq j$, we obtain using the mathematical induction that

$$\left|p_n^{(j+1)}(0)\right| \lesssim \left(\frac{n}{a_n}\right)^j \left|p_n'(0)\right|. \tag{3.79}$$

Therefore, from (3.65) we prove the result easily. □

Proof of Theorem 3.6. We know that from (3.39)

$$\left|p_n''(x_{kn})\right| \leq \left|\frac{B_1^{[0]}(x_{kn})}{B_2^{[0]}(x_{kn})}\right| \left|p_n'(x_{kn})\right| \leq \left(\frac{T(a_n)}{a_n} + \left|Q'(x_{kn})\right| + \frac{1}{|x_{kn}|}\right) \left|p_n'(x_{kn})\right| \tag{3.80}$$

and from (3.44)

$$\left|p_n^{(3)}(x_{kn})\right| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^2 \left|p_n'(x_{kn})\right|. \tag{3.81}$$

Suppose

$$\begin{gathered}\left|p_n^{(2s-1)}(x_{kn})\right| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{2s-2} \left|p_n'(x_{kn})\right|, \\ \left|p_n^{(2s)}(x_{kn})\right| \lesssim \left(\frac{T(a_n)}{a_n} + \left|Q'(x_{kn})\right| + \frac{1}{|x_{kn}|}\right)\left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{2s-2} \left|p_n'(x_{kn})\right|.\end{gathered} \tag{3.82}$$

Then since

$$\begin{gathered}\left|\frac{B_{2s}^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})}\right| \left|p_n^{(2s)}(x_{kn})\right| \lesssim \left(\frac{T(a_n)}{a_n} + |Q'(x_{kn})| + \frac{1}{|x_{kn}|}\right)^2 \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{2s-2} \left|p_n'(x_{kn})\right|, \\ \left|\frac{B_{2s-1}^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})}\right| \left|p_n^{(2s-1)}(x_{kn})\right| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{2s} \left|p_n'(x_{kn})\right|, \\ \left|\frac{B_{s}^{[2s-1]}(x_{kn})}{B_{2s+1}^{[2s-1]}(x_{kn})}\right| \left|p_n^{(s)}(x_{kn})\right| \lesssim \left(\frac{T(a_n)}{a_n} + \left|Q'(x_{kn})\right| + \frac{1}{|x_{kn}|}\right)\left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{2s-1} \left|p_n'(x_{kn})\right|,\end{gathered} \tag{3.83}$$

we have

$$\left|p_n^{(2s+1)}(x_{kn})\right| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{2s} \left|p_n'(x_{kn})\right|. \tag{3.84}$$

Here, we used that $T(a_n)/a_n + |Q'(x_{kn})| + 1/|x_{kn}| \lesssim A_n(x_{kn}) + T(a_n)/a_n$. Similarly, since

$$\left|\frac{B_s^{[2s]}(x_{kn})}{B_{2s+2}^{[2s]}(x_{kn})}\right|\left|p_n^{(s)}(x_{kn})\right| \lesssim \left(\frac{T(a_n)}{a_n} + \left|Q'(x_{kn})\right| + \frac{1}{|x_{kn}|}\right)\left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{2s}\left|p'_n(x_{kn})\right|, \tag{3.85}$$

we have

$$\left|p_n^{(2s+2)}(x_{kn})\right| \lesssim \left(\frac{T(a_n)}{a_n} + \left|Q'(x_{kn})\right| + \frac{1}{|x_{kn}|}\right)\left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{2s}\left|p'_n(x_{kn})\right|. \tag{3.86}$$

□

4. Estimation of the Coefficients of Higher-Order Hermite-Fejér Interpolation

Let l, m be nonnegative integers with $0 \le l < m \le \nu$. For $f \in C^{(l)}(\mathbb{R})$ we define the (l,m)-order Hermite-Fejér interpolation polynomials $L_n(l,m,f;x) \in \mathcal{P}_{mn-1}$ as follows: for each $k = 1,2,\ldots,n$,

$$\begin{aligned} L_n^{(j)}(l,m,f;x_{k,n,\rho}) &= f^{(j)}(x_{k,n,\rho}), \quad j = 0,1,2,\ldots,l, \\ L_n^{(j)}(l,m,f;x_{k,n,\rho}) &= 0, \quad j = l+1, l+2, \ldots, m-1. \end{aligned} \tag{4.1}$$

Especially for each $P \in \mathcal{P}_{mn-1}$ we see $L_n(m-1,m,P;x) = P(x)$. The fundamental polynomials $h_{s,k,n,\rho}(m;x) \in \mathcal{P}_{mn-1}$, $k = 1,2,\ldots,n$ of $L_n(l,m,f;x)$ are defined by

$$h_{s,k,n,\rho}(l,m;x) = l_{k,n,\rho}^m(x)\sum_{i=s}^{m-1} e_{s,i}(l,m,k,n)(x - x_{k,n,\rho})^i. \tag{4.2}$$

Here, $l_{k,n,\rho}(x)$ is fundamental Lagrange interpolation polynomial of degree $n-1$ (cf. [18, page 23]) given by

$$l_{k,n,\rho}(x) = \frac{p_n\left(w_\rho^2;x\right)}{(x - x_{k,n,\rho})p'_n(w_\rho^2;x_{k,n,\rho})}, \tag{4.3}$$

and $h_{s,k,n,\rho}(l,m;x)$ satisfies

$$h_{s,k,n,\rho}^{(j)}(l,m;x_{p,n,\rho}) = \delta_{s,j}\delta_{k,p} \quad j,s = 0,1,\ldots,m-1, p = 1,2,\ldots,n. \tag{4.4}$$

Then

$$L_n(l,m,f;x) = \sum_{k=1}^{n}\sum_{s=0}^{l} f^{(s)}(x_{k,n,\rho})h_{s,k,n,\rho}(l,m;x). \tag{4.5}$$

In this section, we often denote $l_{kn}(x) := l_{k,n,\rho}(x)$ and $h_{skn}(x) := h_{s,k,n,\rho}(x)$ if it does not confuse us. Then we will first estimate $(l_{kn}^m)^{(j)}(x_{kn})$ for $0 \le j \le \nu - 1$. Since we have

$$l_{kn}^{(j)}(x) = \frac{p_n^{(j+1)}(x_{kn})}{(j+1)p_n'(x_{kn})} \tag{4.6}$$

by induction on m, we can estimate $(l_{kn}^m)^{(j)}(x_{kn})$.

Theorem 4.1. *Let $0 \le j \le \nu - 1$. Then one has for $|x_{kn}| \le a_n/2$*

$$\left|(l_{kn}^m)^{(j)}(x_{kn})\right| \le C\left(\frac{n}{a_n}\right)^j. \tag{4.7}$$

In addition, one has that for $|x_{kn}| \le a_n(1+\eta_n)$

$$\left|(l_{kn}^m)^{(j)}(x_{kn})\right| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^j \tag{4.8}$$

and if j is odd, then one has that for $0 < |x_{kn}| \le a_n(1+\eta_n)$

$$\left|(l_{kn}^m)^{(j)}(x_{kn})\right| \lesssim \left(\frac{T(a_n)}{a_n} + \left|Q'(x_{kn})\right| + \frac{1}{|x_{kn}|}\right)\left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{j-1}. \tag{4.9}$$

For $j = 0, 1, \ldots$ define $\phi_j(1) := (2j+1)^{-1}$ and for $k \ge 2$

$$\phi_j(k) := \sum_{r=0}^{j} \frac{1}{2j-2r+1}\binom{2j}{2r}\phi_r(k-1). \tag{4.10}$$

Theorem 4.2 (cf. [10, Lemma 10]). *Let $0 < \alpha < 1/2$ and let $a_n/\alpha n \le |x_{kn}| \le \alpha a_n$. Then for $0 \le 2s \le \nu - 2$ there exists uniquely a sequence $\{\phi_j(m)\}_{j=0}^{\infty}$ of positive numbers*

$$(l_{kn}^m)^{(2s)}(x_{kn}) = (-1)^s \phi_s(m)\beta^s(x_{kn}, n)\left(\frac{n}{a_n}\right)^{2s}(1 + \xi_s(m, \alpha, x_{kn}, n)) \tag{4.11}$$

and $|\xi_s(m, \alpha, x_{kn}, n)| \le C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n))$. Moreover, one has for $1 \le 2s - 1 \le \nu - 1$

$$\left|(l_{kn}^m)^{(2s-1)}(x_{kn})\right| \le C\mu_1(\alpha, n)\left(\frac{n}{a_n}\right)^{2s-1}. \tag{4.12}$$

Theorem 4.3. *Suppose the same assumptions as Theorem 4.2. Given any $\delta > 0$, there exists a small fixed positive constant $0 < \alpha_0(\delta) < 1/2$ such that (4.11) holds satisfying $|\xi_j(m, \alpha, x_{kn}, n)| \le \delta$ and*

$$\left|(l_{kn}^m)^{(2j+1)}(x_{kn})\right| \le \delta\left(\frac{n}{a_n}\right)^{2j+1} \tag{4.13}$$

for $a_n/\alpha_0 n \le |x_{kn}| \le \alpha_0 a_n$.

Theorem 4.4. *Let $0 \le s \le i \le m-1$. Then one has for $|x_{kn}| \le a_n/2$*

$$|e_{s,i}(l,m,k,n)| \le C\left(\frac{n}{a_n}\right)^{i-s}. \tag{4.14}$$

On the other hand, one has for $|x_{kn}| \le a_n(1+\eta_n)$

$$|e_{s,i}(l,m,k,n)| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{i-s}. \tag{4.15}$$

Especially, if $i-s$ is odd, then one has

$$|e_{s,i}(l,m,k,n)| \lesssim \left(\frac{T(a_n)}{a_n} + \left|Q'(x_{kn})\right| + \frac{1}{|x_{kn}|}\right)\left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{i-s-1}. \tag{4.16}$$

Especially, for $f \in C(\mathbb{R})$ we define the m-order Hermite-Fejér interpolation polynomials $L_n(m,f;x) \in \mathcal{P}_{mn-1}$ as the $(0,m)$-order Hermite-Fejér interpolation polynomials $L_n(0,m,f;x)$. Then we know that

$$L_n(m,f;x) = \sum_{k=1}^{n} f(x_{k,n,\rho})h_{k,n,\rho}(m;x), \tag{4.17}$$

where $e_i(m,k,n) := e_{0,i}(0,m,k,n)$ and

$$h_{k,n,\rho}(m;x) = l_{k,n,\rho}^{m}(x)\sum_{i=0}^{m-1} e_i(m,k,n)(x - x_{k,n,\rho})^i. \tag{4.18}$$

Then for the convergence theorem with respect to $L_n(m,f;x)$ we have the following corollary.

Corollary 4.5. *Let $0 \le i \le m-1$. Then one has for $|x_{kn}| \le a_n/2$*

$$|e_i(m,k,n)| \le C\left(\frac{n}{a_n}\right)^{i}. \tag{4.19}$$

On the other hand, one has for $|x_{kn}| \le a_n(1+\eta_n)$

$$|e_i(m,k,n)| \lesssim \left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{i}. \tag{4.20}$$

Especially, if i is odd, then one has

$$|e_i(m,k,n)| \lesssim \left(\frac{T(a_n)}{a_n} + \left|Q'(x_{kn})\right| + \frac{1}{|x_{kn}|}\right)\left(A_n(x_{kn}) + \frac{T(a_n)}{a_n}\right)^{i-1}. \tag{4.21}$$

Proof of Theorem 4.1. Theorem 4.1 is shown by induction with respect to m. The case of $m = 1$ follows from (4.6), Corollary 3.5, and Theorem 3.6. Suppose that for the case of $m - 1$ the results hold. Then from the following relation:

$$\left(l_{kn}^{m}\right)^{(j)}(x_{kn}) = \sum_{r=0}^{j}\binom{j}{r}\left(l_{kn}^{m-1}\right)^{(r)}(x_{kn})l_{kn}^{(j-r)}(x_{kn}), \tag{4.22}$$

we have (4.7) and (4.8). Moreover, we obtain (4.9) from the following: for $1 \leq 2s-1 \leq \nu - 1$

$$\begin{aligned}\left(l_{kn}^{m}\right)^{(2s-1)}(x_{kn}) = &\sum_{r=0}^{s}\binom{2s-1}{2r}\left(l_{kn}^{m-1}\right)^{(2r)}(x_{kn})l_{kn}^{(2s-2r-1)}(x_{kn})\\ &+\sum_{r=0}^{s}\binom{2s-1}{2r+1}\left(l_{kn}^{m-1}\right)^{(2r+1)}(x_{kn})l_{kn}^{(2s-2r-2)}(x_{kn}).\end{aligned} \tag{4.23}$$

□

Proof of Theorem 4.2. Similarly to Theorem 4.1, we use mathematical induction with respect to m. From Theorem 3.3 we know that for $0 \leq 2s \leq \nu - 1$

$$l_{kn}^{(2s)}(x_{kn}) = (-1)^{s}\phi_{s}(1)\beta^{s}(x_{kn}, n)\left(\frac{n}{a_n}\right)^{2s}(1 + \xi_{s}(1, \alpha, x_{kn}, n)) \tag{4.24}$$

and for $1 \leq 2s - 1 \leq \nu - 1$

$$\left|l_{kn}^{(2s-1)}(x_{kn})\right| \leq C\mu_{1}(\alpha, n)\left(\frac{n}{a_n}\right)^{2s-1}, \tag{4.25}$$

where $\xi_{s}(1, \alpha, x_{kn}, n) = \widetilde{\rho}_{2s+1}(\alpha, x_{kn}, n)$ and

$$|\xi_{s}(1, \alpha, x_{kn}, n)| \leq C(\mu_{1}(\alpha, n) + \mu_{2}(\alpha, n) + \mu_{3}(\alpha, n)). \tag{4.26}$$

Then from the following relations:

$$\begin{aligned}\left(l_{kn}^{m}\right)^{(j)}(x_{kn}) = &\sum_{0\leq 2r\leq j}\binom{j}{2r}\left(l_{kn}^{m-1}\right)^{(2r)}(x_{kn})l_{kn}^{(j-2r)}(x_{kn})\\ &+\sum_{1\leq 2r-1\leq j}\binom{j}{2r-1}\left(l_{kn}^{m-1}\right)^{(2r-1)}(x_{kn})l_{kn}^{(j-2r+1)}(x_{kn}).\end{aligned} \tag{4.27}$$

we have the results by induction with respect to m. □

Proof of Theorem 4.3. It is proved by the same reason as the proof of Corollary 3.4. □

Proof of Theorem 4.4. To prove the result, we proceed by induction on i. From (4.2) and (4.4) we know that $e_{s,s}(l,m,k,n) = 1/s!$ and the following recurrence relation; for $s+1 \le i \le m-1$

$$e_{s,i}(l,m,k,n) = -\sum_{p=s}^{i-1} \frac{1}{(i-p)!} e_{s,p}(l,m,k,n) \left(l_{k,n,\rho}^{m}\right)^{(i-p)} (x_{k,n,\rho}). \tag{4.28}$$

When $i = s$, $e_{s,s}(l,\nu,k,n) = 1/s!$ so that (4.14) and (4.15) are satisfied for $i = s$. From (4.7), (4.8), (4.28), and assumption of induction on i, for $s+1 \le i \le m-1$, we have the results easily. When $i-s$ is odd, we know that

$$\begin{aligned} &i-p : \text{odd}, \quad \text{if } p-s : \text{even}, \\ &i-p : \text{even}, \quad \text{if } p-s : \text{odd}. \end{aligned} \tag{4.29}$$

Therefore, similarly we have (4.16) from (4.8), (4.9), (4.28), and assumption of induction on i. □

Proof of Corollary 4.5. Since $e_i(m,k,n) = e_{0,i}(0,m,k,n)$, it is trivial from Theorem 4.4. □

We rewrite the relation (4.10) in the form for $\nu = 1,2,3,\ldots,$

$$\phi_0(\nu) := 1 \tag{4.30}$$

and for $j = 1,2,3,\ldots, \nu = 2,3,4,\ldots,$

$$\phi_j(\nu) - \phi_j(\nu-1) = \frac{1}{2j+1} \sum_{r=0}^{j-1} \binom{2j+1}{2r} \phi_r(\nu-1). \tag{4.31}$$

Now, for every j we will introduce an auxiliary polynomial determined by $\{\Psi_j(y)\}_{j=1}^{\infty}$ as the following lemma.

Lemma 4.6 (see[10, Lemma 11]). (i) *For $j = 0,1,2,\ldots,$ there exists a unique polynomial $\Psi_j(y)$ of degree j such that*

$$\Psi_j(\nu) = \phi_j(\nu), \quad \nu = 1,2,3,\ldots. \tag{4.32}$$

(ii) $\Psi_0(y) = 1$ *and* $\Psi_j(0) = 0$, $j = 1,2,\ldots$.

Since $\Psi_j(y)$ is a polynomial of degree j, we can replace $\phi_j(\nu)$ in (4.10) with $\Psi_j(y)$, that is,

$$\Psi_j(y) = \sum_{r=0}^{j} \frac{1}{2j-2r+1} \binom{2j}{2r} \Psi_r(y-1), \quad j = 0,1,2,\ldots, \tag{4.33}$$

for an arbitrary y and $j = 0, 1, 2, \ldots$. We use the notation $F_{kn}(x, y) = (l_{kn}(x))^y$ which coincides with $l_{kn}^y(x)$ if y is an integer. Since $l_{kn}(x_{kn}) = 1$, we have $F_{kn}(x, t) > 0$ for x in a neighborhood of x_{kn} and an arbitrary real number y.

We can show that $(\partial/\partial x)^j F_{kn}(x_{kn}, y)$ is a polynomial of degree at most j with respect to y for $j = 0, 1, 2, \ldots$, where $(\partial/\partial x)^j F_{kn}(x_{kn}, y)$ is the jth partial derivative of $F_{kn}(x, y)$ with respect to x at (x_{kn}, y) (see [6, page 199]). We prove these facts by induction on j. For $j = 0$ it is trivial. Suppose that it holds for $j \geq 0$. To simplify the notation, let $F(x) = F_{kn}(x, y)$ and $l(x) = l_{kn}(x)$ for a fixed y. Then $F'(x)l(x) = yl'(x)F(x)$. By Leibniz's rule, we easily see that

$$F^{(j+1)}(x_{kn}) = -\sum_{s=0}^{j-1} F^{(s+1)}(x_{kn}) l^{(j-s)}(x_{kn}) + y \sum_{s=0}^{j} l^{(s+1)}(x_{kn}) F^{(j-s)}(x_{kn}), \tag{4.34}$$

which shows that $F^{(j+1)}(x_{kn})$ is a polynomial of degree at most $j + 1$ with respect to y. Let $P_{kn}^{[j]}(y)$, $j = 0, 1, 2, \ldots$ be defined by

$$\left(\frac{\partial}{\partial x}\right)^{2j} F_{kn}(x_{kn}, y) = (-1)^j \beta^j(x_{kn}, n) \left(\frac{n}{a_n}\right)^{2j} \Psi_j(y) + P_{kn}^{[j]}(y). \tag{4.35}$$

Then $P_{kn}^{[j]}(y)$ is a polynomial of degree at most $2j$.

By Theorem 4.2 we have the following.

Lemma 4.7 (see[10, Lemma 12]). *Let $j = 0, 1, 2, \ldots$, and let M be a positive constant. If $a_n/\alpha n \leq |x_{kn}| \leq \alpha a_n$ and $|y| \leq M$, then*

$$\left| \left(\frac{\partial}{\partial y}\right)^s P_{kn}^{[j]}(y) \right| \leq C(\mu_1(\alpha, n) + \mu_2(\alpha, n) + \mu_3(\alpha, n)) \left(\frac{n}{a_n}\right)^{2j}, \quad s = 0, 1, \tag{4.36}$$

$$\left| \left(\frac{\partial}{\partial y}\right)^{2j+1} F_{kn}(x_{kn}, y) \right| \leq C\mu_1(\alpha, n) \left(\frac{n}{a_n}\right)^{2j+1}. \tag{4.37}$$

Lemma 4.8 (see[10, Lemma 13]). *If $y < 0$, then for $j = 0, 1, 2, \ldots$,*

$$(-1)^j \Psi_j(y) > 0. \tag{4.38}$$

Lemma 4.9. *For positive integers s and m with $1 \leq m \leq \nu$*

$$\sum_{r=0}^{s} \binom{2s}{2r} \Psi_r(-m) \phi_{s-r}(m) = 0. \tag{4.39}$$

Proof. If we let $C_s(y) = \sum_{r=0}^{s}\binom{2s}{2r}\Psi_r(-y)\Psi_{s-r}(y)$, then it suffices to show that $C_s(m) = 0$. For every s

$$0 = (l_{kn}^{-m+m})^{2s}(x_{kn}) = \sum_{i=0}^{2s}\binom{2s}{i}(l_{kn}^{-m})^{(i)}(x_{kn})(l_{kn}^{m})^{(2s-i)}(x_{kn})$$

$$= \sum_{r=0}^{s}\binom{2s}{2r}\left(\frac{\partial}{\partial x}\right)^{2r}F_{kn}(x_{kn},-m)(l_{kn}^{m})^{(2s-2r)}(x_{kn}) \tag{4.40}$$

$$+\sum_{r=0}^{s-1}\binom{2s}{2r+1}\left(\frac{\partial}{\partial x}\right)^{2r+1}F_{kn}(x_{kn},-m)(l_{kn}^{m})^{(2s-2r-1)}(x_{kn}).$$

By (4.24), (4.35), and (4.36), we see that the first sum $\sum_{r=0}^{s}$ has the form of

$$\sum_{r=0}^{s} = (-1)^s\beta^s(x_{kn},n)\left(\frac{n}{a_n}\right)^{2s}\left(\sum_{r=0}^{s}\binom{2s}{2r}\Psi_r(-m)\phi_{s-r}(m) + \tilde{\eta}_s(-m,\alpha,x_{kn},n)\right). \tag{4.41}$$

Then since

$$\tilde{\eta}_s(-m,\alpha,x_{kn},n) = \sum_{r=0}^{s}\binom{2s}{2r}\Psi_r(-m)\phi_{s-r}(m)\xi_{s-r}(m,\alpha,x_{kn},n)$$
$$+\sum_{r=0}^{s}\binom{2s}{2r}(-1)^{-r}\beta^{-r}(x_{kn},n)\left(\frac{n}{a_n}\right)^{-2r}\phi_{s-r}(m)P_{kn}^{[j]}(m)(1+\xi_{s-r}(m,\alpha,x_{kn},n)), \tag{4.42}$$

we know that $|\tilde{\eta}_s(-m,\alpha,x_{kn},n)| \le C(\mu_1(\alpha,n)+\mu_2(\alpha,n)+\mu_3(\alpha,n))$. By (4.37) and (4.7), the second sum $\sum_{r=0}^{s-1}$ is bounded by $C\mu_1(\alpha,n)(n/a_n)^{2s}$. Here, we can make $C(\mu_1(\alpha,n)+\mu_2(\alpha,n)+\mu_3(\alpha,n)) < \delta$ for arbitrary positive δ. Therefore, we obtain the following result: for every s

$$0 = \sum_{r=0}^{s}\binom{2s}{2r}\Psi_r(-m)\Psi_{s-r}(m). \tag{4.43}$$

□

Then the following theorem is important to show a divergence theorem with respect to $L_n(m,f;x)$ where m is an odd integer.

Theorem 4.10 (cf. [10, (4.16)] and [15]). *For $j = 0,1,2,\ldots$, there is a polynomial $\Psi_j(x)$ of degree j such that $(-1)^j\Psi_j(-m) > 0$ for $m = 1,3,5,\ldots$ and the following relation holds. Let $0 < \alpha < 1/2$.*

Then one has an expression for $a_n/\alpha n \le |x_{kn}| \le \alpha a_n$, *and* $0 \le 2s \le m-1$:

$$e_{2s}(m,k,n) = (-1)^s \frac{1}{(2s)!}\Psi_s(-m)\beta^s(x_{kn},n)\left(\frac{n}{a_n}\right)^{2s}\left(1+\eta_s(m,\alpha,x_{kn},n)\right), \tag{4.44}$$

where $\eta_s(m,\alpha,x_{kn},n)$ *satisfies that for* $a_n/\alpha n \le |x_{kn}| \le \alpha a_n$ *and for* $s = 0,1,2,\ldots$

$$\left|\eta_s(m,\alpha,x_{kn},n)\right| \le C\left(\mu_1(\alpha,n)+\mu_2(\alpha,n)+\mu_3(\alpha,n)\right). \tag{4.45}$$

Proof. We prove (4.44) by induction on s. Since $e_0(m,k,n)=1$ and $\Psi_0(y)=1$, (4.44) holds for $s=0$. From (4.28) we write $e_{2s}(m,k,n)$ in the form of

$$\begin{aligned} e_{2s}(m,k,n) &= -\sum_{r=0}^{s-1}\frac{1}{(2s-2r)!}e_{2r}(m,k,n)\left(l_{kn}^m\right)^{(2s-2r)}(x_{kn}) \\ &\quad -\sum_{r=1}^{s}\frac{1}{(2s-2r+1)!}e_{2r-1}(m,k,n)\left(l_{kn}^m\right)^{(2s-2r+1)}(x_{kn}) \\ &=: I + II. \end{aligned} \tag{4.46}$$

Then by (4.12) and (4.14), $|II|$ is bounded by $C\mu_1(\alpha,n)(n/a_n)^{2s}$. For $0 \le i < s$ we suppose (4.44) and (4.45). Then we have for I

$$\sum_{r=0}^{s-1} = \frac{(-1)^{s+1}}{(2s)!}\beta^s(x_{kn},n)\left(\frac{n}{a_n}\right)^{2s}\sum_{r=0}^{s-1}\binom{2s}{2r}\Psi_r(-m)\phi_{s-r}(m)(1+\eta_r)(1+\xi_{s-r}), \tag{4.47}$$

where $\xi_{s-r} := \xi_{s-r}(m,\alpha,x_{kn},n)$ and $\eta_r := \eta_r(m,\alpha,x_{kn},n)$ which are defined in (4.11) and (4.44). Then using Lemma 4.9 and $\phi_0(m)=1$ we have the following form:

$$e_{2s}(m,k,n) = \frac{(-1)^s}{(2s)!}\Psi_s(-m)\beta^s(x_{kn},n)\left(\frac{n}{a_n}\right)^{2s}\left(1+\eta_s(m,\alpha,x_{kn},n)\right). \tag{4.48}$$

Here, since

$$\begin{aligned} \eta_s(m,\alpha,x_{kn},n) &= \sum_{r=0}^{s-1}\binom{2s}{2r}\Psi_r(-m)\phi_{s-r}(m)\left(\eta_r+\xi_{s-r}+\eta_r\xi_{s-r}\right) \\ &\quad + (-1)^s\beta^{-s}(x_{kn},n)\left(\frac{n}{a_n}\right)^{-2s} II, \end{aligned} \tag{4.49}$$

we see that $|\eta_s(\nu,\alpha,x_{kn},n)| \le C(\mu_1(\alpha,n)+\mu_2(\alpha,n)+\mu_3(\alpha,n))$. Therefore, we proved the result. □

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