

## Research Article

# Positive Semidefinite Matrices, Exponential Convexity for Majorization, and Related Cauchy Means

M. Anwar,<sup>1</sup> N. Latif,<sup>2</sup> and J. Pečarić<sup>2,3</sup>

<sup>1</sup> Center for Advanced Mathematics and Physics, NUST Campus at College of  
Electrical & Mechanical Engineering, Peshawar Road, Rawalpindi 48000, Pakistan

<sup>2</sup> Abdus Salam School of Mathematical Sciences, GC University, Lahore 5400, Pakistan

<sup>3</sup> Faculty of Textile Technology, University of Zagreb, 1002 Zagreb, Croatia

Correspondence should be addressed to M. Anwar, matloob.t@yahoo.com

Received 26 October 2009; Accepted 10 March 2010

Academic Editor: Panayiotis Siafarikas

Copyright © 2010 M. Anwar et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We prove positive semidefiniteness of matrices generated by differences deduced from majorization-type results which implies exponential convexity and log-convexity of these differences and also obtain Lyapunov's and Dresher's inequalities for these differences. We introduce new Cauchy means and show that these means are monotone.

## 1. Introduction and Preliminaries

Let  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{p} = (p_1, \dots, p_n)$  denote two sequences of positive real numbers with  $\sum_{i=1}^n p_i = 1$ . The well-known Jensen inequality for convex function [1, page 43] gives that, for  $t < 0$  or  $t > 1$ ,

$$\sum_{i=1}^n p_i x_i^t \geq \left( \sum_{i=1}^n p_i x_i \right)^t, \quad (1.1)$$

and vice versa for  $0 < t < 1$ .

In [2], the following generalization of this theorem is given.

**Theorem 1.1.** For  $-\infty < r < s < t < +\infty$ ,

$$\lambda_s^{t-r} \leq \lambda_r^{t-s} \lambda_t^{s-r}, \quad (1.2)$$

where

$$\lambda_t := \begin{cases} \frac{\sum_{i=1}^n p_i x_i^t - (\sum_{i=1}^n p_i x_i)^t}{t(t-1)}, & t \neq 0, 1, \\ \log \left( \sum_{i=1}^n p_i x_i \right) - \sum_{i=1}^n p_i \log x_i, & t = 0, \\ \sum_{i=1}^n p_i x_i \log x_i - \left( \sum_{i=1}^n p_i x_i \right) \log \left( \sum_{i=1}^n p_i x_i \right), & t = 1. \end{cases} \quad (1.3)$$

For fixed  $n \geq 2$  let

$$\mathbf{x} = (x_1, \dots, x_n), \quad \mathbf{y} = (y_1, \dots, y_n) \quad (1.4)$$

denote two  $n$ -tuples. Let

$$\begin{aligned} x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}, & \quad y_{[1]} \geq y_{[2]} \geq \dots \geq y_{[n]}, \\ x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}, & \quad y_{(1)} \leq y_{(2)} \leq \dots \leq y_{(n)} \end{aligned} \quad (1.5)$$

be their ordered components.

*Definition 1.2* (see [1, page 319]).  $\mathbf{y}$  is said to majorize  $\mathbf{x}$  (or  $\mathbf{x}$  is said to be majorized by  $\mathbf{y}$ ), in symbol,  $\mathbf{y} \succ \mathbf{x}$ , if

$$\sum_{i=1}^m x_{[i]} \leq \sum_{i=1}^m y_{[i]} \quad (1.6)$$

holds for  $m = 1, 2, \dots, n-1$  and

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i. \quad (1.7)$$

Note that (1.6) is equivalent to

$$\sum_{i=n-m+1}^n x_{(i)} \leq \sum_{i=n-m+1}^n y_{(i)} \quad (1.8)$$

for  $m = 1, 2, \dots, n-1$ .

The following theorem is well-known as the majorization theorem and a convenient reference for its proof is given by Marshall and Olkin [3, page11] (see also [1, page 320]).

**Theorem 1.3.** Let  $I$  be an interval in  $\mathbb{R}$ , and let  $\mathbf{x}, \mathbf{y}$  be two  $n$ -tuples such that  $x_i, y_i \in I$  ( $i = 1, \dots, n$ ). Then

$$\sum_{i=1}^n \phi(x_i) \leq \sum_{i=1}^n \phi(y_i) \quad (1.9)$$

holds for every continuous convex function  $\phi : I \rightarrow \mathbb{R}$  if and only if  $\mathbf{y} > \mathbf{x}$  holds.

*Remark 1.4* (see [4]). If  $\phi(x)$  is a strictly convex function, then equality in (1.9) is valid iff  $x_{[i]} = y_{[i]}$ ,  $i = 1, \dots, n$ .

The following theorem can be regarded as a generalization of Theorem 1.3 and is proved by Fuchs in [5] (see also [1, page 323]).

**Theorem 1.5.** Let  $\mathbf{x}, \mathbf{y}$  be two decreasing real  $n$ -tuples, and let  $\mathbf{p} = (p_1, \dots, p_n)$  be a real  $n$ -tuple such that

$$\begin{aligned} \sum_{i=1}^k p_i x_i &\leq \sum_{i=1}^k p_i y_i \quad \text{for } k = 1, \dots, n-1, \\ \sum_{i=1}^n p_i x_i &= \sum_{i=1}^n p_i y_i. \end{aligned} \quad (1.10)$$

Then for every continuous convex function  $\phi : I \rightarrow \mathbb{R}$ , one has

$$\sum_{i=1}^n p_i \phi(x_i) \leq \sum_{i=1}^n p_i \phi(y_i). \quad (1.11)$$

*Definition 1.6.* A function  $h : (a, b) \rightarrow \mathbb{R}$  is exponentially convex function if it is continuous and

$$\sum_{i,j=1}^n \xi_i \xi_j h(x_i + x_j) \geq 0 \quad (1.12)$$

for all  $n \in \mathbb{N}$  and all choices  $\xi_i \in \mathbb{R}$  and  $x_i \in (a, b)$ ,  $i = 1, \dots, n$  such that  $x_i + x_j \in (a, b)$ ,  $1 \leq i, j \leq n$ .

The following proposition is given in [6].

**Proposition 1.7.** Let  $h : (a, b) \rightarrow \mathbb{R}$ . The following propositions are equivalent.

- (i)  $h$  is exponentially convex.
- (ii)  $h$  is continuous and

$$\sum_{i,j=1}^n \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0, \quad (1.13)$$

for every  $n \in \mathbb{N}$ , every  $\xi_i \in \mathbb{R}$ , and every  $x_i, x_j \in (a, b)$ ,  $1 \leq i, j \leq n$ .

**Corollary 1.8.** *If  $h$  is exponentially convex, then*

$$\det \left[ h \left( \frac{x_i + x_j}{2} \right) \right]_{i,j=1}^n \geq 0, \quad (1.14)$$

for every  $n \in \mathbb{N}$  and every  $x_i \in (a, b)$ ,  $i = 1, \dots, n$ .

**Corollary 1.9.** *If  $h : (a, b) \rightarrow \mathbb{R}^+$  is exponentially convex function, then  $h$  is a log-convex function in Jensen's sense:*

$$h \left( \frac{x + y}{2} \right) \leq \sqrt{h(x)h(y)}, \quad \forall x, y \in (a, b). \quad (1.15)$$

In this paper, we prove positive semidefiniteness of matrices generated by differences deduced from majorization-type results (1.9), (1.11), (4.2), and (4.5) which implies exponential convexity and log-convexity of these differences and also obtain Lyapunov's and Dresher's inequalities for these differences. In [7], new Cauchy means are introduced. By using these means, a generalization of (1.2) was given (see [7]). In the present paper, we give related results in discrete and indiscrete cases and some new means of the Cauchy type.

## 2. Main Results

**Lemma 2.1.** *Define the function*

$$\varphi_s(x) := \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0, 1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases} \quad (2.1)$$

Then  $\varphi_s'' = x^{s-2}$ , that is,  $\varphi_s$  is convex for  $x > 0$ .

**Definition 2.2.** It is said that a positive function  $f$  is log-convex in the Jensen sense on some interval  $I \subseteq \mathbb{R}$  if

$$f(s)f(t) \geq f^2 \left( \frac{s+t}{2} \right) \quad (2.2)$$

holds for every  $s, t \in I$ .

The following lemma gives an equivalent condition for convexity of function  $f$  [1, page 2].

**Lemma 2.3.** *If  $\phi$  is convex on an interval  $I \subseteq \mathbb{R}$ , then*

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \geq 0 \quad (2.3)$$

holds for every  $s_1 < s_2 < s_3$ ,  $s_1, s_2, s_3 \in I$ .

**Theorem 2.4.** Let  $\mathbf{x}$  and  $\mathbf{y}$  be two positive  $n$ -tuples,  $\mathbf{y} \succ \mathbf{x}$ ,

$$\Lambda_t = \Lambda_t(\mathbf{x}; \mathbf{y}) := \sum_{i=1}^n \varphi_t(y_i) - \sum_{i=1}^n \varphi_t(x_i), \tag{2.4}$$

and all  $x_{[i]}$ 's and  $y_{[i]}$ 's are not equal.

Then the following statements are valid.

(a) For every  $n \in \mathbb{N}$  and  $s_1, \dots, s_n \in \mathbb{R}$ , the matrix  $[\Lambda_{(s_i+s_j)/2}]_{i,j=1}^n$  is a positive semidefinite matrix. Particularly

$$\det [\Lambda_{(s_i+s_j)/2}]_{i,j=1}^k \geq 0 \tag{2.5}$$

for  $k = 1, \dots, n$ .

(b) The function  $s \mapsto \Lambda_s$  is exponentially convex.

(c) The function  $s \mapsto \Lambda_s$  is log-convex on  $\mathbb{R}$  and the following inequality holds for  $-\infty < r < s < t < \infty$ :

$$\Lambda_s^{t-r} \leq \Lambda_r^{t-s} \Lambda_t^{s-r}. \tag{2.6}$$

*Proof.* (a) Consider the function

$$\mu(x) = \sum_{i,j}^k u_i u_j \varphi_{s_{ij}}(x) \tag{2.7}$$

for  $k = 1, \dots, n$ ,  $x > 0$ ,  $u_i \in \mathbb{R}$ ,  $s_{ij} \in \mathbb{R}$ , where  $s_{ij} = (s_i + s_j)/2$  and  $\varphi_{s_{ij}}$  is defined in (2.1).

We have

$$\mu''(x) = \sum_{i,j}^k u_i u_j x^{s_{ij}-2} = \left( \sum_i^k u_i x^{s_i/2-1} \right)^2 \geq 0, \quad x \geq 0. \tag{2.8}$$

This shows that  $\mu$  is a convex function for  $x \geq 0$ .

Using Theorem 1.3,

$$\sum_{m=1}^n \mu(\mathbf{y}_m) - \sum_{m=1}^n \mu(\mathbf{x}_m) \geq 0. \tag{2.9}$$

This implies that

$$\sum_{m=1}^n \left( \sum_{i,j}^k u_i u_j \varphi_{s_{ij}}(\mathbf{y}_m) \right) - \sum_{m=1}^n \left( \sum_{i,j}^k u_i u_j \varphi_{s_{ij}}(\mathbf{x}_m) \right) \geq 0, \tag{2.10}$$

or equivalently

$$\sum_{i,j}^k u_i u_j \Lambda_{s_{ij}} \geq 0. \quad (2.11)$$

From last inequality, it follows that the matrix  $[\Lambda_{(s_i+s_j)/2}]_{i,j=1}^n$  is a positive semidefinite matrix, that is, (2.5) is valid.

(b) Note that  $\Lambda_s$  is continuous for  $s \in \mathbb{R}$ . Then by using Proposition 1.7, we get exponentially convexity of the function  $s \rightarrow \Lambda_s$ .

(c) Since  $\varphi_t(x)$  is continuous and strictly convex function for  $x > 0$  and all  $x_{[i]}$ 's and  $y_{[i]}$ 's are not equal, therefore by Theorem 1.3 with  $\phi = \varphi_t$  we have

$$\sum_{i=1}^n \varphi_t(y_i) > \sum_{i=1}^n \varphi_t(x_i). \quad (2.12)$$

This implies

$$\Lambda_t = \Lambda_t(\mathbf{x}; \mathbf{y}) = \sum_{i=1}^n \varphi_t(y_i) - \sum_{i=1}^n \varphi_t(x_i) > 0, \quad (2.13)$$

that is,  $\Lambda_t$  is a positive-valued function.

A simple consequence of Corollary 1.9 is that  $\Lambda_s$  is log-convex; then by definition

$$\log \Lambda_s^{t-r} \leq \log \Lambda_r^{t-s} + \log \Lambda_t^{s-r}, \quad (2.14)$$

which is equivalent to (2.6). □

**Theorem 2.5.** Let  $\Lambda_t$  be defined as in Theorem 2.4 and  $t, s, u, v \in \mathbb{R}$  such that  $s \leq u$ ,  $t \leq v$ ,  $s \neq t$ , and  $u \neq v$ . Then

$$\left( \frac{\Lambda_t}{\Lambda_s} \right)^{1/(t-s)} \leq \left( \frac{\Lambda_v}{\Lambda_u} \right)^{1/(v-u)}. \quad (2.15)$$

*Proof.* For a convex function  $\varphi$ , a simple consequence of (2.3) is the following inequality [1, page 2]:

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(y_2) - \varphi(y_1)}{y_2 - y_1}, \quad (2.16)$$

with  $x_1 \leq y_1$ ,  $x_2 \leq y_2$ ,  $x_1 \neq x_2$ ,  $y_1 \neq y_2$ . Since by Theorem 2.4(c) and  $\Lambda_t$  is log-convex, we can set in (2.16)  $\varphi(x) = \log \Lambda_t$ ,  $x_1 = s$ ,  $x_2 = t$ ,  $y_1 = u$ , and  $y_2 = v$ . We get

$$\frac{\log \Lambda_t - \log \Lambda_s}{t - s} \leq \frac{\log \Lambda_v - \log \Lambda_u}{v - u}, \quad (2.17)$$

from which (2.15) follows.  $\square$

**Theorem 2.6.** Let  $\mathbf{x}$  and  $\mathbf{y}$  be two positive decreasing  $n$ -tuples, let  $\mathbf{p} = (p_1, \dots, p_n)$  be a real  $n$ -tuple and let

$$\lambda_t = \lambda_t(\mathbf{x}, \mathbf{y}; \mathbf{p}) := \sum_{i=1}^n p_i \varphi_t(y_i) - \sum_{i=1}^n p_i \varphi_t(x_i) \quad (2.18)$$

such that conditions (1.10) are satisfied and  $\lambda_t$  is positive.

Then the following statements are valid.

- (a) For every  $n \in \mathbb{N}$  and  $s_1, \dots, s_n \in \mathbb{R}$ , the matrix  $[\lambda_{(s_i+s_j)/2}]_{i,j=1}^n$  is a positive semidefinite matrix. Particularly

$$\det[\lambda_{(s_i+s_j)/2}]_{i,j=1}^k \geq 0 \quad (2.19)$$

for  $k = 1, \dots, n$ .

- (b) The function  $s \mapsto \lambda_s$  is exponentially convex.  
 (c) The function  $s \mapsto \lambda_s$  is log-convex on  $\mathbb{R}$  and the following inequality holds for  $-\infty < r < s < t < \infty$ :

$$\lambda_s^{t-r} \leq \lambda_r^{t-s} \lambda_t^{s-r}. \quad (2.20)$$

*Proof.* As in the proof of Theorem 2.4, we use Theorem 1.5 instead of Theorem 1.3.  $\square$

**Theorem 2.7.** Let  $\lambda_t$  be defined as in Theorem 2.6 and  $t, s, u, v \in \mathbb{R}$  such that  $s \leq u$ ,  $t \leq v$ ,  $s \neq t$ , and  $u \neq v$ . Then

$$\left(\frac{\lambda_t}{\lambda_s}\right)^{1/(t-s)} \leq \left(\frac{\lambda_v}{\lambda_u}\right)^{1/(v-u)}. \quad (2.21)$$

*Proof.* Similar to the proof of Theorem 2.5.  $\square$

### 3. Cauchy Means

Let us note that (2.15) and (2.21) have the form of some known inequalities between means (e.g., Stolarsky means, Gini means, etc). Here we will prove that expressions on both sides of (2.15) and (2.21) are also means.

**Lemma 3.1.** Let  $f \in C^2(I)$ ,  $I$  interval in  $\mathbb{R}$ , be such that

$$m \leq f''(x) \leq M. \quad (3.1)$$

Consider the functions  $\phi_1, \phi_2$  defined as

$$\begin{aligned} \phi_1(x) &= \frac{Mx^2}{2} - f(x), \\ \phi_2(x) &= f(x) - \frac{mx^2}{2}, \end{aligned} \quad (3.2)$$

then  $\phi_i(x)$  for  $i = 1, 2$  are convex.

*Proof.* Since

$$\begin{aligned} \phi_1''(x) &= M - f''(x) \geq 0, \\ \phi_2''(x) &= f''(x) - m \geq 0, \end{aligned} \quad (3.3)$$

that is,  $\phi_i$  for  $i = 1, 2$  are convex. □

Denote

$$I_1 = [m_1, M_1], \quad \text{where } m_1 = \min\{m_x, m_y\}, \quad M_1 = \max\{M_x, M_y\}. \quad (3.4)$$

In the above expression,  $m_x$  and  $m_y$  are the minimums of  $x$  and  $y$ , respectively. Similarly,  $M_x$  and  $M_y$  are the maximums of  $x$  and  $y$  respectively.

**Theorem 3.2.** Let  $\mathbf{x}$  and  $\mathbf{y}$  be two positive  $n$ -tuples,  $\mathbf{y} \succ \mathbf{x}$ , all  $x_{[i]}$ 's and  $y_{[i]}$ 's are not equal, and  $f \in C^2(I_1)$ , with  $I_1$  being defined as in (3.4), then there exists  $\xi \in I_1$  such that

$$\sum_{i=1}^n f(y_i) - \sum_{i=1}^n f(x_i) = \frac{f''(\xi)}{2} \left\{ \sum_{i=1}^n y_i^2 - \sum_{i=1}^n x_i^2 \right\}. \quad (3.5)$$

*Proof.* Since  $f \in C^2(I_1)$  and  $I_1$  is compact, then  $m_1 \leq f''(x) \leq M_1$  for  $x \in I_1$ . Then by applying  $\phi_1$  and  $\phi_2$  defined in Lemma 3.1 for  $\phi$  in Theorem 1.3, we have

$$\begin{aligned} \sum_{i=1}^n \phi_1(x_i) &\leq \sum_{i=1}^n \phi_1(y_i), \\ \sum_{i=1}^n \phi_2(x_i) &\leq \sum_{i=1}^n \phi_2(y_i), \end{aligned} \quad (3.6)$$



that is,

$$\sum_{i=1}^n f(y_i) - \sum_{i=1}^n f(x_i) \leq \frac{M_1}{2} \left\{ \sum_{i=1}^n y_i^2 - \sum_{i=1}^n x_i^2 \right\}, \quad (3.7)$$

$$\sum_{i=1}^n f(y_i) - \sum_{i=1}^n f(x_i) \geq \frac{m_1}{2} \left\{ \sum_{i=1}^n y_i^2 - \sum_{i=1}^n x_i^2 \right\}. \quad (3.8)$$

By combining (3.7) and (3.8)

$$m_1 \leq 2 \left( \frac{\sum_{i=1}^n f(y_i) - \sum_{i=1}^n f(x_i)}{\sum_{i=1}^n y_i^2 - \sum_{i=1}^n x_i^2} \right) \leq M_1. \quad (3.9)$$

$\sum_{i=1}^n y_i^2 - \sum_{i=1}^n x_i^2 \neq 0$  because  $x_{[i]} \neq y_{[i]}$ , for  $i = 1, \dots, n$  by using Remark 1.4. Using the fact that for  $m_1 \leq \rho \leq M_1$ , there exists  $\xi \in I_1$  such that  $f''(\xi) = \rho$ , we get (3.5).  $\square$

**Theorem 3.3.** Let  $\mathbf{x}$  and  $\mathbf{y}$  be two positive  $n$ -tuples,  $\mathbf{y} > \mathbf{x}$ , all  $x_{[i]}$ 's and  $y_{[i]}$ 's are not equal, and  $f, g \in C^2(I_1)$ , with  $I_1$  being defined as in (3.4), then there exists  $\xi \in I_1$  such that

$$\frac{\sum_{i=1}^n f(y_i) - \sum_{i=1}^n f(x_i)}{\sum_{i=1}^n g(y_i) - \sum_{i=1}^n g(x_i)} = \frac{f''(\xi)}{g''(\xi)}, \quad (3.10)$$

provided that  $g''(x) \neq 0$  for every  $x \in I_1$ .

*Proof.* Let a function  $k \in C^2(I_1)$  be defined as

$$k = c_1 f - c_2 g, \quad (3.11)$$

where  $c_1$  and  $c_2$  are defined as

$$c_1 = \sum_{i=1}^n g(y_i) - \sum_{i=1}^n g(x_i), \quad (3.12)$$

$$c_2 = \sum_{i=1}^n f(y_i) - \sum_{i=1}^n f(x_i).$$

Then, using Theorem 3.2 with  $f = k$ , we have

$$0 = \left( c_1 \frac{f''(\xi)}{2} - c_2 \frac{g''(\xi)}{2} \right) \left\{ \sum_{i=1}^n y_i^2 - \sum_{i=1}^n x_i^2 \right\}. \quad (3.13)$$

Since

$$\sum_{i=1}^n y_i^2 - \sum_{i=1}^n x_i^2 \neq 0, \quad (3.14)$$

therefore, (3.13) gives

$$\frac{c_2}{c_1} = \frac{f''(\xi)}{g''(\xi)}. \quad (3.15)$$

After putting values, we get (3.10). The denominator of left-hand side is nonzero by using  $f = g$  in Theorem 3.2.  $\square$

**Corollary 3.4.** *Let  $\mathbf{x}$  and  $\mathbf{y}$  be two positive  $n$ -tuples such that  $\mathbf{y} > \mathbf{x}$  and all  $x_{[i]}$ 's and  $y_{[i]}$ 's are not equal, then for  $-\infty < s \neq t \neq 0$ ,  $1 \neq s < +\infty$  there exists  $\xi \in I_1$ , with  $I_1$  being defined as in (3.4), such that*

$$\xi^{t-s} = \frac{s(s-1) \sum_{i=1}^n y_i^t - \sum_{i=1}^n x_i^t}{t(t-1) \sum_{i=1}^n y_i^s - \sum_{i=1}^n x_i^s}. \quad (3.16)$$

*Proof.* Setting  $f(x) = x^t$  and  $g(x) = x^s$ ,  $t \neq s \neq 0, 1$  in (3.10), we get (3.16).  $\square$

*Remark 3.5.* Since the function  $\xi \mapsto \xi^{t-s}$  is invertible, then from (3.16) we have

$$m_1 \leq \left\{ \frac{s(s-1) \sum_{i=1}^n y_i^t - \sum_{i=1}^n x_i^t}{t(t-1) \sum_{i=1}^n y_i^s - \sum_{i=1}^n x_i^s} \right\}^{1/(t-s)} \leq M_1. \quad (3.17)$$

In fact, similar result can also be given for (3.10). Namely, suppose that  $f''/g''$  has inverse function. Then from (3.10), we have

$$\xi = \left( \frac{f''}{g''} \right)^{-1} \left( \frac{\sum_{i=1}^n f(y_i) - \sum_{i=1}^n f(x_i)}{\sum_{i=1}^n g(y_i) - \sum_{i=1}^n g(x_i)} \right). \quad (3.18)$$

So, we have that the expression on the right-hand side of (3.18) is also a mean. By the inequality (3.17), we can consider for positive  $n$ -tuples  $\mathbf{x}$  and  $\mathbf{y}$  such that  $\mathbf{y} > \mathbf{x}$ ,

$$M_{t,s} = \left( \frac{\Lambda_t}{\Lambda_s} \right)^{1/(t-s)} \quad (3.19)$$

for  $-\infty < s \neq t < +\infty$ , as means in broader sense. Moreover we can extend these means in other cases. So passing to the limit, we have

$$\begin{aligned} M_{s,s} &= \exp \left( \frac{\sum_{i=1}^n y_i^s \log y_i - \sum_{i=1}^n x_i^s \log x_i}{\sum_{i=1}^n y_i^s - \sum_{i=1}^n x_i^s} - \frac{2s-1}{s(s-1)} \right), \quad s \neq 0, 1, \\ M_{0,0} &= \exp \left( \frac{\sum_{i=1}^n \log^2 y_i - \sum_{i=1}^n \log^2 x_i}{2 [\sum_{i=1}^n \log y_i - \sum_{i=1}^n \log x_i]} + 1 \right), \\ M_{1,1} &= \exp \left( \frac{\sum_{i=1}^n y_i \log^2 y_i - \sum_{i=1}^n x_i \log^2 x_i}{2 [\sum_{i=1}^n y_i \log y_i - \sum_{i=1}^n x_i \log x_i]} - 1 \right). \end{aligned} \quad (3.20)$$

**Theorem 3.6.** Let  $t, s, u, v \in \mathbb{R}$  such that  $t \leq u$ ,  $s \leq v$ , then the following inequality is valid:

$$M_{t,s} \leq M_{u,v}. \quad (3.21)$$

*Proof.* Since  $\Lambda_s$  is log-convex, therefore by (2.15) we get (3.21).  $\square$

**Theorem 3.7.** Let  $\mathbf{x}$  and  $\mathbf{y}$  be two positive decreasing  $n$ -tuples, let  $\mathbf{p}$  be a real  $n$ -tuple such that conditions (1.10) are satisfied,  $\lambda_t$  is positive defined as in Theorem 2.6, and  $f \in C^2(I_1)$ , with  $I_1$  being defined as in (3.4), then there exists  $\xi \in I_1$  such that

$$\sum_{i=1}^n p_i f(y_i) - \sum_{i=1}^n p_i f(x_i) = \frac{f''(\xi)}{2} \left\{ \sum_{i=1}^n p_i y_i^2 - \sum_{i=1}^n p_i x_i^2 \right\}, \quad (3.22)$$

*Proof.* As in the proof of Theorem 3.2, we use Theorem 1.5 instead of Theorem 1.3.  $\square$

**Theorem 3.8.** Let  $\mathbf{x}$  and  $\mathbf{y}$  be two positive decreasing  $n$ -tuples,  $\mathbf{p}$  be a real  $n$ -tuple such that conditions (1.10) are satisfied,  $\lambda_t$  is positive defined as in Theorem 2.6 and  $f, g \in C^2(I_1)$ ,  $I_1$  is defined as in (3.4). Then there exists  $\xi \in I_1$  such that

$$\frac{\sum_{i=1}^n p_i f(y_i) - \sum_{i=1}^n p_i f(x_i)}{\sum_{i=1}^n p_i g(y_i) - \sum_{i=1}^n p_i g(x_i)} = \frac{f''(\xi)}{g''(\xi)}. \quad (3.23)$$

provided that  $g''(x) \neq 0$  for every  $x \in I_1$ .

*Proof.* Similar to the proof of Theorem 3.3.  $\square$

**Corollary 3.9.** Let  $\mathbf{x}$  and  $\mathbf{y}$  be two positive decreasing  $n$ -tuples, let  $\mathbf{p}$  be a real  $n$ -tuple such that conditions (1.10) are satisfied and  $\lambda_t$  is positive defined as in Theorem 2.6, then for  $-\infty < s \neq t \neq 0, 1 \neq s < +\infty$  there exists  $\xi \in I_1$ , with  $I_1$  being defined as in (3.4), such that

$$\xi^{t-s} = \frac{s(s-1) \sum_{i=1}^n p_i y_i^t - \sum_{i=1}^n p_i x_i^t}{t(t-1) \sum_{i=1}^n p_i y_i^s - \sum_{i=1}^n p_i x_i^s}. \quad (3.24)$$

*Proof.* Setting  $f(x) = x^t$  and  $g(x) = x^s$ ,  $t \neq s \neq 0, 1$  in (3.23), we get (3.24).  $\square$

*Remark 3.10.* Since the function  $\xi \mapsto \xi^{t-s}$  is invertible, then from (3.24) we have

$$m_1 \leq \left\{ \frac{s(s-1) \sum_{i=1}^n p_i y_i^t - \sum_{i=1}^n p_i x_i^t}{t(t-1) \sum_{i=1}^n p_i y_i^s - \sum_{i=1}^n p_i x_i^s} \right\}^{1/(t-s)} \leq M_1. \quad (3.25)$$

In fact, similar result can also be given for (3.23). Namely, suppose that  $f''/g''$  has inverse function. Then from (3.23), we have

$$\xi = \left( \frac{f''}{g''} \right)^{-1} \left( \frac{\sum_{i=1}^n p_i f(y_i) - \sum_{i=1}^n p_i f(x_i)}{\sum_{i=1}^n p_i g(y_i) - \sum_{i=1}^n p_i g(x_i)} \right). \quad (3.26)$$

So, we have that the expression on the right-hand side of (3.26) is also a mean. By the inequality (3.25), we can consider for positive  $n$ -tuples  $x$  and  $y$  such that conditions (1.10) are satisfied, and

$$\widetilde{M}_{t,s} = \left( \frac{\lambda_t}{\lambda_s} \right)^{1/(t-s)} \quad (3.27)$$

for  $-\infty < s \neq t < +\infty$ , as means in broader sense. Moreover we can extend these means in other cases. So passing to the limit, we have

$$\begin{aligned} \widetilde{M}_{s,s} &= \exp\left( \frac{\sum_{i=1}^n p_i y_i^s \log y_i - \sum_{i=1}^n p_i x_i^s \log x_i}{\sum_{i=1}^n p_i y_i^s - \sum_{i=1}^n p_i x_i^s} - \frac{2s-1}{s(s-1)} \right), \quad s \neq 0, 1, \\ \widetilde{M}_{0,0} &= \exp\left( \frac{\sum_{i=1}^n p_i \log^2 y_i - \sum_{i=1}^n p_i \log^2 x_i}{2(\sum_{i=1}^n p_i \log y_i - \sum_{i=1}^n p_i \log x_i)} + 1 \right), \\ \widetilde{M}_{1,1} &= \exp\left( \frac{\sum_{i=1}^n p_i y_i \log^2 y_i - \sum_{i=1}^n p_i x_i \log^2 x_i}{2(\sum_{i=1}^n p_i y_i \log y_i - \sum_{i=1}^n p_i x_i \log x_i)} - 1 \right). \end{aligned} \quad (3.28)$$

**Theorem 3.11.** Let  $t, s, u, v \in \mathbb{R}$  such that  $t \leq u, s \leq v$ , then the following inequality is valid:

$$\widetilde{M}_{t,s} \leq \widetilde{M}_{u,v}. \quad (3.29)$$

*Proof.* Since  $\lambda_s$  is log-convex, therefore by (2.21) we get (3.29).  $\square$

#### 4. Some Related Results

Let  $x(\tau), y(\tau)$  be real valued functions defined on an interval  $[a, b]$  such that  $\int_a^s x(\tau) d\tau, \int_a^s y(\tau) d\tau$  both exist for all  $s \in [a, b]$ .

*Definition 4.1* (see [1, page 324]).  $y(\tau)$  is said to majorize  $x(\tau)$ , in symbol,  $y(\tau) \succ x(\tau)$ , for  $\tau \in [a, b]$  if they are decreasing in  $\tau \in [a, b]$  and

$$\int_a^s x(\tau) d\tau \leq \int_a^s y(\tau) d\tau \quad \text{for } s \in [a, b], \quad (4.1)$$

and equality in (4.1) holds for  $s = b$ .

The following theorem can be regarded as a majorization theorem in integral case [1, page 325].

**Theorem 4.2.**  $y(\tau) > x(\tau)$  for  $\tau \in [a, b]$  iff they are decreasing in  $[a, b]$  and

$$\int_a^b \phi(x(\tau))d\tau \leq \int_a^b \phi(y(\tau))d\tau \quad (4.2)$$

holds for every  $\phi$  that is continuous, and convex in  $[a, b]$  such that the integrals exist.

The following theorem is a simple consequence of Theorem 12.14 in [8] (see also [1, page 328]):

**Theorem 4.3.** Let  $x(\tau), y(\tau) : [a, b] \rightarrow \mathbb{R}$ ,  $x(\tau)$  and  $y(\tau)$  are continuous and increasing and let  $G : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation.

(a) If

$$\int_v^b x(\tau)dG(\tau) \leq \int_v^b y(\tau)dG(\tau) \quad \text{for every } v \in [a, b], \quad (4.3)$$

$$\int_a^b x(\tau)dG(\tau) = \int_a^b y(\tau)dG(\tau) \quad (4.4)$$

hold, then for every continuous convex function  $f$ , one has

$$\int_a^b f(x(\tau))dG(\tau) \leq \int_a^b f(y(\tau))dG(\tau). \quad (4.5)$$

(b) If (4.3) holds, then (4.5) holds for every continuous increasing convex function  $f$ .

**Theorem 4.4.** Let  $x(\tau)$  and  $y(\tau)$  be two positive functions defined on an interval  $[a, b]$ , decreasing in  $[a, b]$ ,  $y(\tau) > x(\tau)$ ,

$$\beta_t(x(\tau); y(\tau)) := \int_a^b \varphi_t(y(\tau))d\tau - \int_a^b \varphi_t(x(\tau))d\tau, \quad (4.6)$$

and  $\beta_t$  is positive.

Then the following statements are valid.

(a) For every  $n \in \mathbb{N}$  and  $s_1, \dots, s_n \in \mathbb{R}$ , the matrix  $[\beta_{(s_i+s_j)/2}]_{i,j=1}^n$  is a positive semidefinite matrix. Particularly

$$\det[\beta_{(s_i+s_j)/2}]_{i,j=1}^k \geq 0 \quad (4.7)$$

for  $k = 1, \dots, n$ .

(b) The function  $s \mapsto \beta_s$  is exponentially convex.

(c) The function  $s \mapsto \beta_s$  is log-convex on  $\mathbb{R}$  and the following inequality holds for  $-\infty < r < s < t < \infty$  :

$$\beta_s^{t-r} \leq \beta_r^{t-s} \beta_t^{s-r}. \quad (4.8)$$

*Proof.* As in the proof of Theorem 2.4, we use Theorem 4.2 instead of Theorem 1.3.  $\square$

**Theorem 4.5.** Let  $\beta_t$  be defined as in Theorem 4.4 and  $t, s, u, v \in \mathbb{R}$  such that  $s \leq u, t \leq v, s \neq t$ , and  $u \neq v$ . Then

$$\left(\frac{\beta_t}{\beta_s}\right)^{1/(t-s)} \leq \left(\frac{\beta_v}{\beta_u}\right)^{1/(v-u)}. \quad (4.9)$$

*Proof.* Similar to the proof of Theorem 2.5.  $\square$

Denote

$$I_2 = [m_2, M_2], \quad \text{where } m_2 = \min\{m_{x(\tau)}, m_{y(\tau)}\}, \quad M_2 = \max\{M_{x(\tau)}, M_{y(\tau)}\}. \quad (4.10)$$

In the above expression,  $m_{x(\tau)}$  and  $m_{y(\tau)}$  are the minimums of  $x(\tau)$  and  $y(\tau)$ , respectively. Similarly,  $M_{x(\tau)}$  and  $M_{y(\tau)}$  are the maximums of  $x(\tau)$  and  $y(\tau)$ , respectively.

**Theorem 4.6.** Let  $x(\tau)$  and  $y(\tau)$  be two positive decreasing functions in  $[a, b]$  such that  $y(\tau) > x(\tau)$ ,  $\beta_t$  is positive defined as in Theorem 4.4, and  $f \in C^2(I_2)$ , with  $I_2$  being defined as in (4.10), then there exists  $\xi \in I_2$  such that

$$\int_a^b f(y(\tau))d\tau - \int_a^b f(x(\tau))d\tau = \frac{f''(\xi)}{2} \left\{ \int_a^b y^2(\tau)d\tau - \int_a^b x^2(\tau)d\tau \right\}. \quad (4.11)$$

*Proof.* As in the proof of Theorem 3.2, we use Theorem 4.2 instead of Theorem 1.3.  $\square$

**Theorem 4.7.** Let  $x(\tau)$  and  $y(\tau)$  be two positive decreasing functions in  $[a, b]$  such that  $y(\tau) > x(\tau)$ ,  $\beta_t$  is positive defined as in Theorem 4.4, and  $f, g \in C^2(I_2)$ , with  $I_2$  being defined as in (4.10). Then there exists  $\xi \in I_2$  such that

$$\frac{\int_a^b f(y(\tau))d\tau - \int_a^b f(x(\tau))d\tau}{\int_a^b g(y(\tau))d\tau - \int_a^b g(x(\tau))d\tau} = \frac{f''(\xi)}{g''(\xi)}, \quad (4.12)$$

provided that  $g''(z) \neq 0$  for every  $z \in I_2$ .

*Proof.* Similar to the proof of Theorem 3.3.  $\square$

**Corollary 4.8.** Let  $x(\tau)$  and  $y(\tau)$  be two positive decreasing functions in  $[a, b]$  such that  $y(\tau) > x(\tau)$  and  $\beta_t$  is positive defined as in Theorem 4.4, then for  $-\infty < s \neq t \neq 0, 1 \neq s < +\infty$  there exists  $\xi \in I_2$ , with  $I_2$  being defined as in (4.10), such that

$$\xi^{t-s} = \frac{s(s-1) \int_a^b y^t(\tau) d\tau - \int_a^b x^t(\tau) d\tau}{t(t-1) \int_a^b y^s(\tau) d\tau - \int_a^b x^s(\tau) d\tau}. \quad (4.13)$$

*Proof.* Set  $f(x) = x^t$  and  $g(x) = x^s$ ,  $t \neq s \neq 0, 1$  in (4.12), we get (4.13).  $\square$

*Remark 4.9.* Since the function  $\xi \mapsto \xi^{t-s}$  is invertible, then from (4.13) we have

$$m_2 \leq \left\{ \frac{s(s-1) \int_a^b y^t(\tau) d\tau - \int_a^b x^t(\tau) d\tau}{t(t-1) \int_a^b y^s(\tau) d\tau - \int_a^b x^s(\tau) d\tau} \right\}^{1/(t-s)} \leq M_2. \quad (4.14)$$

In fact, similar result can also be given for (4.12). Namely, suppose that  $f''/g''$  has inverse function. Then from (4.12), we have

$$\xi = \left( \frac{f''}{g''} \right)^{-1} \left( \frac{\int_a^b f(y(\tau)) d\tau - \int_a^b f(x(\tau)) d\tau}{\int_a^b g(y(\tau)) d\tau - \int_a^b g(x(\tau)) d\tau} \right). \quad (4.15)$$

So, we have that the expression on the right-hand side of (4.15) is also a mean. By the inequality (4.14), we can consider for positive functions  $x(\tau)$  and  $y(\tau)$  such that  $y(\tau) > x(\tau)$ , and

$$\widehat{M}_{t,s} = \left( \frac{\beta_t}{\beta_s} \right)^{1/(t-s)} \quad (4.16)$$

for  $-\infty < s \neq t < +\infty$ , as means in broader sense. Moreover we can extend these means in other cases. So passing to the limit, we have

$$\begin{aligned} \widehat{M}_{s,s} &= \exp \left( \frac{\int_a^b y^s(\tau) \log y(\tau) d\tau - \int_a^b x^s(\tau) \log x(\tau) d\tau}{\int_a^b y^s(\tau) d\tau - \int_a^b x^s(\tau) d\tau} - \frac{2s-1}{s(s-1)} \right), \quad s \neq 0, 1, \\ \widehat{M}_{0,0} &= \exp \left( \frac{\int_a^b \log^2 y(\tau) d\tau - \int_a^b \log^2 x(\tau) d\tau}{2 \left[ \int_a^b \log y(\tau) d\tau - \int_a^b \log x(\tau) d\tau \right]} + 1 \right), \\ \widehat{M}_{1,1} &= \exp \left( \frac{\int_a^b y(\tau) \log^2 y(\tau) d\tau - \int_a^b x(\tau) \log^2 x(\tau) d\tau}{2 \left[ \int_a^b y(\tau) \log y(\tau) d\tau - \int_a^b x(\tau) \log x(\tau) d\tau \right]} - 1 \right). \end{aligned} \quad (4.17)$$

**Theorem 4.10.** Let  $t, s, u, v \in \mathbb{R}$  such that  $t \leq u$ ,  $s \leq v$ , then the following inequality is valid:

$$\widehat{M}_{t,s} \leq \widehat{M}_{u,v}. \quad (4.18)$$

*Proof.* Since  $\beta_s$  is log-convex, therefore by (4.9) we get (4.18).  $\square$

**Theorem 4.11.** Let  $x(\tau), y(\tau) : [a, b] \rightarrow \mathbb{R}$ ,  $x(\tau)$  and  $y(\tau)$  are positive, continuous, and increasing and let  $G : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation. Also let

$$\Gamma_t(x(\tau), y(\tau); G(\tau)) := \int_a^b \varphi_t(y(\tau)) dG(\tau) - \int_a^b \varphi_t(x(\tau)) dG(\tau), \quad (4.19)$$

such that conditions (4.3) and (4.4) are satisfied and  $\Gamma_t$  is positive.

Then the following statements are valid.

- (a) For every  $n \in \mathbb{N}$  and  $s_1, \dots, s_n \in \mathbb{R}$ , the matrix  $[\Gamma_{(s_i+s_j)/2}]_{i,j=1}^n$  is a positive semidefinite matrix. Particularly

$$\det [\Gamma_{(s_i+s_j)/2}]_{i,j=1}^k \geq 0 \quad (4.20)$$

for  $k = 1, \dots, n$ .

- (b) The function  $s \mapsto \Gamma_s$  is exponentially convex.  
 (c) The function  $s \mapsto \Gamma_s$  is log-convex on  $\mathbb{R}$  and the following inequality holds for  $-\infty < r < s < t < \infty$ :

$$\Gamma_s^{t-r} \leq \Gamma_r^{t-s} \Gamma_t^{s-r}. \quad (4.21)$$

*Proof.* As in the proof of Theorem 2.4, we use Theorem 4.3 instead of Theorem 1.3.  $\square$

**Theorem 4.12.** Let  $\Gamma_t$  be defined as in Theorem 4.11 and  $t, s, u, v \in \mathbb{R}$  such that  $s \leq u$ ,  $t \leq v$ ,  $s \neq t$ , and  $u \neq v$ . Then

$$\left( \frac{\Gamma_t}{\Gamma_s} \right)^{1/(t-s)} \leq \left( \frac{\Gamma_v}{\Gamma_u} \right)^{1/(v-u)}. \quad (4.22)$$

*Proof.* Similar to the proof of Theorem 2.5.  $\square$

**Theorem 4.13.** Let  $x(\tau)$  and  $y(\tau)$  be positive, continuous, and increasing functions in  $[a, b]$  such that conditions (4.3) and (4.4) are satisfied,  $\Gamma_t$  is positive defined as in Theorem 4.11,  $f \in C^2(I_2)$ , with



$I_2$  being defined as in (4.10), and  $G : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation, then there exists  $\xi \in I_2$  such that

$$\int_a^b f(y(\tau))dG(\tau) - \int_a^b f(x(\tau))dG(\tau) = \frac{f''(\xi)}{2} \left\{ \int_a^b y^2(\tau)dG(\tau) - \int_a^b x^2(\tau)dG(\tau) \right\}. \quad (4.23)$$

*Proof.* As in the proof of Theorem 3.2, we use Theorem 4.3 instead of Theorem 1.3.  $\square$

**Theorem 4.14.** Let  $x(\tau)$  and  $y(\tau)$  be positive, continuous and increasing functions in  $[a, b]$  such that conditions (4.3) and (4.4) are satisfied,  $G : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation,  $\Gamma_t$  is positive defined as in Theorem 4.11, and  $f, g \in C^2(I_2)$ , with  $I_2$  being defined as in (4.10). Then there exists  $\xi \in I_2$  such that

$$\frac{\int_a^b f(y(\tau))dG(\tau) - \int_a^b f(x(\tau))dG(\tau)}{\int_a^b g(y(\tau))dG(\tau) - \int_a^b g(x(\tau))dG(\tau)} = \frac{f''(\xi)}{g''(\xi)}, \quad (4.24)$$

provided that  $g''(z) \neq 0$  for every  $z \in I_2$ .

*Proof.* Similar to the proof of Theorem 3.3.  $\square$

**Corollary 4.15.** Let  $x(\tau)$  and  $y(\tau)$  be positive, continuous, and increasing functions in  $[a, b]$  such that conditions (4.3) and (4.4) be satisfied,  $\Gamma_t$  is positive defined as in Theorem 4.11, and  $G : [a, b] \rightarrow \mathbb{R}$  be a function of bounded variation, then for  $-\infty < s \neq t \neq 0, 1 \neq s < +\infty$  there exists  $\xi \in I_2$ , with  $I_2$  being defined as in (4.10), such that

$$\xi^{t-s} = \frac{s(s-1) \int_a^b y^t(\tau)dG(\tau) - \int_a^b x^t(\tau)dG(\tau)}{t(t-1) \int_a^b y^s(\tau)dG(\tau) - \int_a^b x^s(\tau)dG(\tau)}. \quad (4.25)$$

*Proof.* Setting  $f(x) = x^t$  and  $g(x) = x^s$ ,  $t \neq s \neq 0, 1$  in (4.24), we get (4.25).  $\square$

*Remark 4.16.* Since the function  $\xi \mapsto \xi^{t-s}$  is invertible, then from (4.25) we have

$$m_2 \leq \left\{ \frac{s(s-1) \int_a^b y^t(\tau)dG(\tau) - \int_a^b x^t(\tau)dG(\tau)}{t(t-1) \int_a^b y^s(\tau)dG(\tau) - \int_a^b x^s(\tau)dG(\tau)} \right\}^{1/(t-s)} \leq M_2. \quad (4.26)$$

In fact, similar result can also be given for (4.24). Namely, suppose that  $f''/g''$  has inverse function. Then from (4.24), we have

$$\xi = \left( \frac{f''}{g''} \right)^{-1} \left( \frac{\int_a^b f(y(\tau)) dG(\tau) - \int_a^b f(x(\tau)) dG(\tau)}{\int_a^b g(y(\tau)) dG(\tau) - \int_a^b g(x(\tau)) dG(\tau)} \right). \quad (4.27)$$

So, we have that the expression on the right-hand side of (4.27) is also a mean. By the inequality (4.26), we can consider for positive functions  $x(\tau)$  and  $y(\tau)$  such that conditions (4.3) and (4.4) are satisfied, and

$$\mathcal{M}_{t,s} = \left( \frac{\Gamma_t}{\Gamma_s} \right)^{1/(t-s)} \quad (4.28)$$

for  $-\infty < s \neq t < +\infty$ , as means in broader sense. Moreover we can extend these means in other cases. So passing to the limit, we have

$$\begin{aligned} \mathcal{M}_{s,s} &= \exp \left( \frac{\int_a^b y^s(\tau) \log y(\tau) dG(\tau) - \int_a^b x^s(\tau) \log x(\tau) dG(\tau)}{\int_a^b y^s(\tau) dG(\tau) - \int_a^b x^s(\tau) dG(\tau)} - \frac{2s-1}{s(s-1)} \right), \quad s \neq 0, 1, \\ \mathcal{M}_{0,0} &= \exp \left( \frac{\int_a^b \log^2 y(\tau) dG(\tau) - \int_a^b \log^2 x(\tau) dG(\tau)}{2 \left[ \int_a^b \log y(\tau) dG(\tau) - \int_a^b \log x(\tau) dG(\tau) \right]} + 1 \right), \\ \mathcal{M}_{1,1} &= \exp \left( \frac{\int_a^b y(\tau) \log^2 y(\tau) dG(\tau) - \int_a^b x(\tau) \log^2 x(\tau) dG(\tau)}{2 \left[ \int_a^b y(\tau) \log y(\tau) dG(\tau) - \int_a^b x(\tau) \log x(\tau) dG(\tau) \right]} - 1 \right). \end{aligned} \quad (4.29)$$

**Theorem 4.17.** Let  $t, s, u, v \in \mathbb{R}$  such that  $t \leq u, s \leq v$ , then the following inequality is valid:

$$\mathcal{M}_{t,s} \leq \mathcal{M}_{u,v}. \quad (4.30)$$

*Proof.* Since  $\Gamma_s$  is log-convex, therefore by (4.22) we get (4.30).  $\square$

*Remark 4.18.* Let  $x = \sum_{i=1}^n p_i y_i / \sum_{i=1}^n p_i$  such that  $p_i > 0$  and  $\sum_{i=1}^n p_i = 1$ . If we substitute in Theorem 2.6  $(x_1; x_2; \dots; x_n) = (x; x; \dots; x)$ , we get (1.2). In fact in such results we have that  $y$  is monotonic  $n$ -tuple. But since the weights are positive, our results are also valid for arbitrary  $y$ .

## Acknowledgments

This research work is funded by Higher Education Commission of Pakistan. The research of the third author was supported by the Croatian Ministry of Science, Education and Sports under the Research Grants 117-1170889-0888. The authors thank I. Olkin for many valuable suggestions.

## References

- [1] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, vol. 187 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1992.
- [2] M. Anwar and J. Pečarić, "On logarithmic convexity for differences of power means and related results," *Mathematical Inequalities & Applications*, vol. 12, no. 1, pp. 81–90, 2009.

- [3] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, vol. 143 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1979.
- [4] Z. Kadelburg, D. Đukić, M. Lukić, and I. Matić, "Inequalities of Karamata, Schur and Muirhead, and some applications," *The Teaching of Mathematics*, vol. 8, no. 1, pp. 31–45, 2005.
- [5] L. Fuchs, "A new proof of an inequality of Hardy-Littlewood-Pólya," *Mathematics Tidsskr*, pp. 53–54, 1947.
- [6] M. Anwar, J. Jekšetić, J. Pečarić, and A. ur Rehman, "Exponential convexity, positive semi-definite matrices and fundamental inequalities," to appear in *Journal of Mathematical Inequalities*.
- [7] M. Anwar and J. Pečarić, "New means of Cauchy's type," *Journal of Inequalities and Applications*, vol. 2008, Article ID 163202, 10 pages, 2008.
- [8] J. E. Pečarić, "On some inequalities for functions with nondecreasing increments," *Journal of Mathematical Analysis and Applications*, vol. 98, no. 1, pp. 188–197, 1984.