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Research Article

A Generalized Halanay Inequality for Stability of Nonlinear Neutral Functional Differential Equations

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This paper is devoted to generalize Halanay's inequality which plays an important rule in study of stability of differential equations. By applying the generalized Halanay inequality, the stability results of nonlinear neutral functional differential equations (NFDEs) and nonlinear neutral delay integrodifferential equations (NDIDEs) are obtained.

1. Introduction

In 1966, in order to discuss the stability of the zero solution of

$$u'(t) = -Au(t) + Bu(t - \tau^*), \quad \tau^* > 0, \tag{1.1}$$

Halanay used the inequality as follows.

Lemma 1.1 (Halanay's inequality, see [1]). *If*

$$v'(t) \leq -Av(t) + B \sup_{t-\tau \leq s \leq t} v(s), \quad \text{for } t \geq t_0, \tag{1.2}$$

where $A > B > 0$, then there exist $c > 0$ and $\kappa > 0$ such that

$$v(t) \leq ce^{-\kappa(t-t_0)}, \quad \text{for } t \geq t_0, \tag{1.3}$$

and hence $v(t) \to 0$ as $t \to \infty$.

In 1996, in order to investigate analytical and numerical stability of an equation of the type

$$u'(t) = f\left(t, u(t), u(\eta(t)), \int_{t-\tau(t)}^{t} K(t, s, u(s))ds\right), \quad t \geq t_0,$$
$$y(t) = \phi(t), \quad t \leq t_0, \ \phi \text{ bounded and continuous for } t \leq t_0, \tag{1.4}$$

Baker and Tang [2] give a generalization of Halanay inequality as Lemma 1.2 which can be used for discussing the stability of solutions of some general Volterra functional differential equations.

Lemma 1.2 (see [2]). *Suppose $v(t) > 0$, $t \in (-\infty, +\infty)$, and*

$$v'(t) \leq -A(t)v(t) + B(t) \sup_{t-\tau(t)\leq s\leq t} v(s) \quad (t \geq t_0), \qquad v(t) = |\psi(t)| \quad (t \leq t_0), \tag{1.5}$$

where $\psi(t)$ is bounded and continuous for $t \leq t_0$, $A(t)$, $B(t) > 0$ for $t \in [t_0, +\infty)$, $\tau(t) \geq 0$, and $t - \tau(t) \to +\infty$ as $t \to +\infty$. If there exists $p > 0$ such that

$$-A(t) + B(t) \leq -p < 0, \quad \text{for } t \geq t_0, \tag{1.6}$$

then

$$\text{(i) } v(t) \leq \sup_{t\in(-\infty,t_0]} |\psi(t)|, \quad \text{for } t \geq t_0,$$
$$\text{(ii) } v(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{1.7}$$

In recent years, the Halanay inequality has been extended to more general type and used for investigating the stability and dissipativity of various functional differential equations by several researchers (see, e.g., [3–7]). In this paper, we consider a more general inequality and use this inequality to discuss the stability of nonlinear neutral functional differential equations (NFDEs) and a class of nonlinear neutral delay integrodifferential equations (NDIDEs).

2. Generalized Halanay Inequality

In this section, we first give a generalization of Lemma 1.1.

Theorem 2.1 (generalized Halanay inequality). *Consider*

$$u'(t) \leq -A(t)u(t) + B(t) \max_{s\in[t-\tau,t]} u(s) + C(t) \max_{s\in[t-\tau,t]} w(s),$$
$$w(t) \leq G(t) \max_{s\in[t-\tau,t]} u(s) + H(t) \max_{s\in[t-\tau,t]} w(s), \qquad t \geq t_0, \tag{2.1}$$

where $A(t)$, $B(t)$, $C(t)$, $D(t)$, $G(t)$, and $H(t)$ are nonnegative continuous functions on $[t_0, \infty)$, and the notation $(')$ denotes the conventional derivative or the one-sided derivatives. Suppose that

$$A(t) \geq A_0 > 0, \quad H(t) \leq H_0 < 1, \quad \frac{B(t)}{A(t)} + \frac{C(t)G(t)}{(1-H(t))A(t)} \leq p < 1, \quad \forall t \geq t_0. \tag{2.2}$$

Then for any $\varepsilon > 0$, one has

$$u(t) < (1+\varepsilon)Ue^{\nu^*(t-t_0)}, \qquad w(t) < (1+\varepsilon)We^{\nu^*(t-t_0)}, \tag{2.3}$$

where $U = \max_{s\in[t_0-\tau,t_0]}u(s)$, $W = \max_{s\in[t_0-\tau,t_0]}w(s)$, and $\nu^* < 0$ is defined by the following procedure. Firstly, for every fixed t, let ν denote the maximal real root of the equation

$$\nu + A(t) - B(t)e^{-\nu\tau} - \frac{C(t)G(t)e^{-2\nu\tau}}{1-H(t)e^{-\nu\tau}} = 0. \tag{2.4}$$

Obviously, ν is different for different t, that is to say, ν is a function of t. Then we define ν^* as

$$\nu^* := \sup_{t\geq t_0}\{\nu(t)\}. \tag{2.5}$$

To prove the theorem, we need the following lemmas.

Lemma 2.2. *There exists nontrivial solution $\tilde{u}(t) = \tilde{U}e^{\nu_*(t-t_0)}$, $\tilde{w}(t) = \widetilde{W}e^{\nu_*(t-t_0)}$, $t \geq t_0$, $\nu_* \geq 0$, ($\tilde{U}$ and $\widetilde{W}$ are constants) to systems*

$$\begin{aligned} u'(t) &= -A(t)u(t) + B(t)u(t-\tau) + C(t)w(t-\tau), \\ w(t) &= G(t)u(t-\tau) + H(t)w(t-\tau), \end{aligned} \quad t \geq t_0 \tag{2.6}$$

if and only if for any fixed t characteristic equation (2.4) *has at least one nonnegative root ν.*

Proof. If systems (2.6) have nontrivial solution $\tilde{u}(t) = \tilde{U}e^{\nu_*(t-t_0)}$, $\tilde{w}(t) = \widetilde{W}e^{\nu_*(t-t_0)}$, then ν_* is obviously a nonnegative root of the characteristic equation (2.4). Conversely, if characteristic equation (2.4) has nonnegative root ν for any fixed t, then $\tilde{u}(t) = \tilde{U}e^{\nu_*(t-t_0)}$ and $\tilde{w}(t) = \widetilde{W}e^{\nu_*(t-t_0)}$, $\nu_* = \inf_{t\geq t_0}\{\nu(t)\} \geq 0$, are obviously a nontrivial solution of (2.6). □

Lemma 2.3. *If* (2.2) *holds, then*

(i) *for any fixed t, characteristic equation* (2.4) *does not have any nonnegative root but has a negative root ν;*

(ii) $\nu^* < 0$.

Proof. We consider the following two cases successively.

Case 1 ($\tau = 0$). Obviously, for any fixed t, the root of characteristic equation (2.4) is $\nu = -A(t) + B(t) + C(t)G(t)/(1 - H(t)) < 0$. Now we want to show that $\nu^* < 0$. Suppose this is not true. Take ϵ such that $0 < \epsilon < (1-p)A_0$. Then there exists $t^* \geq t_0$ such that $0 > \nu(t^*) > -\epsilon$. Using condition (2.2), we have

$$\begin{aligned} 0 &= \nu(t^*) + A(t^*) - B(t^*) - \frac{C(t^*)G(t^*)}{1 - H(t^*)} \\ &> -\epsilon + A(t^*) - pA(t^*) \\ &= -\epsilon + (1-p)A(t^*) \\ &\geq -\epsilon + (1-p)A_0 \\ &> 0, \end{aligned} \tag{2.7}$$

which is a contradiction, and therefore $\nu^* < 0$.

Case 2 ($\tau > 0$). In this case, obviously, for any fixed t, 0 is not a root of (2.4). If (2.4) has a positive root ν at a certain fixed t, then it follows from (2.2) and (2.4) that

$$B(t) + \frac{C(t)G(t)}{1 - H(t)} < B(t)e^{-\nu\tau} + \frac{C(t)G(t)e^{-2\nu\tau}}{1 - H(t)e^{-\nu\tau}}, \tag{2.8}$$

that is,

$$\frac{C(t)G(t)}{1 - H(t)} < \frac{C(t)G(t)e^{-2\nu\tau}}{1 - H(t)e^{-\nu\tau}}. \tag{2.9}$$

After simply calculating, we have $H(t) > 1$ which contradicts the assumption. Thus, (2.4) does not have any nonnegative root.

To prove that (2.4) has a negative root ν for any fixed t, we set $\nu_0 = \tau^{-1}\ln H(t)$ and define

$$\mathscr{H}(\nu) = \nu + A(t) - B(t)e^{-\nu\tau} - \frac{C(t)G(t)e^{-2\nu\tau}}{1 - H(t)e^{-\nu\tau}}. \tag{2.10}$$

Then it is easily obtained that

$$\mathscr{H}(0) > 0, \qquad \lim_{\nu \to \nu_0^+} \mathscr{H}(\nu) = -\infty. \tag{2.11}$$

On the other hand, when $\nu \in (\nu_0, 0]$, we have

$$\begin{aligned}\mathscr{H}'(\nu) &= 1 + B(t)\tau e^{-\nu\tau} + \frac{2C(t)G(t)\tau e^{-2\nu\tau}[1 - H(t)e^{-\nu\tau}]}{[1 - H(t)e^{-\nu\tau}]^2} \\ &\quad + \frac{C(t)G(t)e^{-2\nu\tau}H(t)\tau e^{-\nu\tau}}{[1 - H(t)e^{-\nu\tau}]^2} > 0,\end{aligned} \tag{2.12}$$

which implies that $\mathscr{H}(\nu)$ is a strictly monotone increasing function. Therefore, for any fixed t the characteristic equation (2.4) has a negative root $\nu \in (\nu_0, 0)$.

It remains to prove that $\nu^* < 0$. If it does not hold, we arbitrarily take $\tilde{p}$ such that $(1 - H_0)p + H_0 < \tilde{p} < 1$ and fix

$$0 < \epsilon < \min\Big\{(1 - \tilde{p})A_0, (2\tau)^{-1}\big[\ln\tilde{p} - \ln\big((1 - H_0)p + H_0\big)\big]\Big\}. \tag{2.13}$$

Then there exists $t^* \geq t_0$ such that $0 > \nu(t^*) > -\epsilon$. Since

$$\begin{aligned}e^{\epsilon\tau}H(t^*) &\leq H_0 e^{\epsilon\tau} \leq H_0\left[\frac{\tilde{p}}{(1 - H_0)p + H_0}\right]^{1/2} < 1, \\ \frac{1}{1 - H(t^*)e^{\epsilon\tau}} &\leq \frac{1 - H_0}{(1 - H_0 e^{\epsilon\tau})(1 - H(t^*))},\end{aligned} \tag{2.14}$$

we have

$$\begin{aligned}0 &= \nu(t^*) + A(t^*) - B(t^*)e^{-\nu(t^*)\tau} - \frac{C(t^*)G(t^*)e^{-2\nu(t^*)\tau}}{1 - H(t^*)e^{-\nu(t^*)\tau}} \\ &> -\epsilon + A(t^*) - B(t^*)e^{\epsilon\tau} - \frac{C(t^*)G(t^*)e^{2\epsilon\tau}}{1 - H(t^*)e^{\epsilon\tau}} \\ &> -\epsilon + A(t^*) - \frac{e^{2\epsilon\tau}(1 - H_0)}{1 - H_0 e^{\epsilon\tau}}\left[B(t^*) + \frac{C(t^*)G(t^*)}{1 - H(t^*)}\right] \\ &\geq -\epsilon + A(t^*) - \frac{e^{2\epsilon\tau}(1 - H_0)}{1 - H_0 e^{\epsilon\tau}}pA(t^*) \\ &> -\epsilon + A(t^*) - \tilde{p}A(t^*) \\ &= -\epsilon + (1 - \tilde{p})A(t^*) \\ &\geq -\epsilon + (1 - \tilde{p})A_0 \\ &> 0,\end{aligned} \tag{2.15}$$

which is a contradiction, and therefore $\nu^* < 0$. □

Lemma 2.4. *If* (2.6) *has a solution with exponential form* $\tilde{u}(t) = \widetilde{U}e^{\nu^*(t-t_0)}$, $\widetilde{w}(t) = \widetilde{W}e^{\nu^*(t-t_0)}$, $t \geq t_0$, $\nu^* < 0$, *then for any* $\varepsilon > 0$, *any nontrivial solution* $u(t)$, $w(t)$ *of* (2.1) *satisfies* (2.3).

Proof. The required result follows at once when $t \in [t_0 - \tau, t_0]$. If there exists t_* such that when $t < t_*$,

$$u(t) < (1+\varepsilon)Ue^{\nu^*(t-t_0)}, \qquad w(t) < (1+\varepsilon)We^{\nu^*(t-t_0)} \tag{2.16}$$

with $u(t_*) = (1+\varepsilon)Ue^{\nu^*(t_*-t_0)}$ or $w(t_*) = (1+\varepsilon)We^{\nu^*(t_*-t_0)}$, then for $t \leq t_*$, we can find that

$$\begin{aligned}
u(t) &\leq e^{-\int_{t_0}^{t} A(x)dx}u(t_0) + \int_{t_0}^{t} e^{-\int_{r}^{t} A(x)dx}\left[B(r)\max_{s\in[r-\tau,r]} u(s) + C(r)\max_{s\in[r-\tau,r]} w(s)\right]dr \\
&< e^{-\int_{t_0}^{t} A(x)dx}(1+\varepsilon)U + \int_{t_0}^{t} e^{-\int_{r}^{t} A(x)dx}\left[B(r)(1+\varepsilon)Ue^{\nu^*(r-\tau-t_0)} + C(r)(1+\varepsilon)We^{\nu^*(r-\tau-t_0)}\right]dr \\
&= \tilde{u}(t) = (1+\varepsilon)Ue^{\nu^*(t-t_0)}, \\
w(t) &< G(t)\max_{s\in[t-\tau,t]}(1+\varepsilon)Ue^{\nu^*(s-t_0)} + H(t)\max_{s\in[t-\tau,t]}(1+\varepsilon)We^{\nu^*(s-t_0)} \\
&= \widetilde{w}(t) = (1+\varepsilon)We^{\nu^*(t-t_0)},
\end{aligned} \tag{2.17}$$

a contradiction proving the lemma. □

Proof of Theorem 2.1. By Lemma 2.3, we can find that for any fixed t, characteristic equation (2.4) only has negative root and $\nu^* < 0$. Thus from Lemma 2.2 we know that systems (2.6) have not nontrivial solution with the form $\tilde{u}(t) = \widetilde{U}e^{\nu_*(t-t_0)}$, $\widetilde{w}(t) = \widetilde{W}e^{\nu_*(t-t_0)}$, $t \geq t_0$, $\nu_* \geq 0$. However, it is easily verified that systems (2.6) have nontrivial solution $\tilde{u}(t) = \widetilde{U}e^{\nu^*(t-t_0)}$, $\widetilde{w}(t) = \widetilde{W}e^{\nu^*(t-t_0)}$, $t \geq t_0$, $\nu^* < 0$. The result now follows from Lemma 2.4. □

Corollary 2.5. *If* (2.1) *and* (2.2) *hold, then*

$$\begin{aligned}
&\text{(i) } u(t) \leq \max_{s\in[t_0-\tau,t_0]} u(s), \qquad w(t) \leq \max_{s\in[t_0-\tau,t_0]} w(s); \\
&\text{(ii) } \lim_{t\to+\infty} u(t) = 0, \qquad \lim_{t\to+\infty} w(t) = 0.
\end{aligned} \tag{2.18}$$

Proof. (i) follows at once from the arbitrariness of ε. Since $\nu^* < 0$, (ii) is an immediate consequence of Theorem 2.1. □

Corollary 2.6 (see [3]). *Suppose that* $A = \inf_{t \geq t_0} A(t)$, $B = \sup_{t \geq t_0} B(t)$, $C = \sup_{t \geq t_0} C(t)$, $G = \sup_{t \geq t_0} G(t)$, *and* $H = \sup_{t \geq t_0} H(t)$. *Then when*

$$A > 0, \quad H < 1, \quad -A + B + \frac{CG}{1 - H} < 0, \tag{2.19}$$

equation (2.3) *holds for any* $\varepsilon > 0$, *where* $\nu^* < 0$ *is defined by*

$$\nu^* := \max\left\{ \nu : \mathscr{H}(\nu) = \nu + A - Be^{-\nu\tau} - \frac{CGe^{-2\nu\tau}}{1 - He^{-\nu\tau}} = 0 \right\}. \tag{2.20}$$

3. Applications of the Halanay Inequality

In this section, we consider several simple applications of Theorem 2.1 to the study of stability for nonlinear neutral functional differential equations (NFDEs) and nonlinear neutral delay-integrodifferential equations (NDIDEs).

3.1. Stability of Nonlinear NFDEs

Neutral functional differential equations (NFDEs) are frequently encountered in many fields of science and engineering, including communication network, manufacturing systems, biology, electrodynamics, number theory, and other areas (see, e.g., [8–11]). During the last two decades, the problem of stability of various neutral systems has been the subject of considerable research efforts. Many significant results have been reported in the literature. For the recent progress, the reader is referred to the work of Gu et al. [12] and Bellen and Zennaro [13]. However, these studies were devoted to the stability of linear systems and nonlinear systems with special form, and there exist few results available in the literature for general nonlinear NFDEs. Therefore, deriving some sufficient conditions for the stability of nonlinear NFDEs motivates the present study.

Let $\mathbf{X}$ be a real or complex Banach space with norm $\|\cdot\|$. For any given closed interval $[a, b] \subset \mathbf{R}$, let the symbol $C_{\mathbf{X}}[a, b]$ denote a Banach space consisting of all continuous mappings $x : [a, b] \to \mathbf{X}$, on which the norm is defined by $\|x\|_{[a,b]} = \max_{t \in [a,b]} \|x(t)\|$.

Our investigations will center on the stability of nonlinear NFDEs

$$\begin{gathered} \dot{y}(t) = f\left(t, y(t), y_t, \dot{y}_t\right), \quad t \geq t_0, \\ y_{t_0} = \phi, \qquad \dot{y}_{t_0} = \dot{\phi}, \end{gathered} \tag{3.1}$$

where the derivative $(\dot{\ })$ is the conventional derivative, $y_t(\theta) = y(t + \theta)$, $-\tau \leq \theta \leq 0$, $\tau \geq 0$ and t_0 are constants, $\phi : [t_0 - \tau, t_0] \to \mathbf{X}$ is a given continuously differentiable mapping, and $f : \mathbf{R} \times \mathbf{X} \times C_{\mathbf{X}}[-\tau, 0] \times C_{\mathbf{X}}[-\tau, 0] \to \mathbf{X}$ is a given continuous mapping and satisfies the following conditions:

$$\begin{aligned} &[1-\alpha(t)\lambda] G_f(0, t, y_1, y_2, \chi, \psi) \\ &\quad \leq G_f(\lambda, t, y_1, y_2, \chi, \psi), \quad \forall \lambda \geq 0,\ t \geq t_0,\ y_1, y_2 \in \mathbf{X},\ \chi, \psi \in C_{\mathbf{X}}[-\tau, 0], \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\| f(t, y_1, \chi_1, \psi_1) - f(t, y_2, \chi_2, \psi_2) \| \\ &\quad \leq L(t) \| y_1 - y_2 \| + \beta(t) \| \chi_1 - \chi_2 \|_{[t-\tau, t]} + \gamma(t) \| \psi_1 - \psi_2 \|_{[t-\tau, t]}, \\ &\qquad \forall t \geq t_0,\ y_1, y_2 \in \mathbf{X},\ \chi_1, \psi_1, \chi_2, \psi_2 \in C_{\mathbf{X}}[-\tau, 0], \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} G_f(\lambda, t, y_1, y_2, \chi, \psi) &:= \| y_1 - y_2 - \lambda [f(t, y_1, \chi, \psi) - f(t, y_2, \chi, \psi)] \|, \\ &\forall \lambda \in \mathbf{R},\ t \geq t_0,\ y_1, y_2 \in \mathbf{X},\ \chi, \psi \in C_{\mathbf{X}}[-\tau, 0], \end{aligned} \tag{3.4}$$

and throughout this paper, $\alpha(t)$, $L(t)$, $\beta(t)$ and $\gamma(t) < 1$, for all $t \geq t_0$, denote continuous functions. The existence of a unique solution on the interval $[t_0, \infty)$ of (3.1) will be assumed.

To study the stability of (3.1), we need to consider a perturbed problem

$$\begin{aligned} \dot{z}(t) &= f(t, z(t), z_t, \dot{z}_t), \quad t \geq t_0, \\ z_{t_0} &= \varphi, \qquad \dot{z}_{t_0} = \dot{\varphi}, \end{aligned} \tag{3.5}$$

where we assume the initial function $\varphi(t)$ is also a given continuously differentiable mapping, but it may be different from $\phi(t)$ in problem (3.1).

To prove our main results in this section, we need the following lemma.

Lemma 3.1 (cf. Li [14]). *If the abstract function $\omega(t) : \mathbf{R} \to \mathbf{X}$ has a left-hand derivative at point $t = t^*$, then the function $\|\omega(t)\|$ also has the left-hand derivative at point $t = t^*$, and the left-hand derivative is*

$$D_-(\|\omega(t^*)\|) = \lim_{\xi \to -0} \frac{\|\omega(t^*) + \xi \omega'(t^* - 0)\| - \|\omega(t^*)\|}{\xi}. \tag{3.6}$$

If $\omega(t)$ has a right-hand derivative at point $t = t^*$, then the function $\|\omega(t)\|$ also has the right-hand derivative at point $t = t^*$, and the right-hand derivative is

$$D_+(\|\omega(t^*)\|) = \lim_{\xi \to +0} \frac{\|\omega(t^*) + \xi \omega'(t^* + 0)\| - \|\omega(t^*)\|}{\xi}. \tag{3.7}$$

Theorem 3.2. *Let the continuous mapping f satisfy* (3.2) *and* (3.3). *Suppose that*

$$\alpha(t) \le \alpha_0 < 0, \quad \gamma(t) \le \gamma_0 < 1, \quad \frac{\gamma(t)L(t) + \beta(t)}{-[1-\gamma(t)]\alpha(t)} \le p < 1, \quad \forall t \ge t_0. \tag{3.8}$$

Then for any $\varepsilon > 0$, one have

$$\begin{aligned}\|y(t) - z(t)\| &< (1+\varepsilon) \max_{s\in[t_0-\tau,t_0]} \|\phi(s) - \varphi(s)\| e^{\nu^{\#}(t-t_0)},\\ \|\dot{y}(t) - \dot{z}(t)\| &< (1+\varepsilon) \max_{s\in[t_0-\tau,t_0]} \|\dot{\phi}(s) - \dot{\varphi}(s)\| e^{\nu^{\#}(t-t_0)},\end{aligned} \tag{3.9}$$

where $\nu^{\#} < 0$ is defined by the following procedure. Firstly, for every fixed t, let ν denote the maximal real root of the equation

$$\nu - \alpha(t) - \beta(t)e^{-\nu\tau} - \frac{\gamma(t)[L(t) + \beta(t)]e^{-2\nu\tau}}{1 - \gamma(t)e^{-\nu\tau}} = 0. \tag{3.10}$$

Since ν is a function of t, then one defines $\nu^{\#}$ as $\nu^{\#} := \sup_{t\ge t_0}\{\nu(t)\}$. Furthermore, one has

$$\begin{aligned}&\|y(t) - z(t)\| \le \max_{s\in[t_0-\tau,t_0]} \|\phi(s) - \varphi(s)\|, \qquad \|\dot{y}(t) - \dot{z}(t)\| \le \max_{s\in[t_0-\tau,t_0]} \|\dot{\phi}(s) - \dot{\varphi}(s)\|,\\ &\lim_{t\to+\infty} \|y(t) - z(t)\| = 0, \qquad \lim_{t\to+\infty} \|\dot{y}(t) - \dot{z}(t)\| = 0.\end{aligned} \tag{3.11}$$

Proof. Let us define $\Upsilon(t) = \|y(t) - z(t)\|$ and $\widetilde{\Upsilon}(t) = \|\dot{y}(t) - \dot{z}(t)\|$. By means of

$$\begin{aligned}\|y(t) - z(t) - \lambda[\dot{y}(t) - \dot{z}(t)]\| \ge{}& \|y(t) - z(t) - \lambda[f(t, y(t), y_t, \dot{y}_t) - f(t, z(t), y_t, \dot{y}_t)]\|\\ &- \lambda\Big[\beta(t)\|y - z\|_{[t-\tau,t]} + \gamma(t)\|\dot{y} - \dot{z}\|_{[t-\tau,t]}\Big], \quad \lambda \ge 0,\end{aligned} \tag{3.12}$$

from Lemma 3.1, we have

$$\begin{aligned}D_-(\Upsilon(t)) &= \lim_{\lambda\to+0} \frac{\|y(t) - z(t) - \lambda[\dot{y}(t) - \dot{z}(t)]\| - \|y(t) - z(t)\|}{-\lambda}\\ &\le \lim_{\lambda\to+0}\left[\frac{G_f(0) - G_f(\lambda)}{\lambda} + \beta(t)\|y - z\|_{[t-\tau,t]} + \gamma(t)\|\dot{y} - \dot{z}\|_{[t-\tau,t]}\right]\\ &\le \lim_{\lambda\to+0}\left\{\frac{[1 - (1-\alpha(t)\lambda)]G_f(0)}{\lambda} + \beta(t)\|y - z\|_{[t-\tau,t]} + \gamma(t)\|\dot{y} - \dot{z}\|_{[t-\tau,t]}\right\}\\ &= \alpha(t)\Upsilon(t) + \beta(t)\|y - z\|_{[t-\tau,t]} + \gamma(t)\|\dot{y} - \dot{z}\|_{[t-\tau,t]}.\end{aligned} \tag{3.13}$$

On the other hand, it is easily obtained from (3.3) that

$$\widetilde{Y}(t) \le L(t)Y(t) + \beta(t)\|y - z\|_{[t-\tau,t]} + \gamma(t)\|\dot{y} - \dot{z}\|_{[t-\tau,t]}, \quad t \ge t_0. \tag{3.14}$$

Thus, the application of Theorem 2.1 and Corollary 2.5 to (3.13) and (3.14) leads to Theorem 3.2. □

Remark 3.3. In Theorem 3.2, the derivative $(\dot{})$ can be understood as the right-hand derivative and the same results can be obtained. In fact, defining

$$M(\theta, t) := \frac{\partial}{\partial y} f(t, (1-\theta)z(t) + \theta y(t), y_t, \dot{y}_t), \quad \theta \in [0,1],\ t \ge t_0, \tag{3.15}$$

we have

$$\begin{aligned}
D_+(Y(t)) &= \lim_{\lambda \to +0} \frac{\|y(t) - z(t) + \lambda[\dot{y}(t) - \dot{z}(t)]\| - \|y(t) - z(t)\|}{\lambda} \\
&\le \lim_{\lambda \to +0} \frac{1}{\lambda}\left[\left\|\left(I + \lambda \int_0^1 M(\theta,t)d\theta\right)[y(t) - z(t)]\right\| - \|y(t) - z(t)\|\right] \\
&\quad + \beta(t)\|y - z\|_{[t-\tau,t]} + \gamma(t)\|\dot{y} - \dot{z}\|_{[t-\tau,t]} \\
&\le \lim_{\lambda \to +0} \frac{1}{\lambda}\left[\left\|I + \lambda \int_0^1 M(\theta,t)d\theta\right\| - 1\right]\|y(t) - z(t)\| \\
&\quad + \beta(t)\|y - z\|_{[t-\tau,t]} + \gamma(t)\|\dot{y} - \dot{z}\|_{[t-\tau,t]} \\
&\le \mu\left[\int_0^1 M(\theta,t)d\theta\right]Y(t) + \beta(t)\|y - z\|_{[t-\tau,t]} + \gamma(t)\|\dot{y} - \dot{z}\|_{[t-\tau,t]} \\
&\le \alpha(t)Y(t) + \beta(t)\|y - z\|_{[t-\tau,t]} + \gamma(t)\|\dot{y} - \dot{z}\|_{[t-\tau,t]},
\end{aligned} \tag{3.16}$$

where I denotes the identity matrix, and $\mu[\cdot]$ denotes the logarithmic norm induced by $\langle\cdot,\cdot\rangle$.

Remark 3.4. From (3.9), we know that $\|y(t) - z(t)\|$ and $\|\dot{y}(t) - \dot{z}(t)\|$ have an exponential asymptotic decay when the conditions of Theorem 3.2 are satisfied.

Not that for special case where $\mathbf{X}$ is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$, condition (3.2) is equivalent to a one-sided Lipschitz condition (cf. Li [14])

$$\begin{aligned}
&\operatorname{Re}\langle y_1 - y_2, f(t, y_1, \chi, \varphi) - f(t, y_2, \chi, \varphi)\rangle \\
&\quad \le \alpha(t)\|y_1 - y_2\|, \quad \forall t \ge t_0,\ y_1, y_2 \in \mathbf{X},\ \chi, \varphi \in C_{\mathbf{X}}[-\tau, 0].
\end{aligned} \tag{3.17}$$

Example 3.5. Consider neutral delay differential equations with maxima (see [15])

$$\begin{gathered}\dot{y}(t)=\hat{f}\left(t,y(t),y(\eta_0(t)),\max_{t-h\leq s\leq\eta_1(t)}y(s),\dot{y}(\zeta_0(t)),\max_{t-h\leq s\leq\zeta_1(t)}\dot{y}(s)\right),\qquad t\geq[0,T]\\ t-h\leq\eta_i(t),\ \zeta_i(t)\leq t,\ i=0,1,\\ y(t)=\phi(t),\qquad \dot{y}(t)=\dot{\phi}(t),\quad t\in[-\tau,0].\end{gathered}\tag{3.18}$$

Since it can be equivalently written in the pattern of IVP (3.1) in NFDEs, on the basis of Theorem 3.2, we can assert that the system is exponentially stable if the assumptions of Theorem 3.2 are satisfied.

Example 3.6. As a specific example, consider the following nonlinear system:

$$\begin{gathered}\dot{y}_1(t)=\cos t-2y_1(t)+0.4y_2(t)+0.1\sin y_2(\eta_1(t))+\sin t\int_{t-1}^{t}\frac{0.3\dot{y}_1(\theta)}{1+\dot{y}_1^2(\theta)}d\theta,\quad t\geq0,\\ \dot{y}_2(t)=\sin t+0.4y_1(t)-2y_2(t)-0.2\cos y_1(\eta_2(t))+\cos t\int_{t-1}^{t}\frac{0.3\dot{y}_2(\theta)}{1+\dot{y}_2^2(\theta)}d\theta,\quad t\geq0,\\ y_1(t)=\phi_1(t),\quad y_2(t)=\phi_2(t),\quad t\leq0,\end{gathered}\tag{3.19}$$

where there exists a constant τ such that $t-\tau\leq\eta_i(t)\leq t$ $(i=1,2)$. It is easy to verify that $\alpha(t)=-1.6$, $\beta(t)=0.2$, $\gamma(t)=0.3$, and $L(t)=2.4$. Then, according to Theorem 3.2 presented in this paper, we can assert that the system (3.19) is exponentially stable.

3.2. Asymptotic Stability of Nonlinear NDIDEs

Consider neutral Volterra delay-integrodifferential equations

$$\begin{gathered}\dot{y}(t)=\tilde{f}\left(t,y(t),y(t-\tau(t)),\dot{y}(t-\tau(t)),\int_{t-\tau(t)}^{t}K(t,\theta,y(\theta))d\theta\right),\quad t\geq t_0,\\ y(t)=\phi(t),\quad \dot{y}(t)=\dot{\phi}(t),\quad t\in[t_0-\tau,t_0].\end{gathered}\tag{3.20}$$

Since (3.20) is a special case of (3.1), we can directly obtain a sufficient condition for stability of (3.20).

Theorem 3.7. *Let the continuous mapping* $\tilde{f}$ *in* (3.20) *satisfy*

$$\begin{aligned}&[1-\alpha(t)\lambda]\widetilde{G}_{\tilde{f}}(0,t,y_1,y_2,u,v,w)\\ &\quad\leq\widetilde{G}_{\tilde{f}}(\lambda,t,y_1,y_2,u,v,w),\quad \forall\lambda\geq0,\ t\geq t_0,\ y_1,y_2,u,v,w\in\mathbf{X},\end{aligned}\tag{3.21}$$

$$\begin{aligned}&\left\|\tilde{f}(t,y_1,u_1,v_1,w_1)-\tilde{f}(t,y_2,u_2,v_2,w_2)\right\|\\&\quad\le L(t)\left\|y_1-y_2\right\|+\beta(t)\|u_1-u_2\|\\&\qquad+\gamma(t)\|v_1-v_2\|\\&\qquad+\mu(t)\|w_1-w_2\|,\quad \forall t\ge t_0,\ y_1,y_2,u_1,u_2,v_1,v_2,w_1,w_2\in\mathbf{X},\end{aligned}\tag{3.22}$$

$$\left\|K(t,\theta,y_1)-K(t,\theta,y_2)\right\|\le L_K(t)\left\|y_1-y_2\right\|,\quad (t,\theta)\in\mathbb{D},\ y_1,y_2\in\mathbf{X},\tag{3.23}$$

where $\mathbb{D}=\{(t,\theta):\ t\in[0,+\infty),\ \theta\in[-\tau,t]\}$,

$$\begin{aligned}\tilde{G}_{\tilde{f}}(\lambda,t,y_1,y_2,u,v,w):=\left\|y_1-y_2-\lambda\left[\tilde{f}(t,y_1,u,v,w)-\tilde{f}(t,y_2,u,v,w)\right]\right\|,\\ \forall\lambda\in\mathbf{R},\ t\ge t_0,\ y_1,y_2,u,v,w\in\mathbf{X}.\end{aligned}\tag{3.24}$$

Then if

$$\alpha(t)\le\alpha_0<0,\quad \gamma(t)\le\gamma_0<1,\quad \forall t\ge t_0,\tag{3.25}$$

$$\frac{\gamma(t)L(t)+\beta(t)+\tau\mu(t)L_K(t)}{-\left[1-\gamma(t)\right]\alpha(t)}\le p<1,\quad \forall t\ge t_0,\tag{3.26}$$

one has (3.9) *and* (3.11).

Our main objective in this subsection is to apply Corollary 2.5 to (3.20) and give another sufficient condition for the asymptotical stability of the solution to (3.20). We will assume that (3.21) and (3.23) are satisfied. We also assume that the continuous mapping $\tilde{f}$ in (3.20) satisfies

$$\begin{aligned}&\left\|\tilde{f}(t,y,u,v_1,w_1)-\tilde{f}(t,y,u,v_2,w_2)\right\|\\&\quad\le\gamma(t)\|v_1-v_2\|+\mu(t)\|w_1-w_2\|,\quad \forall t\ge t_0,\ y,u,v_1,v_2,w_1,w_2\in\mathbf{X},\\&\left\|\mathcal{F}(t,y,u_1,v,w,r,s)-\mathcal{F}(t,y,u_2,v,w,r,s)\right\|\\&\quad\le\sigma(t)\|u_1-u_2\|,\quad \forall t\ge t_0+\tau,\ y,u_1,u_2,v,w,r,s\in\mathbf{X},\end{aligned}\tag{3.27}$$

where $\mathcal{F}$ is defined as

$$\mathcal{F}(t,y,u,v,w,r,s):=\tilde{f}\Big(t,y,u,\tilde{f}(t-\tau(t),u,v,w,r),s\Big).\tag{3.28}$$

The mappings $\eta^{(\nu)}(t)$, $\nu = 1, 2, \ldots$, which are frequently used in that following analysis, are defined recursively by

$$\eta^{(1)}(t) = \eta(t) = t - \tau(t), \qquad \eta^{(2)}(t) = \eta\left(\eta^{(1)}(t)\right) = \eta(\eta(t)), \qquad \eta^{(\nu)}(t) = \eta\left(\eta^{(\nu-1)}(t)\right). \tag{3.29}$$

Theorem 3.8. *Let the continuous mapping $\tilde{f}$ in* (3.20) *satisfy* (3.21), (3.23), *and* (3.27). *Suppose that* (3.25) *and*

$$\frac{\sigma(t) + \tau\mu(t)L_K(t)}{-[1-\gamma(t)]\alpha(t)} \leq p < 1, \quad \forall t \geq t_0, \tag{3.30}$$

are satisfied. Then one has

$$\lim_{t \to +\infty} \|y(t) - z(t)\| = 0. \tag{3.31}$$

Furthermore, if $\tilde{f}$ satisfies

$$\left\|\tilde{f}(t, y_1, u, v, w) - \tilde{f}(t, y_2, u, v, w)\right\| \leq L\|y_1 - y_2\|, \quad \forall t \geq t_0,\ y_1, y_2, u, v, v, w, w \in \mathbf{X}, \tag{3.32}$$

where L is a constant, then one has

$$\lim_{t \to +\infty} \|\dot{y}(t) - \dot{z}(t)\| = 0. \tag{3.33}$$

Proof. Define

$$\begin{aligned} \Phi(t) = \Bigg\| \tilde{f}\left(t, z(t), y(\eta(t)), \dot{y}(\eta(t)), \int_{\eta(t)}^{t} K(t, s, y(s))ds\right) \\ -\tilde{f}\left(t, z(t), z(\eta(t)), \dot{z}(\eta(t)) \int_{\eta(t)}^{t} K(t, s, z(s))ds\right)\Bigg\|. \end{aligned} \tag{3.34}$$

Then it follows that

$$Y'(t) \leq \alpha(t)Y(t) + \Phi(t), \quad t \geq t_0. \tag{3.35}$$

It is easily obtained from (3.17) and (3.27) that

$$\begin{aligned}
\Phi(t) = \Bigg\| &\tilde{f}\left(t, z(t), y(\eta(t)), \tilde{f}\left(\eta(t), y(\eta(t)), y\left(\eta^{(2)}(t)\right), \dot{y}\left(\eta^{(2)}(t)\right), \int_{\eta^{(2)}(t)}^{\eta(t)} K(t,s,y(s))ds\right),\right. \\
&\left.\int_{\eta(t)}^{t} K(t,s,y(s))ds\right) \\
&- \tilde{f}\left(t, z(t), y(\eta(t)), \tilde{f}\left(\eta(t), y(\eta(t)), y\left(\eta^{(2)}(t)\right), \dot{y}\left(\eta^{(2)}(t)\right), \int_{\eta^{(2)}(t)}^{\eta(t)} K(t,s,y(s))ds\right),\right. \\
&\left.\int_{\eta(t)}^{t} K(t,s,y(s))ds\right)\Bigg\| \\
&\le \sigma(t)Y(\eta(t)) + \gamma(t)\Phi(\eta(t)) + \mu(t)\tau \max_{s\in[t-\tau,t]} \|K(t,s,y(s)) - K(t,s,z(s))\| \\
&\le \sigma(t)Y(\eta(t)) + \gamma(t)\Phi(\eta(t)) + \mu(t)\tau \max_{s\in[t-\tau,t]} L_K(t)Y(s) \\
&\le \gamma(t)\Phi(\eta(t)) + [\sigma(t) + \mu(t)\tau L_K(t)] \max_{s\in[t-\tau,t]} Y(s), \quad t \ge t_0 + \tau.
\end{aligned} \tag{3.36}$$

By virtue of Corollary 2.5, from (3.35)-(3.36) it is sufficient to prove (3.31) and

$$\lim_{t\to\infty} \Phi(t) = 0. \tag{3.37}$$

Since

$$\|\dot{y}(t) - \dot{z}(t)\| \le L\|y(t) - z(t)\| + \Phi(t), \quad t \ge t_0, \tag{3.38}$$

the last assertion follows. □

3.3. Comparison with the Existing Results

(i) In 2004, Wang and Li [16] were among the first who studied IVP in nonlinear NDDEs with a single delay $\tau(t)$ in a finite dimensional space $\mathbf{C}^n$, that is,

$$\begin{gathered}
\dot{z}(t) = f(t, y, y(t-\tau(t)), \dot{y}(t-\tau(t))), \quad t \ge t_0, \\
y(t) = \phi(t), \quad \dot{y}(t) = \dot{\phi}(t), \quad t \le t_0.
\end{gathered} \tag{3.39}$$

They obtained the asymptotic stability result (3.31) for the cases of (3.25), (3.26) and (3.25), and (3.30) under the following assumptions:

(a) there exists a constant $\tau_0 > 0$ such that

$$\tau(t) \ge \tau_0, \quad \forall t \ge t_0; \tag{3.40}$$

(b) $t - \tau(t)$ is a strictly increasing function on the interval $[t_0, +\infty)$;

(c) $\lim_{t\to+\infty}(t - \tau(t)) = +\infty$.

From Theorems 3.7 and 3.8 of the present paper, we can obtain the asymptotic stability results (3.31) for NDDEs (3.39), which do not require the above severe conditions (a) and (b) to be satisfied but require $0 \le \tau(t) \le \tau$.

(ii) In 2004, using a generalized Halanay inequality proved by Baker and Tang [2], Zhang and Vandewalle [17, 18] proved the contractility and asymptotic stability of solution to Volterra delay-integrodifferential equations with a constant delay

$$\dot{y}(t) = f\left(t, y(t), y(t-\tau), \int_{t-\tau}^{t} K(t, \theta, y(\theta))\, d\theta\right), \quad t \ge t_0,$$
$$y(t) = \phi(t), \quad t \in [t_0 - \tau, t_0], \tag{3.41}$$

in finite-dimensional space for the case of

$$\frac{\beta + \tau\mu L_K}{-\alpha} \le p < 1, \tag{3.42}$$

where $\alpha = \sup_{t\ge t_0}\alpha(t)$, $\beta = \sup_{t\ge t_0}\beta(t)$, $\mu = \sup_{t\ge t_0}\mu(t)$, and $L_K = \sup_{t\ge t_0}L_K(t)$.

Note that in this case, $\gamma(t) \equiv 0$, and condition (3.26) is equivalent to condition (3.30). Since Theorem 3.7 or Theorem 3.8 of the present paper can be applied to (3.41) with a variable delay $\tau(t)$, $0 \le \tau(t) \le \tau$, and (3.9), (3.11) can be obtained under condition (3.26), the results of these two theorems are more general and deeper than these obtained by Zhang and Vandewalle mentioned above.

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