Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 546015, 4 pages
doi:10.1155/2010/546015

*Research Article*

# Alon-Babai-Suzuki's Conjecture Related to Binary Codes in Nonmodular Version

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Received 23 August 2009; Accepted 22 January 2010

Academic Editor: Ram N. Mohapatra



Let $K = \{k_1, k_2, \ldots, k_r\}$ and $L = \{l_1, l_2, \ldots, l_s\}$ be sets of nonnegative integers. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a family of subsets of $[n]$ with $[F_i] \in K$ for each $i$ and $|F_i \cap F_j| \in L$ for any $i \neq j$. Every subset $F_e$ of $[n]$ can be represented by a binary code $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ such that $a_i = 1$ if $i \in F_e$ and $a_i = 0$ if $i \notin F_e$. Alon et al. made a conjecture in 1991 in modular version. We prove Alon-Babai-Sukuki's Conjecture in nonmodular version. For any $K$ and $L$ with $n \geq s + \max k_i$, $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.

## 1. Introduction

In this paper, $\mathcal{F}$ stands for a family of subsets of $[n] = \{1, 2, \ldots, n\}$, $K = \{k_1, \ldots, k_r\}$, and $L = \{l_1, \ldots, l_s\}$, where $|F_i| \in K$ for all $F_i \in \mathcal{F}$, $|F_i \cap F_j| \in L$ for all $F_i, F_j \in \mathcal{F}, i \neq j$. The variable $x$ will stand as a shorthand for the $n$-dimensional vector variable $(x_1, x_2, \ldots, x_n)$. Also, since these variables will take the values only 0 and 1, all the polynomials we will work with will be reduced modulo the relation $x_i^2 = x_i$. We define the characteristic vector $v_i = (v_{i1}, v_{i2}, \ldots, v_{in})$ of $F_i$ such that $v_{ij} = 1$ if $j \in F_i$ and $v_{ij} = 0$ if $j \notin F_i$. We will present some results in this paper that give upper bounds on the size of $\mathcal{F}$ under various conditions. Below is a list of related results by others.

**Theorem 1.1** (Ray-Chaudhuri and Wilson [1]). *If $K = \{k\}$, and $L$ is any set of nonnegative integers with $k > \max l_j$, then $|\mathcal{F}| \leq \binom{n}{s}$.*

**Theorem 1.2** (Alon et al. [2]). *If $K$ and $L$ are two sets of nonnegative integers with $k_i > s - r$, for every $i$, then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

**Theorem 1.3** (Snevily [3]). *If $K$ and $L$ are any sets such that* $\min k_i > \max l_j$, *then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}$.

**Theorem 1.4** (Snevily [4]). *Let $K$ and $L$ be sets of nonnegative integers such that* $\min k_i > \max l_j$. *Then,* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2r+1}$.

**Conjecture 1.5** (Snevily [5]). *For any $K$ and $L$ with* $\min k_i > \max l_j$, $|\mathcal{F}| \le \binom{n}{s}$.

In the same paper in which he stated the above conjecture, Snevily mentions that it seems hard to prove the above bound and states the following weaker conjecture.

**Conjecture 1.6** (Snevily [5]). *For any $K$ and $L$ with* $\min k_i > \max l_j$, $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.

Hwang and Sheikh [6] proved the bound of Conjecture 1.6 when $K$ is a consecutive set. The second theorem we prove is a special case of Conjecture 1.6 with the extra condition that $\bigcap_{i=1}^{m} F_i \ne \emptyset$. These two theorems are stated hereunder.

**Theorem 1.7** (Hwang and Sheikh [6]). *Let $K = \{k_1, k_2, \ldots, k_r\}$ where $k_i = k_1 + i - 1$, $k_1 > s - r$, and $L = \{l_1, l_2, \ldots, l_s\}$. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be such that $|F_i| \in K$ for each $i$, $|F_i| \notin L$, and $|F_i \cap F_j| \in L$ for any $i \ne j$. Then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.

**Theorem 1.8** (Hwang and Sheikh [6]). *Let $K = \{k_1, k_2, \ldots, k_r\}$, $L = \{l_1, l_2, \ldots, l_s\}$, and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be such that $|F_i| \in K$ for each $i$, $|F_i \cap F_j| \in L$ for any $i \ne j$, and $k_i > s - r$. If $\bigcap_{i=1}^{m} F_i \ne \emptyset$, then* $|\mathcal{F}| \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-r}$.

**Theorem 1.9** (Alon et al. [2]). *Let $K$ and $L$ be subsets of $\{0, 1, ..., p-1\}$ such that $K \cap L = \emptyset$, where $p$ is a prime and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ a family of subsets of $[n]$ such that $|F_i| (\bmod\ p) \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| (\bmod\ p) \in L$ for $i \ne j$. If $r(s - r + 1) \le p - 1$, and $n \ge s + \max k_i$, then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

**Conjecture 1.10** (Alon et al. [2]). *Let $K$ and $L$ be subsets of $\{0, 1, \ldots, p-1\}$ such that $K \cap L = \emptyset$, where $p$ is a prime and $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ a family of subsets of $[n]$ such that $|F_i| (\bmod p) \in K$ for all $F_i \in \mathcal{F}$ and $|F_i \cap F_j| (\bmod\ p) \in L$ for $i \ne j$. If $n \ge s + \max k_i$, then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

In [2], Alon et al. proved their conjectured bound under the extra conditions that $r(s - r + 1) \le p - 1$ and $n \ge s + \max k_i$. Qian and Ray-Chaudhuri [7] proved that if $n > 2s - r$ instead of $n \ge s + \max k_i$, then the above bound holds.

We prove an Alon-Babai-Suzuki's conjecture in non-modular version.

**Theorem 1.11.** *Let $K = \{k_1, k_2, \ldots, k_r\}$, $L = \{l_1, l_2, \ldots, l_s\}$ be two sets of nonnegative integers and let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be such that $|F_i| \in K$ for each $i$, $|F_i \cap F_j| \in L$ for any $i \ne j$, and $n \ge s + \max_i |F_i|$. then* $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

## 2. Proof of Theorem

*Proof of Theorem 1.11.* For each $F_i \in \mathcal{F}$, consider the polynomial

$$f_i(x) = \prod_{\substack{j \\ l_j<|F_i|}} \left(v_i \cdot x - \left(k_i - l_j\right)\right), \tag{2.1}$$

where $v_i$ is the characteristic vector of $F_i$ and $v_i^*$ is the characteristic vector of $F_i^* = F_i - \{1\}$. Let $\overline{v_i}$ the characteristic vector of $F_i^c$, and $\overline{v_i}^*$ be the characteristic vector of $(F_i^c)^*$.

We order $\{F_i\}$ by size of $F_i$, that is, $|F_j| \leq |F_k|$ if $j < k$. We substitute the characteristic vector $\overline{v_i}$ of $F_i^c$ by order of size of $F_i$. Clearly, $f_i(\overline{v_i}) \neq 0$ for $1 \leq i \leq m$ and $f_i(\overline{v_j}) = 0$ for $1 \leq j < i \leq m$. Assume that

$$\sum_i \alpha_i f_i(x) = 0. \tag{2.2}$$

We prove that $\{f_i(x)\}$ is linearly independent. Assume that this is false. Let $i_0$ be the smallest index such that $\alpha_{i_0} \neq 0$. We substitute $\overline{v_{i_0}}$ into the above equation. Then we get $\alpha_{i_0} f_{i_0}(\overline{v_{i_0}}) = 0$. We get a contradiction. So $\{f_i(x)\}$ is linearly independent. Let $\mathcal{E} = \{E_1, \ldots, E_e\}$ be the family of subsets of $[n]$ with size at most $s - r$, which is ordered by size, that is, $|E_i| \leq |E_j|$ if $i < j$, where $e = \sum_{i=0}^{s-r} \binom{n}{i}$. Let $u_i$ denote the characteristic vector of $E_i$. We define the multilinear polynomial $g_i$ in $n$ variables for each $E_i$:

$$g_i(x) = \prod_{l=1}^{r} \left(\sum_{t=1}^{n} x_t - (n - k_l)\right) \prod_{j \in E_i} x_j. \tag{2.3}$$

We prove that $\{g_i(x)\}$ is linearly independent. Assume that

$$\sum_i \beta_i g_i(x) = 0. \tag{2.4}$$

Choose the smallest size of $E_i$. Let $u_i$ be the characteristic vector of $E_i$. We substitute $u_i$ into the above equation. We know that $g_i(u_i) \neq 0$ and $g_j(u_i) = 0$ for any $i < j$. Since $n \geq s + \max k_i$, we get $\beta_i = 0$. If we follow the same process, then the family $\{g_i(x)\}$ is linearly independent. Next, we prove that $\{f_i(x), g_i(x)\}$ is linearly independent. Now, assume that

$$\sum_i \alpha_i f_i(x) + \sum_i \beta_i g_i(x) = 0. \tag{2.5}$$

Let $F_1$ be the smallest size of $F_i$. We substitute the characteristic vector $\overline{v_1}$ of $F_1^c$ into the above equation. Since $|F_i^c| = n - k_l$, $g_i(\overline{v_1}) = 0$ for all $i$. We only get $\alpha_1 f_1(\overline{v_1}) = 0$. So $\alpha_1 = 0$. By the same way, choose the smallest size from $\{F_i\}$ after deleting $F_1$. We do the same process. We also can get $\alpha_2 = 0$. By the same process, we prove that all $\alpha_i = 0$. We prove that $\{f_i(x), g_i(x)\}$ is linearly independent.

Any polynomial in the set $\{f_i(x), g_i(x)\}$ can be represented by a linear combination of multilinear monomials of degree $\le s$. The space of such multilinear polynomials has dimension $\sum_{i=0}^{s}\binom{n}{i}$. We found $|\mathcal{F}| + \sum_{i=0}^{s-r}\binom{n}{i}$ linearly independent polynomials with degree at most $s$. So $|\mathcal{F}| + \sum_{i=0}^{s-r}\binom{n}{i} \le \sum_{i=0}^{s}\binom{n}{i}$. Thus $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$. □

## Acknowledgments

The authors thank Zoltán Füredi for encouragement to write this paper. The present research has been conducted by the research grant of the Kwangwoon University in 2009.

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