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Research Article

$A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$-Weighted Inequalities with Lipschitz and BMO Norms

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We first define a new kind of $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ two-weight, then obtain some two-weight integral inequalities with Lipschitz norm and BMO norm for Green's operator applied to differential forms.

1. Introduction

Green's operator G is often applied to study the solutions of various differential equations and to define Poisson's equation for differential forms. Green's operator has been playing an important role in the study of PDEs. In many situations, the process to study solutions of PDEs involves estimating the various norms of the operators. Hence, we are motivated to establish some Lipschitz norm inequalities and BMO norm inequalities for Green's operator in this paper.

In the meanwhile, there have been generally studied about $A_r(\Omega)$-weighted [1, 2] and $A_r^{\lambda}(\Omega)$-weighted [3, 4] different inequalities and their properties. Results for more applications of the weight are given in [5, 6]. The purpose of this paper is to derive the new weighted inequalities with the Lipschitz norm and BMO norm for Green's operator applied to differential forms. We will introduce $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$-weight, which can be considered as a further extension of the $A_r^{\lambda}(\Omega)$-weight.

We keep using the traditional notation.

Let Ω be a connected open subset of $\mathbf{R}^n$, let $e_1, e_2, \dots, e_n$ be the standard unit basis of $\mathbf{R}^n$, and let $\bigwedge^l = \bigwedge^l(\mathbf{R}^n)$ be the linear space of l-covectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \dots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, $l = 0, 1, \dots, n$. We let $\mathbf{R} = \mathbf{R}^1$. The Grassman algebra $\bigwedge = \oplus \bigwedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \bigwedge$ and $\beta = \sum \beta^I e_I \in \bigwedge$, the inner product in $\bigwedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$.

We define the Hodge star operator $\star : \bigwedge \to \bigwedge$ by the rule $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle(\star 1)$ for all $\alpha, \beta \in \bigwedge$. The norm of $\alpha \in \bigwedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \bigwedge^0 = \mathbf{R}$. The Hodge star is an isometric isomorphism on $\bigwedge$ with $\star : \bigwedge^l \to \bigwedge^{n-l}$ and $\star\star\, (-1)^{l(n-l)} : \bigwedge^l \to \bigwedge^l$.

Balls are denoted by B and ρB is the ball with the same center as B and with $\operatorname{diam}(\rho B) = \rho \operatorname{diam}(B)$. We do not distinguish balls from cubes throughout this paper.

The n-dimensional Lebesgue measure of a set $E \subseteq \mathbf{R}^n$ is denoted by $|E|$. We call $w(x)$ a weight if $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $w > 0$ a.e. For $0 < p < \infty$ and a weight $w(x)$, we denote the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p,E,w^\alpha} = \left(\int_E |f(x)|^p w^\alpha dx \right)^{1/p}, \tag{1.1}$$

where α is a real number.

Differential forms are important generalizations of real functions and distributions. Specially, a differential l-form ω on Ω is a de Rham current [7, Chapter III] on Ω with values in $\bigwedge^l(\mathbf{R}^n)$; note that a 0-form is the usual function in $\mathbf{R}^n$. A differential l-form ω on Ω is a Schwartz distribution on ω with values in $\bigwedge^l(\mathbf{R}^n)$. We use $D'(\Omega, \bigwedge^l)$ to denote the space of all differential l-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$. We write $L^p(\Omega, \bigwedge^l)$ for the l-forms with $\omega_I \in L^p(\Omega, \mathbf{R})$ for all ordered l-tuples I. Thus $L^p(\Omega, \bigwedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx \right)^{1/p} = \left(\int_\Omega \left(\sum |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}. \tag{1.2}$$

For $\omega \in D'(\Omega, \bigwedge^l)$ the vector-valued differential form

$$\nabla\omega = \left(\frac{\partial\omega}{\partial x_1}, \ldots, \frac{\partial\omega}{\partial x_n} \right) \tag{1.3}$$

consists of differential forms

$$\frac{\partial\omega}{\partial x_i} \in D'\left(\Omega, \bigwedge^l \right), \tag{1.4}$$

where the partial differentiations are applied to the coefficients of ω.

As usual, $W^{1,p}(\Omega, \bigwedge^l)$ is used to denote the Sobolev space of l-forms, which equals $L^p(\Omega, \bigwedge^l) \cap L^p_1(\Omega, \bigwedge^l)$ with norm

$$\|\omega\|_{W^{1,p}(\Omega,\bigwedge^l)} = \|\omega\|_{W^{1,p}(\Omega,\bigwedge^l)} = \operatorname{diam}(\Omega)^{-1} \|\omega\|_{p,\Omega} + \|\nabla\omega\|_{p,\Omega}. \tag{1.5}$$

The notations $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{R})$ and $W^{1,p}_{\mathrm{loc}}(\Omega, \bigwedge^l)$ are self-explanatory. For $0 < p < \infty$ and a weight $w(x)$, the weighted norm of $\omega \in W^{1,p}(\Omega, \bigwedge^l)$ over Ω is denoted by

$$\|\omega\|_{W^{1,p}(\Omega,\bigwedge^l),w^\alpha} = \|\omega\|_{W^{1,p}(\Omega,\bigwedge^l),w^\alpha} = \operatorname{diam}(\Omega)^{-1}\|\omega\|_{p,\Omega,w^\alpha} + \|\nabla\omega\|_{p,\Omega,w^\alpha}, \tag{1.6}$$

where α is a real number.

We denote the exterior derivative by $d : D'(\Omega, \bigwedge^l) \to D'(\Omega, \bigwedge^{l+1})$ for $l = 0, 1, \ldots, n$. Its formal adjoint operator $d^\star : D'(\Omega, \bigwedge^{l+1}) \to D'(\Omega, \bigwedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \bigwedge^{l+1})$, $l = 0, 1, \ldots, n$. Let $\bigwedge^l\Omega$ be the lth exterior power of the cotangent bundle and let $C^\infty(\bigwedge^l\Omega)$ be the space of smooth l-forms on Ω. We set $W(\bigwedge^l\Omega) = \{u \in L^1_{\mathrm{loc}}(\bigwedge^l\Omega) : u \text{ has generalized gradient}\}$. The harmonic l-fields are defined by $H(\bigwedge^l\Omega) = \{u \in W(\bigwedge^l\Omega) : du = d^\star u = 0, u \in L^p \text{ for some } 1 < p < \infty\}$. The orthogonal complement of H in L^1 is defined by $H^\perp = \{u \in L^1 : \langle u, h\rangle = 0 \text{ for all } h \in H\}$. Then, Green's operator G is defined as $G : C^\infty(\bigwedge^l\Omega) \to H^\perp \cap C^\infty(\bigwedge^l\Omega)$ by assigning $G(u)$ to be the unique element of $H^\perp \cap C^\infty(\bigwedge^l\Omega)$ satisfying Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator that maps $C^\infty(\bigwedge^l\Omega)$ onto H, so that $H(u)$ is the harmonic part of u. See [8] for more properties of Green's operator.

The nonlinear elliptic partial differential equation $d^\star A(x, du) = 0$ is called the homogeneous A-harmonic equation or the A-harmonic equation, and the differential equation

$$d^\star A(x, du) = B(x, du) \tag{1.7}$$

is called the nonhomogeneous A-harmonic equation for differential forms, where $A : \Omega \times \bigwedge^l(\mathbf{R}^n) \to \bigwedge^l(\mathbf{R}^n)$ and $B : \Omega \times \bigwedge^l(\mathbf{R}^n) \to \bigwedge^{l-1}(\mathbf{R}^n)$ satisfy the following conditions:

$$|A(x,\xi)| \le a|\xi|^{p-1}, \qquad \langle A(x,\xi), \xi\rangle \ge |\xi|^p, \qquad |B(x,\xi)| \le b|\xi|^{p-1} \tag{1.8}$$

for almost every $x \in \Omega$ and all $\xi \in \bigwedge^l(\mathbf{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.7). A solution to (1.7) is an element of the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \bigwedge^{l-1})$ such that

$$\int_\Omega A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi = 0 \tag{1.9}$$

for all $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega, \bigwedge^{l-1})$ with compact support.

Let $A : \Omega \times \bigwedge^l(\mathbf{R}^n) \to \bigwedge^l(\mathbf{R}^n)$ be defined by $A(x,\xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then, A satisfies the required conditions and $d^\star A(x, du) = 0$ becomes the p-harmonic equation

$$d^\star\left(du|du|^{p-2}\right) = 0 \tag{1.10}$$

for differential forms. If u is a function (a 0-form), (1.10) reduces to the usual p-harmonic equation $\operatorname{div}(\nabla u|\nabla u|^{p-2}) = 0$ for functions. We should notice that if the operator B equals 0 in

(1.7), then (1.7) reduces to the homogeneous A-harmonic equation. Some results have been obtained in recent years about different versions of the A-harmonic equation; see [9–11].

Let $u \in L^1_{\mathrm{loc}}(\Omega, \bigwedge^l)$, $l = 0, 1, \ldots, n$. We write $u \in \operatorname{loc}\operatorname{Lip}_k(\Omega, \bigwedge^l)$, $0 \le k \le 1$, if

$$\|u\|_{\operatorname{loc}\operatorname{Lip}_k,\Omega} = \sup_{\sigma Q \subset \Omega} |Q|^{-(n+k)/n} \|u - u_Q\|_{1,Q} < \infty \tag{1.11}$$

for some $\sigma \ge 1$. Further, we write $\operatorname{Lip}_k(\Omega, \bigwedge^l)$ for those forms whose coefficients are in the usual Lipschitz space with exponent k and write $\|u\|_{\operatorname{Lip}_k,\Omega}$ for this norm. Similarly, for $u \in L^1_{\mathrm{loc}}(\Omega, \bigwedge^l)$, $l = 0, 1, \ldots, n$, we write $u \in \mathrm{BMO}(\Omega, \bigwedge^l)$ if

$$\|u\|_{\star,\Omega} = \sup_{\sigma Q \subset \Omega} |Q|^{-1} \|u - u_Q\|_{1,Q} < \infty \tag{1.12}$$

for some $\sigma \ge 1$. When u is a 0-form, (1.12) reduces to the classical definition of $\mathrm{BMO}(\Omega)$.

Based on the above results, we discuss the weighted Lipschitz and BMO norms. For $u \in L^1_{\mathrm{loc}}(\Omega, \bigwedge^l, w^\alpha)$, $l = 0, 1, \ldots, n$, we write $u \in \operatorname{loc}\operatorname{Lip}_k(\Omega, \bigwedge^l, w^\alpha)$, $0 \le k \le 1$, if

$$\|u\|_{\operatorname{loc}\operatorname{Lip}_k,\Omega,w^\alpha} = \sup_{\sigma Q \subset \Omega} (\mu(Q))^{-(n+k)/n} \|u - u_Q\|_{1,Q,w^\alpha} < \infty \tag{1.13}$$

for some $\sigma > 1$, where Ω is a bounded domain, the Radon measure μ is defined by $d\mu = w(x)^\alpha dx$, w is a weight and, α is a real number. For convenience, we shall write the following simple notation $\operatorname{loc}\operatorname{Lip}_k(\Omega, \bigwedge^l)$ for $\operatorname{loc}\operatorname{Lip}_k(\Omega, \bigwedge^l, w^\alpha)$. Similarly, for $u \in L^1_{\mathrm{loc}}(\Omega, \bigwedge^l, w^\alpha)$, $l = 0, 1, \ldots, n$, we write $u \in \mathrm{BMO}(\Omega, \bigwedge^l, w^\alpha)$ if

$$\|u\|_{\star,\Omega,w^\alpha} = \sup_{\sigma Q \subset \Omega} (\mu(Q))^{-1} \|u - u_Q\|_{1,Q,w^\alpha} < \infty \tag{1.14}$$

for some $\sigma > 1$, where the Radon measure μ is defined by $d\mu = w(x)^\alpha dx$, w is a weight and α is a real number. Again, we use $\mathrm{BMO}(\Omega, \bigwedge^l)$ to replace $\mathrm{BMO}(\Omega, \bigwedge^l, w^\alpha)$ whenever it is clear that the integral is weighted.

2. Preliminary Knowledge and Lemmas

Definition 2.1. We say that the weight $(w_1(x), w_2(x))$ satisfies the $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ condition for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$; let $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$, if $w_1(x) > 0$, $w_2(x) > 0$ a.e. and

$$\sup_B \left(\frac{1}{|B|} \int_B w_1^{\lambda_1} dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{w_2} \right)^{\lambda_2/(r-1)} dx \right)^{\lambda_3(r-1)} < \infty \tag{2.1}$$

for any ball $B \subset \Omega$.

If we choose $w_1 = w_2 = w$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in Definition 2.1, we will obtain the usual $A_r(\Omega)$-weight. If we choose $w_1 = w_2 = w$, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda$ in Definition 2.1, we will obtain the $A_r^{\lambda}(\Omega)$-weight [3]. If we choose $w_1 = w_2 = w$, $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_3 = 1$ in Definition 2.1, we will obtain the $A_r(\lambda, \Omega)$-weight [12].

Lemma 2.2 (see [1]). *If $w \in A_r(\Omega)$, then there exist constants $\beta > 1$ and C, independent of w, such that*

$$\|w\|_{\beta,B} \leq C|B|^{(1-\beta)/\beta}\|w\|_{1,B} \tag{2.2}$$

for all balls $B \subset R^n$.

We need the following generalized Hölder inequality.

Lemma 2.3. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbf{R}^n$, then*

$$\|fg\|_{s,E} \leq \|f\|_{\alpha,E} \cdot \|g\|_{\beta,E} \tag{2.3}$$

for any $E \subset \mathbf{R}^n$.

The following version of weak reverse Hölder inequality appeared in [13].

Lemma 2.4. *Suppose that u is a solution to the nonhomogeneous A-harmonic equation* (1.7) *in Ω, $\sigma > 1$ and $q > 0$. There exists a constant C, depending only on σ, n, p, a, b and q, such that*

$$\|du\|_{p,Q} \leq C|Q|^{(q-p)/pq}\|du\|_{q,\sigma Q} \tag{2.4}$$

for all Q with $\sigma Q \subset \Omega$.

Lemma 2.5 (see [14]). *Let $du \in L^s(\Omega, \bigwedge^l)$ be a smooth form and let G be Green's operator, $l = 1, \ldots, n$, and $1 < s < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|G(u) - (G(u))_B\|_{s,B} \leq C|B| \operatorname{diam}(B)\|du\|_{s,B} \tag{2.5}$$

for all balls $B \subset \Omega$.

We need the following Lemma 2.6 (Caccioppoli inequality) that was proved in [8].

Lemma 2.6 (see [8]). *Let $u \in D'(\Omega, \bigwedge^l)$ be a solution to the nonhomogeneous A-harmonic equation* (1.7) *in Ω and let $\sigma > 1$ be a constant. Then, there exists a constant C, independent of u, such that*

$$\|du\|_{p,B} \leq C \operatorname{diam}(B)^{-1}\|u - c\|_{p,\sigma B} \tag{2.6}$$

for all balls or cubes B with $\sigma B \subset \Omega$ and all closed forms c. Here $1 < p < \infty$.

Lemma 2.7 (see [14]). *Let $du \in L^s(\Omega, \bigwedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a smooth form in a domain Ω. Then, there exists a constant C, independent of u, such that*

$$\|G(u)\|_{\text{loc Lip}_k,\Omega} \leq C\|du\|_{s,\Omega}, \tag{2.7}$$

where k is a constant with $0 \leq k \leq 1$.

Lemma 2.8 (see [14]). *Let $du \in L^s(\Omega, \bigwedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a smooth form in a bounded domain Ω and let G be Green's operator. Then, there exists a constant C, independent of u, such that*

$$\|G(u)\|_{\star,\Omega} \leq C\|du\|_{s,\Omega}. \tag{2.8}$$

3. Main Results and Proofs

Theorem 3.1. *Let $du \in L^s(\Omega, \bigwedge^l, v)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation* (1.7) *in a bounded domain Ω and let G be Green's operator, where the Radon measures μ and v are defined by $d\mu = w_1^{\alpha\lambda_1}(x)$, $dv = w_2^{\alpha\lambda_2\lambda_3}(x)$. Assume that $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$, $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|G(u) - (G(u))_B\|_{1,B,w_1^{\alpha\lambda_1}} \leq C|B| \operatorname{diam}(B)\|du\|_{s,\sigma B,w_2^{\alpha\lambda_2\lambda_3}}, \tag{3.1}$$

where k is a constant with $0 \leq k \leq 1$, and α is a constant with $0 < \alpha < 1$.

Proof. Choose $t = s/(1-\alpha)$ where $0 < \alpha < 1$; then $1 < s < t$ and $\alpha t/(t-s) = 1$. Since $1/s = 1/t + (t-s)/st$, by Lemmas 2.3 and 2.5, we have

$$\begin{aligned}
\|G(u) - (G(u))_B\|_{s,B,w_1^{\alpha\lambda_1}} &= \left(\int_B |G(u) - (G(u))_B|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \\
&\leq \left(\int_B |G(u) - (G(u))_B|^t dx\right)^{1/t} \left(\int_B \left(w_1^{\alpha\lambda_1/s}\right)^{st/(t-s)} dx\right)^{(t-s)/st} \\
&= \|G(u) - (G(u))_B\|_{t,B} \left\|w_1^{\lambda_1}\right\|_{1,B}^{\alpha/s} \\
&\leq C_1|B| \operatorname{diam}(B)\|du\|_{t,B} \left\|w_1^{\lambda_1}\right\|_{1,B}^{\alpha/s}
\end{aligned} \tag{3.2}$$

for all ball $B \subset \Omega$. Choosing $m = s/(\alpha\lambda_3(r-1)+1)$, then $m < s$. From Lemma 2.4, we have

$$\|du\|_{t,B} \leq C_2|B|^{(m-t)/mt}\|du\|_{m,\sigma B}, \tag{3.3}$$

where $\sigma > 1$ and $\sigma B \subset \Omega$. Using Hölder inequality with $1/m = 1/s + \alpha\lambda_3(r-1)/s$, we have

$$\begin{aligned}\|du\|_{m,\sigma B} &= \left(\int_{\sigma B}\left(|du| w_2^{\alpha\lambda_2\lambda_3/s} w_2^{-\alpha\lambda_2\lambda_3/s}\right)^m dx\right)^{1/m}\\ &\leq \left(\int_{\sigma B}|du|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s}\left(\int_{\sigma B}\left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\alpha\lambda_3(r-1)/s}\\ &= \|du\|_{s,\sigma B, w_2^{\alpha\lambda_2\lambda_3}}\left\|\left(\frac{1}{w_2}\right)^{\lambda_2}\right\|_{1/(r-1),\sigma B}^{\alpha\lambda_3/s}.\end{aligned} \tag{3.4}$$

Since $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$, then

$$\begin{aligned}&\left\|w_1^{\lambda_1}\right\|_{1,B}^{\alpha/s}\cdot\left\|\left(\frac{1}{w_2}\right)^{\lambda_2}\right\|_{1/(r-1),\sigma B}^{\alpha\lambda_3/s}\\ &\quad\leq\left[\left(\int_{\sigma B} w_1^{\lambda_1}dx\right)\left(\int_{\sigma B}\left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)}dx\right)^{\lambda_3(r-1)}\right]^{\alpha/s}\\ &\quad=\left[|\sigma B|^{\lambda_3(r-1)+1}\left(\frac{1}{|\sigma B|}\int_{\sigma B} w_1^{\lambda_1}dx\right)\left(\frac{1}{|\sigma B|}\int_{\sigma B}\left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)}dx\right)^{\lambda_3(r-1)}\right]^{\alpha/s}\\ &\quad\leq C_3|\sigma B|^{\alpha\lambda_3(r-1)/s+\alpha/s}\\ &\quad\leq C_4|B|^{\alpha\lambda_3(r-1)/s+\alpha/s}.\end{aligned} \tag{3.5}$$

Since $(m-t)/mt + (\alpha\lambda_3(r-1)+\alpha)/s = 0$, combining with (3.2), (3.3), (3.4), and (3.5), we have

$$\begin{aligned}\|G(u)-(G(u))_B\|_{s,B,w_1^{\alpha\lambda_1}} &\leq C_1|B|\operatorname{diam}(B) C_2|B|^{(m-t)/mt}\|du\|_{s,\sigma B,w_2^{\alpha\lambda_2\lambda_3}} C_4|B|^{\alpha\lambda_3(r-1)/s+\alpha/s}\\ &= C_5|B|\operatorname{diam}(B)\|du\|_{s,\sigma B,w_2^{\alpha\lambda_2\lambda_3}}.\end{aligned} \tag{3.6}$$

Notice that $|\Omega| < \infty$, $1-1/s > 0$; from (3.6) and the Hölder inequality with $1 = 1/s + (s-1)/s$, we find that

$$\begin{aligned}\|G(u)-(G(u))_B\|_{1,B,w_1^{\alpha\lambda_1}} &= \int_B |G(u)-(G(u))_B| w_1^{\alpha\lambda_1}dx\\ &\leq\left(\int_B |G(u)-(G(u))_B|^s w_1^{\alpha\lambda_1}dx\right)^{1/s}\left(\int_B 1^{s/(s-1)} w_1^{\alpha\lambda_1}dx\right)^{(s-1)/s}\\ &= (\mu(B))^{(s-1)/s}\|G(u)-(G(u))_B\|_{s,B,w_1^{\alpha\lambda_1}}\\ &\leq |\Omega|^{1-1/s} C_5|B|\operatorname{diam}(B)\|du\|_{s,\sigma B,w_2^{\alpha\lambda_2\lambda_3}}\\ &\leq C_6|B|\operatorname{diam}(B)\|du\|_{s,\sigma B,w_2^{\alpha\lambda_2\lambda_3}}.\end{aligned} \tag{3.7}$$

We have completed the proof of Theorem 3.1. □

Remark. Specially, choosing $\lambda_2\lambda_3 = \lambda_1$ and $w_1 = w_2$ in Theorem 3.1, we have

$$\|G(u) - (G(u))_B\|_{1,B,w_1^{\alpha\lambda_1}} \le C_6|B|\operatorname{diam}(B)\|du\|_{s,\sigma B,w_1^{\alpha\lambda_1}}. \tag{3.8}$$

Next, we will establish the following weighted norm comparison theorem between the Lipschitz and the BMO norms.

Theorem 3.2. *Let $u \in L^s(\Omega, \bigwedge^l, v)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation (1.7) in a bounded domain Ω and let G be Green's operator, where the Radon measures μ and v are defined by $d\mu = w_1^{\alpha\lambda_1}(x)$, $dv = w_2^{\alpha\lambda_2\lambda_3/s}(x)$. Assume that $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$, $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$ with $w_1(x) \ge \varepsilon > 0$ for any $x \in \Omega$. Then, there exists a constant C, independent of u, such that*

$$\|G(u)\|_{\operatorname{loc}\operatorname{Lip}_k,\Omega,w_1^{\alpha\lambda_1}} \le C\|u\|_{\star,\Omega,w_2^{\alpha\lambda_2\lambda_3/s}}, \tag{3.9}$$

where k is a constant with $0 \le k \le 1$, and α is a constant with $0 < \alpha < 1$.

Proof. Choose $t = s/(1-\alpha)$ where $0 < \alpha < 1$; then $1 < s < t$ and $\alpha t/(t-s) = 1$. Since $1/s = 1/t + (t-s)/st$, by Lemma 2.3, we have

$$\begin{aligned} \|du\|_{s,\sigma_1 B,w_1^{\alpha\lambda_1}} &= \left(\int_{\sigma_1 B} |du|^s w_1^{\alpha\lambda_1}\,dx\right)^{1/s} \\ &\le \left(\int_{\sigma_1 B} |du|^t dx\right)^{1/t} \left(\int_{\sigma_1 B} \left(w_1^{\alpha\lambda_1/s}\right)^{st/(t-s)} dx\right)^{(t-s)/st} \\ &= \|du\|_{t,\sigma_1 B} \left\|w_1^{\lambda_1}\right\|_{1,\sigma_1 B}^{\alpha/s} \end{aligned} \tag{3.10}$$

for any ball B and some constant $\sigma_1 > 1$ with $\sigma_1 B \subset \Omega$. Choosing $c = u_B$ in Lemma 2.6, we find that

$$\|du\|_{t,\sigma_1 B} \le C_1 \operatorname{diam}(B)^{-1}\|u - u_B\|_{t,\sigma_2 B}, \tag{3.11}$$

where $\sigma_2 > \sigma_1$ is a constant and $\sigma_2 B \subset \Omega$. Combining (3.8), (3.10), and (3.11), it follows that

$$\begin{aligned} \|G(u) - (G(u))_B\|_{1,B,w_1^{\alpha\lambda_1}} &\le C_2|B|\operatorname{diam}(B)\|du\|_{s,\sigma_1 B,w_1^{\alpha\lambda_1}} \\ &\le C_2|B|\operatorname{diam}(B)\left\|w_1^{\lambda_1}\right\|_{1,\sigma_1 B}^{\alpha/s} C_1 \operatorname{diam}(B)^{-1}\|u - u_B\|_{t,\sigma_2 B} \\ &= C_3|B|\|u - u_B\|_{t,\sigma_2 B}\left\|w_1^{\lambda_1}\right\|_{1,\sigma_1 B}^{\alpha/s}. \end{aligned} \tag{3.12}$$

Choosing $m = s/(\alpha\lambda_3(r-1)+s)$, then $m < s < t$. Applying the weak reverse Hölder inequality for the solutions of the nonhomogeneous A-harmonic equation, we obtain

$$\|u - u_B\|_{t,\sigma_2 B} \le C_4|B|^{(m-t)/mt}\|u - u_B\|_{m,\sigma_3 B}, \tag{3.13}$$

where $\sigma_3 > \sigma_2$ is a constant and $\sigma_3 B \subset \Omega$. Substituting (3.13) into (3.12), we have

$$\begin{aligned}\|G(u) - (G(u))_B\|_{1,B,w_1^{\alpha\lambda_1}} &\le C_3|B|C_4|B|^{(m-t)/mt}\|u - u_B\|_{m,\sigma_3 B}\left\|w_1^{\lambda_1}\right\|_{1,\sigma_1 B}^{\alpha/s} \\ &= C_5|B|^{1+(m-t)/mt}\|u - u_B\|_{m,\sigma_3 B}\left\|w_1^{\lambda_1}\right\|_{1,\sigma_1 B}^{\alpha/s}.\end{aligned} \tag{3.14}$$

Using Hölder inequality with $1/m = 1/1 + \alpha\lambda_3(r-1)/s$, we have

$$\begin{aligned}\|u - u_B\|_{m,\sigma_3 B} &= \left(\int_{\sigma_3 B}\left(|u - u_B|w_2^{\alpha\lambda_2\lambda_3/s}w_2^{-\alpha\lambda_2\lambda_3/s}\right)^m dx\right)^{1/m} \\ &\le \left(\int_{\sigma_3 B}|u - u_B|w_2^{\alpha\lambda_2\lambda_3/s}dx\right)\left(\int_{\sigma_3 B}\left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)}dx\right)^{\alpha\lambda_3(r-1)/s} \\ &= \|u - u_B\|_{1,\sigma_3 B,w_2^{\alpha\lambda_2\lambda_3/s}}\left\|\left(\frac{1}{w_2}\right)^{\lambda_2}\right\|_{1/(r-1),\sigma_3 B}^{\alpha\lambda_3/s}.\end{aligned} \tag{3.15}$$

Since $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$, then

$$\begin{aligned}&\left\|w_1^{\lambda_1}\right\|_{1,\sigma_1 B}^{\alpha/s}\cdot\left\|\left(\frac{1}{w_2}\right)^{\lambda_2}\right\|_{1/(r-1),\sigma_3 B}^{\alpha\lambda_3/s} \\ &\quad\le \left[\left(\int_{\sigma_3 B}w_1^{\lambda_1}dx\right)\left(\int_{\sigma_3 B}\left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)}dx\right)^{\lambda_3(r-1)}\right]^{\alpha/s} \\ &\quad= \left[|\sigma_3 B|^{\lambda_3(r-1)+1}\left(\frac{1}{|\sigma_3 B|}\int_{\sigma_3 B}w_1^{\lambda_1}dx\right)\left(\frac{1}{|\sigma_3 B|}\int_{\sigma_3 B}\left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)}dx\right)^{\lambda_3(r-1)}\right]^{\alpha/s} \\ &\quad\le C_6|\sigma_3 B|^{\alpha\lambda_3(r-1)/s+\alpha/s} \\ &\quad\le C_7|B|^{\alpha\lambda_3(r-1)/s+\alpha/s}.\end{aligned} \tag{3.16}$$

Since $(m-t)/mt + (\alpha\lambda_3(r-1)+\alpha)/s + 1 = 1/s$, combining with (3.14), (3.15), and (3.16), we have

$$\begin{aligned}\|G(u)-(G(u))_B\|_{1,B,w_1^{\alpha\lambda_1}} &\le C_5|B|^{1+(m-t)/mt}C_7|B|^{\alpha\lambda_3(r-1)/s+\alpha/s}\|u-u_B\|_{1,\sigma_3B,w_2^{\alpha\lambda_2\lambda_3/s}} \\ &= C_8|B|^{1/s}\|u-u_B\|_{1,\sigma_3B,w_2^{\alpha\lambda_2\lambda_3/s}}.\end{aligned} \tag{3.17}$$

Since $\mu(B)=\int_B w_1^{\alpha\lambda_1}dx \ge \int_B \varepsilon^{\alpha\lambda_1}dx = C_9|B|$, we have

$$\frac{1}{\mu(B)} \le \frac{C_{10}}{|B|} \tag{3.18}$$

for all ball B. Notice that $1-k/n>0$ and $|\Omega|<\infty$; from (3.17), we have

$$\begin{aligned}\|G(u)\|_{\mathrm{loc\,Lip}_k,\Omega,w_1^{\alpha\lambda_1}} &= \sup_{\sigma_4B\subset\Omega}(\mu(B))^{-(n+k)/n}\|G(u)-(G(u))_B\|_{1,B,w_1^{\alpha\lambda_1}} \\ &\le C_8\sup_{\sigma_4B\subset\Omega}(\mu(B))^{-1/s-k/n}|B|^{1/s}\|u-u_B\|_{1,\sigma_3B,w_2^{\alpha\lambda_2\lambda_3/s}} \\ &\le C_{11}\sup_{\sigma_4B\subset\Omega}|B|^{-1/s-k/n}|B|^{1+1/s}|B|^{-1}\|u-u_B\|_{1,\sigma_3B,w_2^{\alpha\lambda_2\lambda_3/s}} \\ &\le C_{11}\sup_{\sigma_4B\subset\Omega}|\Omega|^{1-k/n}|B|^{-1}\|u-u_B\|_{1,\sigma_3B,w_2^{\alpha\lambda_2\lambda_3/s}} \\ &\le C_{12}\sup_{\sigma_4B\subset\Omega}|B|^{-1}\|u-u_B\|_{1,\sigma_3B,w_2^{\alpha\lambda_2\lambda_3/s}} \\ &= C_{12}\|u\|_{\star,\Omega,w_2^{\alpha\lambda_2\lambda_3/s}},\end{aligned} \tag{3.19}$$

where $\sigma_4>\sigma_3$ is a constant and $\sigma_4B\subset\Omega$. We have completed the proof of Theorem 3.2. □

Now, we will prove the following weighted inequality between the BMO norm and the Lipschitz norm for Green's operator.

Theorem 3.3. *Let $u\in L^s(\Omega,\bigwedge^l,\upsilon)$, $l=1,2,\ldots,n$, $1<s<\infty$, be a solution of the nonhomogeneous A-harmonic equation* (1.7) *in a bounded domain Ω and let G be Green's operator, where the Radon measures μ and υ are defined by $d\mu=w_1^{\alpha\lambda_1}(x)$, $d\upsilon=w_2^{\alpha\lambda_2\lambda_3/s}(x)$. Assume that $w_1^{\lambda_1}(x)\in A_r(\Omega)$ and $(w_1(x),w_2(x))\in A_r^{\lambda_3}(\lambda_1,\lambda_2,\Omega)$ for some $r>1$, $0<\lambda_1,\lambda_2,\lambda_3<\infty$ with $w_1(x)\ge\varepsilon>0$ for any $x\in\Omega$. Then, there exists a constant C, independent of u, such that*

$$\|G(u)\|_{\star,\Omega,w_1^{\alpha\lambda_1}} \le C\|u\|_{\mathrm{loc\,Lip}_k,\Omega,w_2^{\alpha\lambda_2\lambda_3/s}}, \tag{3.20}$$

where α is a constant with $0<\alpha<1$.

Proof. Since $w_1^{\lambda_1} \in A_r(\Omega)$, using Lemma 2.2, there exist constants $\beta > 1$ and $C_1 > 0$, such that

$$\left\|w_1^{\lambda_1}\right\|_{\beta,B} \le C_1 |B|^{(1-\beta)/\beta} \left\|w_1^{\lambda_1}\right\|_{1,B} \tag{3.21}$$

for any ball $B \subset \mathbf{R}^n$.

Since $1 = 1/s + (s-1)/s$, by Lemma 2.3, we have

$$\begin{aligned}\|G(u) - G(u)_B\|_{1,B,w_1^{\alpha\lambda_1}} &= \int_B |G(u) - G(u)_B| w_1^{\alpha\lambda_1} dx \\ &\le \left(\int_B |G(u) - G(u)_B|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \left(\int_B w_1^{\alpha\lambda_1} dx\right)^{(s-1)/s} \\ &= \mu(B)^{(s-1)/s} \|G(u) - G(u)_B\|_{s,B,w_1^{\alpha\lambda_1}}.\end{aligned} \tag{3.22}$$

Choose $t = s/(1-\alpha/\beta)$ where $0 < \alpha < 1$, $\beta > 1$; then $1 < s < t$ and $\alpha t/(t-s) = \beta$. Since $1/s = 1/t + (t-s)/st$, by Lemma 2.3 and (3.21), we have

$$\begin{aligned}\|G(u) - G(u)_B\|_{s,B,w_1^{\alpha\lambda_1}} &= \left(\int_B \left(|G(u) - G(u)_B| w_1^{\alpha\lambda_1/s}\right)^s dx\right)^{1/s} \\ &\le \left(\int_B |G(u) - G(u)_B|^t dx\right)^{1/t} \left(\int_B w_1^{\lambda_1\beta} dx\right)^{\alpha/(\beta s)} \\ &= \|G(u) - G(u)_B\|_{t,B} \cdot \left\|w_1^{\lambda_1}\right\|_{\beta,B}^{\alpha/s} \\ &\le \|G(u) - G(u)_B\|_{t,B} \cdot C_2 |B|^{(1-\beta)\alpha/(\beta s)} \left\|w_1^{\lambda_1}\right\|_{1,B}^{\alpha/s}.\end{aligned} \tag{3.23}$$

From Lemmas 2.5 and 2.6 with $c = u_B$, we have

$$\begin{aligned}\|G(u) - (G(u))_B\|_{t,B} &\le C_3 |B| \operatorname{diam}(B) \|du\|_{t,B} \\ &\le C_3 |B| \operatorname{diam}(B) C_4 \operatorname{diam}(B)^{-1} \|u - u_B\|_{t,\sigma_1 B} \\ &= C_5 |B| \|u - u_B\|_{t,\sigma_1 B},\end{aligned} \tag{3.24}$$

where $\sigma_1 > 1$ is a constant and $\sigma_1 B \subset \Omega$. Applying the weak reverse Hölder inequality for the solutions of the nonhomogeneous A-harmonic equation, we obtain

$$\|u - u_B\|_{t,\sigma_1 B} \le C_6 |B|^{(m-t)/mt} \|u - u_B\|_{m,\sigma_2 B}, \tag{3.25}$$

where $\sigma_2 > \sigma_1$ is a constant and $\sigma_2 B \subset \Omega$. Choosing $m = s/(\alpha\lambda_3(r-1)+s)$, then $m < 1 < s$. Using Hölder inequality with $1/m = 1/1 + \alpha\lambda_3(r-1)/s$, we have

$$\begin{aligned}
\|u-u_B\|_{m,\sigma_2 B} &= \left(\int_{\sigma_2 B}\left(|u-u_B| w_2^{\alpha\lambda_2\lambda_3/s} w_2^{-\alpha\lambda_2\lambda_3/s}\right)^m dx\right)^{1/m} \\
&\le \left(\int_{\sigma_2 B}|u-u_B| w_2^{\alpha\lambda_2\lambda_3/s} dx\right)\left(\int_{\sigma_2 B}\left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\alpha\lambda_3(r-1)/s} \\
&= \|u-u_B\|_{1,\sigma_2 B, w_2^{\alpha\lambda_2\lambda_3/s}} \left\|\left(\frac{1}{w_2}\right)^{\lambda_2}\right\|_{1/(r-1),\sigma_2 B}^{\alpha\lambda_3/s}.
\end{aligned} \tag{3.26}$$

Since $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ and $(m-t)/mt + \alpha\lambda_3(r-1)/s + \alpha/s + (s-1)/s + (1-\beta)\alpha/(\beta s) = 0$, combining with (3.22), (3.23), (3.24), and (3.25), we have

$$\begin{aligned}
&\|G(u)-G(u)_B\|_{1,B,w_1^{\alpha\lambda_1}} \\
&\quad \le \mu(B)^{(s-1)/s} C_5 |B| C_6 |B|^{(m-t)/mt} C_2 |B|^{(1-\beta)\alpha/(\beta s)} C_7 |B|^{\alpha\lambda_3(r-1)/s+\alpha/s} \|u-u_B\|_{1,\sigma_2 B, w_2^{\alpha\lambda_2\lambda_3/s}} \\
&\quad \le C_8 |B| |B|^{(1-\beta)\alpha/(\beta s)} |B|^{(m-t)/mt+\alpha\lambda_3(r-1)/s+\alpha/s+(s-1)/s} \|u-u_B\|_{1,\sigma_2 B, w_2^{\alpha\lambda_2\lambda_3/s}} \\
&\quad = C_8 |B| \|u-u_B\|_{1,\sigma_2 B, w_2^{\alpha\lambda_2\lambda_3/s}}
\end{aligned} \tag{3.27}$$

From the definitions of the Lipschitz and BMO norms, we obtain

$$\begin{aligned}
\|G(u)\|_{\star,\Omega,w_1^{\alpha\lambda_1}} &= \sup_{\sigma_3 B\subset\Omega} |B|^{-1} \|G(u)-G(u)_B\|_{1,B,w_1^{\alpha\lambda_1}} \\
&= \sup_{\sigma_3 B\subset\Omega} |B|^{k/n} |B|^{-(n+k)/n} \|G(u)-G(u)_B\|_{1,B,w_1^{\alpha\lambda_1}} \\
&\le \sup_{\sigma_3 B\subset\Omega} |M|^{k/n} |B|^{-(n+k)/n} \|G(u)-G(u)_B\|_{1,B,w_1^{\alpha\lambda_1}} \\
&\le C_9 \sup_{\sigma_3 B\subset\Omega} |B|^{-(n+k)/n} \|G(u)-G(u)_B\|_{1,B,w_1^{\alpha\lambda_1}}.
\end{aligned} \tag{3.28}$$

for all balls B with $\sigma_3 > \sigma_2$ and $\sigma_3 B \subset \Omega$. Substituting (3.27) into (3.28), we have

$$\begin{aligned}
\|G(u)\|_{\star,\Omega,w_1^{\alpha\lambda_1}} &\le C_9 \sup_{\sigma_3 B\subset\Omega} |B|^{-(n+k)/n} \|G(u)-G(u)_B\|_{1,B,w_1^{\alpha\lambda_1}} \\
&\le C_9 \sup_{\sigma_3 B\subset\Omega} |B|^{-(n+k)/n} C_8 |B| \|u-u_B\|_{1,\sigma_2 B, w_2^{\alpha\lambda_2\lambda_3/s}} \\
&\le C_{10} \sup_{\sigma_3 B\subset\Omega} |B|^{-(n+k)/n} \|u-u_B\|_{1,\sigma_2 B, w_2^{\alpha\lambda_2\lambda_3/s}} \\
&= C_{10} \|u\|_{\mathrm{loc\,Lip}_k,\Omega, w_2^{\alpha\lambda_2\lambda_3/s}}.
\end{aligned} \tag{3.29}$$

We have completed the proof of Theorem 3.3. □

Using the same methods, and by Lemmas 2.7 and 2.8, we can estimate Lipschitz norm $\|\cdot\|_{\mathrm{loc\,Lip}_k,\Omega,w^\alpha}$ and BMO norm $\|\cdot\|_{\star,\Omega,w^\alpha}$ of Green's operator in terms of L^s norm.

Theorem 3.4. *Let $du \in L^s(\Omega, \bigwedge^l, \mu)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation* (1.7) *in a bounded domain Ω and let G be Green's operator, where the Radon measures μ and υ are defined by $d\mu = w_1^{\alpha\lambda_1}(x)$, $d\upsilon = w_2^{\alpha\lambda_2\lambda_3/s}(x)$. Assume that $w_1^{\lambda_1}(x) \in A_r(\Omega)$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$, $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$ with $w_1(x) \geq \varepsilon > 0$ for any $x \in \Omega$. Then, there exists a constant C, independent of u, such that*

$$\|G(u)\|_{\mathrm{loc\,Lip}_k,\Omega,w_1^{\alpha\lambda_1}} \leq C\|du\|_{s,\Omega,w_2^{\alpha\lambda_2\lambda_3/s}}, \tag{3.30}$$

where k is a constant with $0 \leq k \leq 1$, and α is a constant with $0 < \alpha < 1$.

Theorem 3.5. *Let $du \in L^s(\Omega, \bigwedge^l, \mu)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a solution of the nonhomogeneous A-harmonic equation* (1.7) *in a bounded domain Ω and let G be Green's operator, where the Radon measures μ and υ are defined by $d\mu = w_1^{\alpha\lambda_1}(x)$, $d\upsilon = w_2^{\alpha\lambda_2\lambda_3/s}(x)$. Assume that $w_1^{\lambda_1}(x) \in A_r(\Omega)$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$, $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$ with $w_1(x) \geq \varepsilon > 0$ for any $x \in \Omega$. Then, there exists a constant C, independent of u, such that*

$$\|G(u)\|_{\star,\Omega,w_1^{\alpha\lambda_1}} \leq C\|du\|_{s,\Omega,w_2^{\alpha\lambda_2\lambda_3/s}}, \tag{3.31}$$

where α is a constant with $0 < \alpha < 1$.

Remark. Note that the differentiable functions are special differential forms (0-forms). Hence, the usual p-harmonic equation $\operatorname{div}(\nabla u|\nabla u|^{p-2}) = 0$ for functions is the special case of the A-harmonic equation for differential forms. Therefore, all results that we have proved for solutions of the A-harmonic equation in this paper are still true for p-harmonic functions.

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