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*Research Article*

# Generalized $q$-Euler Numbers and Polynomials of Higher Order and Some Theoretic Identities

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We give a new construction of the $q$-Euler numbers and polynomials of higher order attached to Dirichlet's character $\chi$. We derive some theoretic identities involving the generalized $q$-Euler numbers and polynomials of higher order.

## 1. Introduction

Let $\mathbb{C}$ be the complex number field. We assume that $q \in \mathbb{C}$ with $|q| < 1$ and the $q$-number is defined by $[x]_q = (1-q^x)/(1-q)$ in this paper. The $q$-factorial is given by $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ and the $q$-binomial formulae are known as

$$(x:q)_n = \prod_{i=1}^{n}\left(1-xq^{i-1}\right) = \sum_{i=0}^{n}\binom{n}{i}_q q^{\binom{i}{2}}(-x)^i,$$
$$\frac{1}{(x:q)_n} = \prod_{i=1}^{n}\left(\frac{1}{1-xq^{i-1}}\right) = \sum_{i=0}^{\infty}\binom{n+i-1}{i}_q x^i, \tag{1.1}$$

where $\binom{n}{i}_q = [n]_q!/[n-i]_q![i]_q! = [n]_q[n-1]_q\cdots[n-i+1]_q/[i]_q!$ (see [1–3]).

After Carlitz had constructed the $q$-Bernoulli numbers and polynomials, many mathematicians have studied for $q$-Bernoulli and $q$-Euler numbers and polynomials (see [1–29]). Since the $q$-extensions of Euler numbers and polynomials contain interesting properties to study various fields of mathematical physics and number theory, many researchers considered and investigated the $q$-Euler numbers and polynomials, and derived some identities from them (see [2–5, 8–19]). The purpose of this paper is to give a new approach to the $q$-Euler numbers and polynomials of higher order attached to Dirichlet's character $\chi$. From this, we will derive some theoretic identities involving generalized $q$-Euler numbers and polynomials of higher order.

In Section 2, we present new generating functions which are related to $q$-Euler numbers and polynomials of higher order attached to $\chi$. We obtain distribution relations for the $q$-Euler polynomials attached to $\chi$, and have some identities involving these $q$-Euler polynomials. Using the Cauchy residue theorem, we show that these $q$-extensions of the $q$-$l$-function of order $r$ attached to $\chi$ interpolate the $q$-Euler polynomials of order $r$ at negative integers.

## 2. $q$-Euler Polynomials of Higher Order Attached to $\chi$

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For $d \in \mathbb{N}$ with $d \equiv 1(\text{mod}2)$, let $\chi$ be Dirichlet's character with conductor $d$. For $r \in \mathbb{Z}$ and $h \in \mathbb{Z}$, we will study the generalized $q$-Euler and $(h,q)$-Euler polynomials and numbers of order $r$ attached to $\chi$, respectively.

It is known that the Euler polynomials are defined by $(2/(e^t+1))e^{xt} = \sum_{n=0}^{\infty} E_n(x)(t^n/n!)$, for $|t| < \pi$. In the special case $x = 0, E_n = E_n(0)$ are called the $n$th Euler numbers (see [28, 29]).

First, we define the generalized $q$-Euler polynomials attached to $\chi$ as follows:

$$\sum_{n=0}^{\infty} E_{n,\chi,q}(x)\frac{t^n}{n!} = 2\sum_{m=0}^{\infty}\chi(m)(-1)^m e^{[m+x]_q t}, \tag{2.1}$$

where $E_{n,\chi,q}(x)$ are called the $n$th generalized $q$-Euler polynomials attached to $\chi$. In the special case $x=0$, $E_{n,\chi,q}(=E_{n,\chi,q}(0))$ are called the $n$th generalized $q$-Euler numbers attached to $\chi$.

By (2.1), we see that

$$\begin{aligned} E_{n,\chi,q}(x) &= 2\sum_{m=0}^{\infty}\chi(m)(-1)^m[m+x]_q^n \\ &= \frac{2}{(1-q)^n}\sum_{a=0}^{d-1}\chi(a)(-1)^a\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{l(x+a)}}{1+q^{ld}}. \end{aligned} \tag{2.2}$$

Now we consider the $q$-Euler polynomials of order $r$ attached to $\chi$ as follows:

$$\begin{aligned} F_{q,\chi}^{(r)}(t,x) &= 2^r\sum_{m_1,\ldots,m_r=0}^{\infty}\left(\prod_{i=1}^{r}\chi(m_i)\right)(-1)^{m_1+\cdots+m_r}e^{[m_1+\cdots+m_r+x]_q t} \\ &= \sum_{n=0}^{\infty}E_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!}, \end{aligned} \tag{2.3}$$

where $E_{n,\chi,q}^{(r)}(x)$ are called the $n$th generalized $q$-Euler polynomials of order $r$ attached to $\chi$. In the special case $x=0$, $E_{n,\chi,q}^{(r)}(=E_{n,\chi,q}^{(r)}(0))$ are called the $n$th generalized $q$-Euler numbers of order $r$ attached to $\chi$.

From (2.3), we note that

$$E_{n,\chi,q}^{(r)}(x) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty}\left(\prod_{i=1}^{r}\chi(m_i)\right)(-1)^{m_1+\cdots+m_r}[m_1+\cdots+m_r+x]_q^n$$
$$= \frac{2^r}{(1-q)^n}\sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-1)^{a_1+\cdots+a_r}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{l(x+\sum_{j=1}^{r}a_j)}}{(1+q^{ld})^r}. \tag{2.4}$$

Thus we have

$$E_{n,\chi,q}^{(r)}(x) = 2^r \sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-1)^{a_1+\cdots+a_r}$$
$$\times \sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m[x+a_1+\cdots+a_r+dm]_q^n. \tag{2.5}$$

That is,

$$F_{q,\chi}^{(r)}(t,x) = 2^r \sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-1)^{a_1+\cdots+a_r}\times\sum_{m=0}^{\infty}\binom{m+r-1}{m}e^{[x+a_1+\cdots+a_r+dm]_q t}. \tag{2.6}$$

In the viewpoint of $h$-extension of $E_{n,\chi,q}^{(r)}(x)$, we can define the generalized $(h,q)$-Euler polynomials of order $r$ attached to $\chi$ as follows:

$$F_{q,\chi}^{(h,r)}(t,x) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty}\left(\prod_{i=1}^{r}\chi(m_i)\right)(-1)^{m_1+\cdots+m_r}q^{\sum_{j=1}^{r}(h-j)m_j}e^{[m_1+\cdots+m_r+x]_q t}$$
$$= \sum_{n=0}^{\infty}E_{n,\chi,q}^{(h,r)}(x)\frac{t^n}{n!}, \tag{2.7}$$

where $E_{n,\chi,q}^{(h,r)}(x)$ are called the $n$th generalized $(h,q)$-Euler polynomials of order $r$ attached to $\chi$. In the special case $x=0$, $E_{n,\chi,q}^{(h,r)}(=E_{n,\chi,q}^{(h,r)}(0))$ are called the $n$th generalized $(h,q)$-Euler numbers of order $r$ attached to $\chi$.

By (2.7), we see that

$$E_{n,\chi,q}^{(h,r)}(x) = 2^r[d]_q^n\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q(-1)^m q^{d(h-r)m}$$
$$\times \sum_{a_1,\ldots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-1)^{a_1+\cdots+a_r}q^{\sum_{j=1}^{r}(h-j)a_j}\left[m+\frac{x+a_1+\cdots+a_r}{d}\right]_{q^d}^n. \tag{2.8}$$

That is,

$$F_{q,\chi}^{(h,r)}(t,x)=2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q(-1)^m q^{d(h-r)m}\times\sum_{a_1,\dots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-1)^{a_1+\cdots+a_r}q^{\sum_{j=1}^{r}(h-j)a_j}e^{[x+a_1+\cdots+a_r+dm]_q t}. \tag{2.9}$$

Let $h=r$. Then we have

$$\begin{aligned}E_{n,\chi,q}^{(r,r)}(x)&=\frac{2^r}{(1-q)^n}\sum_{a_1,\dots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-1)^{\sum_{j=1}^{r}a_j}q^{\sum_{j=1}^{r}a_j(r-j)}\times\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{l(x+\sum_{j=1}^{r}a_j)}}{(-q^{ld}:q^d)_r}\\&=2^r[d]_q^n\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q(-1)^m\\&\quad\times\sum_{a_1,\dots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)(-1)^{\sum_{j=1}^{r}a_j}q^{\sum_{j=1}^{r}(r-j)a_j}\left[m+\frac{x+a_1+\cdots+a_r}{d}\right]_{q^d}^n.\end{aligned} \tag{2.10}$$

By (2.3), (2.9), and (2.10), we obtain the following equations:

$$\begin{aligned}\frac{2^r q^{mx}\sum_{a_1,\dots,a_r=0}^{d-1}\left(\prod_{i=1}^{r}\chi(a_i)\right)q^{\sum_{j=1}^{r}(m-j)a_j}(-1)^{\sum_{j=1}^{r}a_j}}{(-q^{d(m-r)}:q^d)_r}&=\sum_{l=0}^{m}\binom{m}{l}(q-1)^l E_{l,\chi,q}^{(0,r)}(x),\\ q^{d(h-1)}E_{n,\chi,q}^{(h,r)}(x+d)+E_{n,\chi,q}^{(h,r)}(x)&=2\sum_{l=0}^{d-1}\chi(l)(-1)^l E_{n,q}^{(h-1,r-1)}(x).\end{aligned} \tag{2.11}$$

In the special case $r=1$, we note that

$$F_{q,\chi}^{(h,1)}(t,x)=\sum_{n=0}^{\infty}E_{n,\chi,q}^{(h,1)}(x)\frac{t^n}{n!}. \tag{2.12}$$

By (2.12), we see that

$$F_{q,\chi}^{(h,1)}(t,x)=2\sum_{n=0}^{\infty}\chi(n)q^{(h-1)n}(-1)^n e^{[n+x]_q t}. \tag{2.13}$$

Hence

$$\begin{aligned}E_{n,\chi,q}^{(h,1)}(x)&=2\sum_{m=0}^{\infty}\chi(m)q^{(h-1)m}(-1)^m[m+x]_q^n\\&=\frac{2}{(1-q)^n}\sum_{a=0}^{d-1}\chi(a)(-1)^a\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{l(x+a)}}{1+q^{ld}}.\end{aligned} \tag{2.14}$$

For $s \in \mathbb{R}$ and $x \in \mathbb{C}$ with $\Re(x) > 0$, we have

$$\frac{1}{\Gamma(s)} \int_0^\infty F_{q,\chi}^{(r)}(-t, x) t^{s-1} dt = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} q^{\sum_{j=1}^{r}(h-j)m_j} \left(\prod_{i=1}^{r} \chi(m_i)\right)}{[m_1 + \cdots + m_r + x]_q^s}. \quad (2.15)$$

By (2.15), we can define the following $q$-$l$-function of order $r$.

*Definition 2.1.* For $s \in \mathbb{C}$, $x \in \mathbb{R}$ with $\Re(x) > 0$, we define the $q$-$l$-function as

$$l_q^{(h,r)}(s, x \mid \chi) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} q^{\sum_{j=1}^{r}(h-j)m_j} \left(\prod_{i=1}^{r} \chi(m_i)\right)}{[m_1 + \cdots + m_r + x]_q^s}. \quad (2.16)$$

Note that $l_q^{(h,r)}(s, x \mid \chi)$ is analytic in whole complex $s$-plane. By (2.7), (2.15), and the Cauchy residue theorem, we obtain the following theorem.

**Theorem 2.2.** *For $n \in \mathbb{Z}_+$, one has*

$$l_q^{(h,r)}(-n, x \mid \chi) = E_{n,\chi,q}^{(h,r)}(x). \quad (2.17)$$

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