Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2010, Article ID 393025, 18 pages
doi:10.1155/2010/393025

Research Article

A New Hilbert-Type Linear Operator with a Composite Kernel and Its Applications

Wuyi Zhong

Department of Mathematics, Guangdong Institute of Education, Guangzhou, Guangdong 510303, China

Correspondence should be addressed to Wuyi Zhong, zwy@gdei.edu.cn

Received 20 April 2010; Accepted 31 October 2010

Academic Editor: Ondřej Došlý

A new Hilbert-type linear operator with a composite kernel function is built. As the applications, two new more accurate operator inequalities and their equivalent forms are deduced. The constant factors in these inequalities are proved to be the best possible.

1. Introduction

In 1908, Weyl [1] published the well-known Hilbert's inequality as follows:

if $a_n, b_n \geq 0$ are real sequences, $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right)^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible.

Under the same conditions, there are the classical inequalities [2]

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \pi\left(\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right)^{1/2}, \tag{1.2}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(m/n)a_m b_n}{m-n} < \pi^2\left(\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right)^{1/2}, \tag{1.3}$$

where the constant factors π and π^2 are the best possible also. Expression (1.2) is called *a more accurate form of* (1.1). Some more accurate inequalities were considered by [3–5]. In 2009, Zhong [5] gave a more accurate form of (1.3).

Set (p, q), (s, r) as two pairs of conjugate exponents, and $p > 1$, $s > 1$, $\alpha \geq 1/2$, and a_n, $b_n \geq 0$, such that $0 < \sum_{n=0}^{\infty}(n+\alpha)^{p(1-\lambda/r)-1} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty}(n+\alpha)^{q(1-\lambda/s)-1} b_n^q < \infty$, then it has

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\alpha))a_m b_n}{(m+\alpha)^{\lambda}-(n+\alpha)^{\lambda}} < \left\{\sum_{n=0}^{\infty}(n+\alpha)^{p(1-\lambda/r)-1}a_n^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}(n+\alpha)^{q(1-\lambda/s)-1}b_n^q\right\}^{1/q}. \tag{1.4}$$

Letting $\phi(x) := (x+\alpha)^{p(1-\lambda/r)-1}$, $\varphi(x) := (x+\alpha)^{q(1-\lambda/s)-1}$, $\ell_{\phi}^{p} := \{a; a = \{a_n\}_{n=0}^{\infty}, \|a\|_{p,\phi} := \{\sum_{n=0}^{\infty}\phi(n)|a_n|^p\}^{1/p} < \infty\}$, $\ell_{\varphi}^{q} := \{b; b = \{b_n\}_{n=0}^{\infty}$, and $\|b\|_{q,\varphi} := \{\sum_{n=0}^{\infty}\varphi(n)|b_n|^q\}^{1/q} < \infty\}$, the expression (1.4) can be rewritten as

$$(Ta, b) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln(m+\alpha)/(n+\alpha)a_m b_n}{(m+\alpha)^{\lambda}-(n+\alpha)^{\lambda}} < k_{\lambda}(s)\|a\|_{p,\phi}\|b\|_{q,\varphi}, \tag{1.5}$$

where $T : \ell_{\phi}^{p} \to \ell_{\varphi}^{p}$ is a linear operator, $k_{\lambda}(s) = \|T\|$. $\|a\|_{p,\phi}$ is the norm of the sequence a with a weight function ϕ. (Ta, b) is a formal inner product of the sequences $Ta(n)(:= \sum_{m=0}^{\infty}(\ln(m+\alpha)/(n+\alpha)a_m)/((m+\alpha)^{\lambda}-(n+\alpha)^{\lambda}))$ and b.

By setting two monotonic increasing functions $u(x)$ and $v(x)$, a new Hilbert-type inequality, which is with a composite kernel function $K(u(x), v(y))$, and its equivalent are built in this paper. As the applications, two new more accurate Hilbert-type inequalities incorporating the linear operator and the norm are deduced.

Firstly, the improved Euler-Maclaurin's summation formula [6] is introduced.

Set $f \in C^4[m, \infty)(m \in N_0)$. If $(-1)^i f^{(i)}(x) > 0$, $f^{(i)}(\infty) = 0$ $(i = 0, 1, 2, 3, 4)$, then it has

$$\sum_{n=m}^{\infty} f(n) < \int_m^{\infty} f(x)dx + \frac{1}{2}f(m) - \frac{1}{12}f'(m),$$
$$\sum_{n=m}^{\infty} f(n) > \int_m^{\infty} f(x)dx + \frac{1}{2}f(m). \tag{1.6}$$

2. Lemmas

Lemma 2.1. *Set (r, s) as a pair of conjugate exponents, $s > 1$, $\beta \geq \alpha \geq e^{7/12}$, $0 < \lambda \leq 1$, and define*

$$f(y) := \frac{1}{ln^{\lambda}\alpha m + ln^{\lambda}\beta y}\frac{ln^{\lambda/r}\alpha m}{(ln^{1-\lambda/s}\beta y)y}, \quad y \in [1, \infty),\ m \in N, \tag{2.1}$$

$$g(y) := \frac{1}{ln^{\lambda}\alpha y + ln^{\lambda}\beta n}\frac{ln^{\lambda/s}\beta n}{(ln^{1-\lambda/r}\alpha y)y}, \quad y \in [1, \infty),\ n \in N, \tag{2.2}$$

$$R_\lambda(m,s) := \int_0^{\ln\beta/\ln\alpha m} \frac{u^{\lambda/s-1}}{1+u^\lambda}du - \frac{1}{2}f(1) + \frac{1}{12}f'(1), \tag{2.3}$$

$$\widetilde{R}_\lambda(n,r) := \int_0^{\ln\alpha/\ln\beta n} \frac{u^{\lambda/r-1}}{1+u^\lambda}du - \frac{1}{2}g(1) + \frac{1}{12}g'(1), \tag{2.4}$$

$$\eta_\lambda(m,s) := \int_0^{\ln\beta/\ln\alpha m} \frac{u^{\lambda/s-1}}{1+u^\lambda}du - \frac{1}{2}f(1). \tag{2.5}$$

Then, it has the following.

(1) *The functions* $f(y)$, $g(y)$ *satisfy the conditions of* (1.6). *It means that*

$$\begin{gathered}(-1)^i F^{(i)}(y) > 0 \quad (F = f, g,\ y \in [1,\infty)),\\ F^{(i)}(\infty) = 0 \quad (F = f, g,\ i = 0,1,2,3,4),\end{gathered} \tag{2.6}$$

(2)

$$R_\lambda(m,s) > 0, \quad \widetilde{R}_\lambda(n,r) > 0, \tag{2.7}$$

(3)

$$0 < \eta_\lambda(m,s) = O\left(\left[\frac{1}{\ln\alpha m}\right]^\rho\right) \quad (\rho > 0, m \longrightarrow \infty). \tag{2.8}$$

Proof. (1) For $\beta \geq \alpha \geq e^{7/12}$, $y \geq 1$, $m \in N$, $0 < \lambda \leq 1$, and $s > 1$, set

$$f_1(y) := \frac{1}{\ln^\lambda\alpha m + \ln^\lambda\beta y}, \qquad f_2(y) := \frac{1}{\ln^{1-\lambda/s}\beta y}, \qquad f_3(y) := \frac{1}{y}. \tag{2.9}$$

It has

$$f(y) = \ln^{\lambda/r}\alpha m f_1(y) f_2(y) f_3(y) \tag{2.10}$$

when $y \geq 1$. It is easy to find that

$$\begin{gathered}(-1)^i f_j^{(i)}(y) > 0, \quad f_j^{(i)}(\infty) = 0 \quad (y \in [1,\infty),\ j = 1,2,3,\ i = 0,1,2,3,4),\\ (-1)^i f^{(i)}(y) > 0, \quad f^{(i)}(\infty) = 0 \quad (y \in [1,\infty),\ i = 0,1,2,3,4).\end{gathered} \tag{2.11}$$

Similarly, it can be shown that $(-1)^i g^{(i)}(y) > 0$, $g^{(i)}(\infty) = 0$ $(y \geq 1, i = 0, 1, 2, 3, 4)$. These tell us that (2.6) holds and the functions $f(y)$, $g(y)$ satisfy the conditions of (1.6).

(2) Set $t = u^{\lambda}$. With the partial integration, it has

$$\begin{aligned}\int_0^{\ln\beta/\ln\alpha m} \frac{u^{\lambda/s-1}}{1+u^{\lambda}} du &= \frac{1}{\lambda}\int_0^{(\ln\beta/\ln\alpha m)^{\lambda}} \frac{t^{1/s-1}}{1+t} dt = \frac{s}{\lambda}\int_0^{(\ln\beta/\ln\alpha m)^{\lambda}} \frac{dt^{1/s}}{1+t} \\ &= \frac{s}{\lambda}\frac{(\ln\beta/\ln\alpha m)^{\lambda/s}}{1+(\ln\beta/\ln\alpha m)^{\lambda}} + \frac{s}{\lambda}\int_0^{(\ln\beta/\ln\alpha m)^{\lambda}} \frac{t^{1/s}}{(1+t)^2} dt \\ &= \frac{s}{\lambda}\frac{\ln^{\lambda}\beta}{\ln^{\lambda}\alpha m + \ln^{\lambda}\beta}\left(\frac{\ln\alpha m}{\ln\beta}\right)^{\lambda/r} + \frac{s^2}{\lambda(1+s)}\int_0^{(\ln\beta/\ln\alpha m)^{\lambda}} \frac{dt^{1/s+1}}{(1+t)^2} \\ &\geq \frac{s}{\lambda}\frac{\ln^{\lambda}\beta}{\ln^{\lambda}\alpha m + \ln^{\lambda}\beta}\left(\frac{\ln\alpha m}{\ln\beta}\right)^{\lambda/r} + \frac{s^2}{\lambda(1+s)}\frac{\ln^{2\lambda}\beta}{(\ln^{\lambda}\alpha m + \ln^{\lambda}\beta)^2}\left(\frac{\ln\alpha m}{\ln\beta}\right)^{\lambda/r}.\end{aligned} \tag{2.12}$$

By (2.1), it has

$$f(1) = \left(\frac{\ln\alpha m}{\ln\beta}\right)^{\lambda/r}\frac{\ln^{\lambda-1}\beta}{\ln^{\lambda}\alpha m + \ln^{\lambda}\beta}, \tag{2.13}$$

$$f'(1) = \left(\frac{\ln\alpha m}{\ln\beta}\right)^{\lambda/r}\left[-\frac{\lambda\ln^{2\lambda-2}\beta}{(\ln^{\lambda}\alpha m + \ln^{\lambda}\beta)^2} - \frac{(1-\lambda/s)\ln^{\lambda-2}\beta}{\ln^{\lambda}\alpha m + \ln^{\lambda}\beta} - \frac{\ln^{\lambda-1}\beta}{\ln^{\lambda}\alpha m + \ln^{\lambda}\beta}\right]. \tag{2.14}$$

In view of (2.12)~(2.14), it has

$$\begin{aligned}R_{\lambda}(m,s) \geq \left(\frac{\ln\alpha m}{\ln\beta}\right)^{\lambda/r}&\left\{\frac{\ln^{\lambda-1}\beta}{\ln^{\lambda}\alpha m + \ln^{\lambda}\beta}\left[-\frac{1-\lambda/s}{12\ln\beta} - \frac{7}{12} + \frac{s}{\lambda}\ln\beta\right]\right. \\ &\left.+\frac{\ln^{2\lambda}\beta}{(\ln^{\lambda}\alpha m + \ln^{\lambda}\beta)^2}\left[\frac{s^2}{\lambda(1+s)} - \frac{\lambda}{12\ln^2\beta}\right]\right\}.\end{aligned} \tag{2.15}$$

As $\ln\beta \geq 7/12$, $s > 1 (r > 1)$, and $0 < \lambda \leq 1$, it has

$$\begin{aligned}-\frac{1-\lambda/s}{12\ln\beta} - \frac{7}{12} + \frac{s}{\lambda}\ln\beta &\geq \frac{7s}{12\lambda}\left(1-\frac{\lambda}{s}\right) - \frac{1}{12\ln\beta}\left(1-\frac{\lambda}{s}\right) \\ &= \left(1-\frac{\lambda}{s}\right)\left(\frac{7s}{12\lambda} - \frac{1}{12\ln\beta}\right) > \left(1-\frac{\lambda}{s}\right)\left(\frac{7}{12} - \frac{1}{12\ln\beta}\right) > 0, \\ \frac{s^2}{\lambda(1+s)} - \frac{\lambda}{12\ln^2\beta} &> \frac{s}{4} - \frac{s}{12\ln^2\beta} = \frac{s}{4}\left(1 - \frac{1}{3\ln^2\beta}\right) > 0.\end{aligned} \tag{2.16}$$

It means that $R_{\lambda}(m,s) > 0$. Similarly, it can be shown that $\widetilde{R}_{\lambda}(n,r) > 0$. The expression (2.7) holds.

(3) By (2.5), (2.12), (2.13), and $0 < \lambda < s$, $\beta \geq e^{7/12}$, it has

$$\eta_\lambda(m,s) \geq \frac{s}{\lambda}\frac{\ln^\lambda \beta}{\ln^\lambda \alpha m + \ln^\lambda \beta}\left(\frac{\ln \alpha m}{\ln \beta}\right)^{\lambda/r} - \frac{1}{2}\frac{\ln^{\lambda-1}\beta}{\ln^\lambda \alpha m + \ln^\lambda \beta}\left(\frac{\ln \alpha m}{\ln \beta}\right)^{\lambda/r}$$
$$= \left(\frac{\ln \alpha m}{\ln \beta}\right)^{\lambda/r}\frac{\ln^{\lambda-1}\beta}{\ln^\lambda \alpha m+\ln^\lambda \beta}\left(\frac{s\ln\beta}{\lambda} - \frac{1}{2}\right) > \left(\frac{\ln \alpha m}{\ln \beta}\right)^{\lambda/r}\frac{\ln^{\lambda-1}\beta}{\ln^\lambda \alpha m + \ln^\lambda \beta}\left(\ln\beta - \frac{1}{2}\right) > 0,$$
$$\eta_\lambda(m,s) < \frac{1}{\lambda}\int_0^{(\ln\beta/\ln\alpha m)^\lambda} u^{1/s-1}du = \frac{s}{\lambda}\left(\frac{\ln\beta}{\ln\alpha m}\right)^{\lambda/s}. \tag{2.17}$$

The expression (2.8) holds, and Lemma 2.1 is proved. □

Lemma 2.2. *Set (r,s) as a pair of conjugate exponents, $s > 1$, $\beta \geq \alpha \geq 1/2$, and $0 < \lambda \leq 1$, and define*

$$f_1(y) := \frac{1}{\lambda(m+\alpha)}\frac{\ln((y+\beta)/(m+\alpha))^\lambda}{((y+\beta)/(m+\alpha))^\lambda - 1}\left(\frac{y+\beta}{m+\alpha}\right)^{\lambda/s-1}, \quad y \in [0,\infty),\ m \in N_0,$$
$$g_1(y) := \frac{1}{\lambda(n+\beta)}\frac{\ln((y+\alpha)/(n+\beta))^\lambda}{((y+\alpha)/(n+\beta))^\lambda - 1}\left(\frac{y+\alpha}{n+\beta}\right)^{\lambda/r-1}, \quad y \in [0,\infty),\ n \in N_0,$$
$$R_\lambda(m,s) := \frac{1}{\lambda^2}\int_0^{(\beta/m+\alpha)^\lambda}\frac{\ln u}{u-1}u^{1/s-1}du - \frac{1}{2}f_1(0) + \frac{1}{12}f_1'(0), \tag{2.18}$$
$$\widetilde{R}_\lambda(n,r) := \frac{1}{\lambda^2}\int_0^{(\alpha/n+\beta)^\lambda}\frac{\ln u}{u-1}u^{1/r-1}du - \frac{1}{2}g_1(0) + \frac{1}{12}g_1'(0),$$
$$\eta_\lambda(m,s) := \frac{1}{\lambda^2}\int_0^{(\beta/m+\alpha)^\lambda}\frac{\ln u}{u-1}u^{1/s-1}du - \frac{1}{2}f_1(0).$$

Then, it has

(1) *The functions $f_1(y)$, $g_1(y)$ satisfy the conditions of* (1.6). *It means that*

$$(-1)^i F^{(i)}(y) > 0 \quad (F = f_1, g_1,\ y \in [0,\infty)),$$
$$F^{(i)}(\infty) = 0 \quad (F = f_1, g_1,\ i = 0,1,2,3,4), \tag{2.19}$$

(2)

$$R_\lambda(m,s) > 0, \qquad \widetilde{R}_\lambda(n,r) > 0, \tag{2.20}$$

(3)

$$0 < \eta_\lambda(m,s) = O\left(\left[\frac{1}{m+\alpha}\right]^\rho\right) \quad (\rho > 0, m \longrightarrow \infty). \tag{2.21}$$

Proof. (1) Letting $h(u) := \ln u/(u-1)$, $u = ((y+\beta)/(m+\alpha))^\lambda$, it can be proved that $f_1(y) = (1/\lambda(m+\alpha))h(u)u^{1/s-1/\lambda}$ satisfy (2.19) as in [5]. Similarly, it can be shown that $g_1(y)$ satisfy (2.19) also.

(2) Setting $u_0 := (\beta/(m+\alpha))^\lambda$, by $u' = \lambda((y+\beta)/(m+\alpha))^{\lambda-1}(1/(m+\alpha)) = (\lambda/(y+\beta))((y+\beta)/(m+\alpha))^\lambda$,$u'(0) = (\lambda/\beta)u_0$, and $h''(u) > 0$, it has

$$\begin{aligned}
\frac{1}{\lambda^2}\int_0^{(\beta/(m+\alpha))^\lambda} \frac{\ln u}{u-1} u^{1/s-1} du &= \frac{1}{\lambda^2}\int_0^{u_0} h(u)u^{1/s-1}du \\
&= \frac{s}{\lambda^2}\int_0^{u_0} h(u)du^{1/s} = \frac{s}{\lambda^2}\left[h(u_0)u_0^{1/s} - \int_0^{u_0} u^{1/s}h'(u)du\right] \\
&\geq \frac{s}{\lambda^2}\left[h(u_0)u_0^{1/s} - h'(u_0)\int_0^{u_0} u^{1/s}du\right] \\
&= \frac{s}{\lambda^2}\left[h(u_0)u_0^{1/s} - \frac{s}{s+1}h'(u_0)u_0^{1/s+1}\right],
\end{aligned} \tag{2.22}$$

$$f_1(0) = \frac{1}{\lambda(m+\alpha)}\frac{\ln(\beta/(m+\alpha))^\lambda}{(\beta/(m+\alpha))^\lambda - 1}\left(\frac{\beta}{m+\alpha}\right)^{\lambda/s-1} = \frac{1}{\lambda\beta}h(u_0)u_0^{1/s}, \tag{2.23}$$

$$f_1'(0) = \frac{1}{\beta^2}u_0^{1/s}\left[h'(u_0)u_0 + \left(\frac{1}{s} - \frac{1}{\lambda}\right)h(u_0)\right]. \tag{2.24}$$

With (2.22)~(2.24), it has

$$R_\lambda(m,s) \geq h(u_0)u_0^{1/s}\left[\frac{s}{\lambda^2} - \frac{1}{2\lambda\beta} + \frac{1}{12\beta^2}\left(\frac{1}{s} - \frac{1}{\lambda}\right)\right] - h'(u_0)u_0^{1/s+1}\left[\frac{s}{1+s} - \frac{1}{12\beta^2}\right]. \tag{2.25}$$

By $h(u_0) > 0$, $h'(u_0) < 0$, and $\beta \geq 1/2$, $s > 1$, $0 < \lambda \leq 1$, it has

$$\begin{aligned}
&\frac{s}{\lambda^2} - \frac{1}{2\lambda\beta} + \frac{1}{12\beta^2}\left(\frac{1}{s} - \frac{1}{\lambda}\right) = \frac{6\beta s(2\beta s - \lambda) - \lambda(s-\lambda)}{12\beta^2 s\lambda^2} > 0, \\
&\frac{s}{1+s} - \frac{1}{12\beta^2} = \frac{12\beta^2 s - (1+s)}{12\beta^2(1+s)} = \frac{(4\beta^2 s - s) + (8\beta^2 s - 1)}{12\beta\lambda(1+s)} > 0.
\end{aligned} \tag{2.26}$$

So $R_\lambda(m,s) > 0$ holds. Similarly, it can be shown that $\widetilde{R}_\lambda(n,r) > 0$.

(3) In view of (2.22), (2.23), by $h(u) > 0$, $h'(u) < 0$, it has

$$\eta_\lambda(m,s) > h(u_0)u_0^{1/s}\left[\frac{s}{\lambda^2} - \frac{1}{2\lambda\beta}\right] = h(u_0)u_0^{1/s}\frac{2\beta s - \lambda}{2\lambda^2\beta} > 0, \tag{2.27}$$

and by $\lim_{u\to 0^+}(\ln u/(u-1))u^{1/2s} = 0$, so there exists a constant $L > 0$, such that $|(\ln u/(u-1))u^{1/2s}| < L$ $(u \in (0,(\beta/(m+\alpha))^\lambda))$. Then it has

$$\eta_\lambda(m,s) < \frac{1}{\lambda^2}\int_0^{(\beta/(m+\alpha))^\lambda} \frac{\ln u}{u-1}u^{1/s-1}du < \frac{L}{\lambda^2}\int_0^{(\beta/(m+\alpha))^\lambda} u^{1/2s-1}du = \frac{2sL}{\lambda^2}\left(\frac{\beta}{m+\alpha}\right)^{\lambda/2s}. \tag{2.28}$$

It means that (2.21) holds. The proof for Lemma 2.2 is finished. □

3. Main Results

Set $\lambda \in R$, $p > 1$, $r > 1$, (p,q), and (r,s) as two pairs of conjugate exponents. $K(x,y) \geq 0((x,y) \in (0,\infty)\times(0,\infty))$ is a measurable kernel function. Both $u(x)$ and $v(x)$ are strictly monotonic increasing differentiable functions in $[n_0,\infty)$ such that $U(n_0) > 0$, $U(\infty) = \infty (U = u, v)$. Give some notations as follows:

(1)

$$\begin{aligned}\phi(x) &:= [u(x)]^{p(1-\lambda/r)-1}\left[u'(x)\right]^{1-p},\\ \varphi(x) &:= [v(x)]^{q(1-\lambda/s)-1}\left[v'(x)\right]^{1-q},\\ \psi(x) &:= \left[\varphi(x)\right]^{1-p} = [v(x)]^{p\lambda/s-1}v'(x) \quad (x \in [n_0,\infty)),\end{aligned} \tag{3.1}$$

(2) set

$$\ell_\phi^p := \left\{a; a = \{a_n\}_{n=n_0}^\infty, \|a\|_{p,\phi} := \left\{\sum_{n=n_0}^\infty \phi(n)|a_n|^p\right\}^{1/p} < \infty\right\}, \tag{3.2}$$

and call ℓ_ϕ^p *a real space of sequences*, where

$$\|a\|_{p,\phi} = \left\{\sum_{n=n_0}^\infty \phi(n)|a_n|^p\right\}^{1/p} \tag{3.3}$$

is called *the norm of the sequence with a weight function* ϕ. Similarly, *the real spaces of sequences* ℓ_φ^q, ℓ_ψ^p and *the norm* $\|b\|_{q,\varphi}$ can be defined as well,

(3) define *a Hilbert-type linear operator* $T : \ell^p_\phi \to \ell^p_\psi$, for all $a \in \ell^p_\phi$,

$$(Ta)(n) := C_n := \sum_{m=n_0}^{\infty} K(u(m), v(n)) a_m \quad (n \geq n_0), \tag{3.4}$$

(4) for all $a \in \ell^p_\phi$, $b \in \ell^q_\varphi$, define *the formal inner product of* Ta *and* b as

$$(Ta, b) := \sum_{n=n_0}^{\infty} \left(\sum_{m=n_0}^{\infty} K(u(m), v(n)) a_m \right) b_n = \sum_{n=n_0}^{\infty} \sum_{m=n_0}^{\infty} K(u(m), v(n)) a_m b_n, \tag{3.5}$$

(5) define two *weight coefficients* $\omega(m, s)$ and $\vartheta(n, r)$ as

$$\begin{gathered} \omega_\lambda(m, s) := \sum_{n=n_0}^{\infty} K(u(m), v(n)) \frac{[u(m)]^{\lambda/r}}{[v(n)]^{1-\lambda/s}} v'(n), \\ \vartheta_\lambda(n, r) := \sum_{m=n_0}^{\infty} K(u(m), v(n)) \frac{[v(n)]^{\lambda/s}}{[u(m)]^{1-\lambda/r}} u'(m), \quad m, n \geq n_0. \end{gathered} \tag{3.6}$$

Then it has some results in the following theorems.

Theorem 3.1. *Suppose that* $a_n \geq 0$, $U'(x)/U(x) > 0 (U = u, v)$, *and* $0 < \sum_{n=n_0}^{\infty} (v'(n)/[v(n)]^{1+\varepsilon}) \leq \sum_{n=n_0}^{\infty} (u'(n)/[u(n)]^{1+\varepsilon}) < \infty (\varepsilon > 0)$. *If there exists a positive number* k_λ, *such that*

$$0 < \omega_\lambda(m, s) < k_\lambda, \quad 0 < \vartheta_\lambda(n, r) < k_\lambda \quad (m, n \geq n_0), \tag{3.7}$$

$$k_\lambda \left(1 - O\left(\frac{1}{[u(m)]^\rho} \right) \right) \leq \omega_\lambda(m, s) \quad (\rho > 0, m \longrightarrow \infty), \tag{3.8}$$

then for all $a \in \ell^p_\phi$ *and* $\|a\|_{p,\phi} > 0$, *it has the following:*

(1)

$$Ta = C = \{C_n\}_{n=n_0}^{\infty} \in \ell^p_\psi, \tag{3.9}$$

It means that $T : \ell^p_\phi \to \ell^p_\psi$,

(2) *T is a bounded linear operator and*

$$\|T\|_{p,\psi} := \sup_{a\in \ell^p_\phi (a\neq\theta)} \frac{\|Ta\|_{p,\psi}}{\|a\|_{p,\phi}} = k_\lambda, \tag{3.10}$$

where C_n*,* T *are defined by* (3.4)*,* $\|Ta\|_{p,\psi} = \|C\|_{p,\psi}$ *is defined as* (3.3).

Proof. By using Hölder's inequality [7] and (3.6), (3.7), it has $C_n \geq 0$ and

$$\begin{aligned}
C_n^p &= \left\{ \sum_{m=n_0}^{\infty} K(u(m), v(n)) \left[\frac{[u(m)]^{(1-\lambda/r)/q} [v'(n)]^{1/p}}{[v(n)]^{(1-\lambda/s)/p} [u'(m)]^{1/q}} a_m \right] \left[\frac{[v(n)]^{(1-\lambda/s)/p} [u'(m)]^{1/q}}{[u(m)]^{(1-\lambda/r)/q} [v'(n)]^{1/p}} \right] \right\}^p \\
&\leq \left\{ \sum_{m=n_0}^{\infty} K(u(m), v(n)) \frac{[u(m)]^{(p-1)(1-\lambda/r)} v'(n)}{[v(n)]^{1-\lambda/s} [u'(m)]^{p-1}} a_m^p \right\} \\
&\quad \times \left\{ \sum_{m=n_0}^{\infty} K(u(m), v(n)) \frac{[v(n)]^{(q-1)(1-\lambda/s)} u'(m)}{[u(m)]^{1-\lambda/r} [v'(n)]^{q-1}} \right\}^{p-1} \\
&= \left\{ \sum_{m=n_0}^{\infty} K(u(m), v(n)) \frac{[u(m)]^{(p-1)(1-\lambda/r)} v'(n)}{[v(n)]^{1-\lambda/s} [u'(m)]^{p-1}} a_m^p \right\} \{\vartheta_\lambda(n, r)\varphi(n)\}^{p-1} \\
&< k_\lambda^{p-1} \left\{ \sum_{m=n_0}^{\infty} K(u(m), v(n)) \frac{[u(m)]^{(p-1)(1-\lambda/r)} v'(n)}{[v(n)]^{1-\lambda/s} [u'(m)]^{p-1}} a_m^p \right\} \left\{ \varphi^{p-1}(n) \right\}.
\end{aligned} \tag{3.11}$$

And by $\psi(n) = \varphi^{1-p}(n)$, it follows that

$$\begin{aligned}
\|Ta\|_{p,\psi}^p &= \sum_{n=n_0}^{\infty} \psi(n) C_n^p < k_\lambda^{p-1} \left\{ \sum_{m=n_0}^{\infty} \sum_{n=n_0}^{\infty} K(u(m), v(n)) \frac{[u(m)]^{(p-1)(1-\lambda/r)} v'(n)}{[v(n)]^{1-\lambda/s} [u'(m)]^{p-1}} a_m^p \right\} \\
&= k_\lambda^{p-1} \sum_{m=n_0}^{\infty} \omega_\lambda(m, s) \phi(m) a_m^p < k_\lambda^p \|a\|_{p,\phi}^p < \infty.
\end{aligned} \tag{3.12}$$

This means that $C = \{C_n\}_{n=0}^{\infty} \in \ell^p_\psi$, $\|Ta\|_{p,\psi} \leq k_\lambda \|a\|_{p,\phi}$, and $\|T\|_{p,\psi} \leq k_\lambda$. T is a bounded linear operator.

If there exists a constant $K < k_\lambda$, such that $\|T\|_{p,\psi} \leq K$, then for $\varepsilon > 0$, setting $\tilde{a}_m := [u(m)]^{\lambda/r-\varepsilon/p-1} u'(m)$, $\tilde{b}_n := [v(n)]^{\lambda/s-\varepsilon/q-1} v'(n)$, it has $\tilde{a} = \{\tilde{a}_m\}_{m=n_0}^{\infty} \in \ell^p_\phi$, $\tilde{b} = \{\tilde{b}_n\}_{n=n_0}^{\infty} \in \ell^q_\varphi$, and

$$\left(T\tilde{a}, \tilde{b}\right) \leq \|T\|_{p,\psi} \|\tilde{a}\|_{p,\phi} \left\|\tilde{b}\right\|_{q,\varphi} \leq K \left\{ \sum_{m=n_0}^{\infty} \frac{u'(m)}{[u(m)]^{1+\varepsilon}} \right\}^{1/p} \left\{ \sum_{n=n_0}^{\infty} \frac{v'(n)}{[v(n)]^{1+\varepsilon}} \right\}^{1/q} \leq K \sum_{m=n_0}^{\infty} \frac{u'(m)}{[u(m)]^{1+\varepsilon}}. \tag{3.13}$$

But on the other side, by (3.8), it has

$$\begin{aligned}\left(T\widetilde{a},\widetilde{b}\right) &= \sum_{m=n_0}^{\infty}\sum_{n=n_0}^{\infty}K(u(m),v(n))\frac{[u(m)]^{\lambda/r-\varepsilon/p-1}}{[v(n)]^{1-\lambda/s+\varepsilon/q}}u'(m)v'(n)\\ &= \sum_{m=n_0}^{\infty}\frac{u'(m)}{[u(m)]^{1+\varepsilon}}\sum_{n=n_0}^{\infty}K(u(m),v(n))\frac{[u(m)]^{\lambda/r+\varepsilon/q}}{[v(n)]^{1-\lambda/s+\varepsilon/q}}v'(n).\end{aligned} \tag{3.14}$$

By the strictly monotonic increase of $v(x)$ and $v(n_0) > 0, v(\infty) = \infty$, there exists $n_1 > n_0$ such that $v(n) > 1$ for all $n > n_1$. So it has

$$\begin{aligned}0 &< \sum_{n=n_0}^{\infty}K(u(m),v(n))\frac{[u(m)]^{\lambda/r+\varepsilon/q}}{[v(n)]^{1-\lambda/s+\varepsilon/q}}v'(n)\\ &= [u(m)]^{\varepsilon/q}\left\{\sum_{n=n_1}^{\infty}K(u(m),v(n))\frac{[u(m)]^{\lambda/r}v'(n)}{[v(n)]^{1-\lambda/s+\varepsilon/q}}+\sum_{n=n_0}^{n_1-1}K(u(m),v(n))\frac{[u(m)]^{\lambda/r}v'(n)}{[v(n)]^{1-\lambda/s+\varepsilon/q}}\right\}\\ &\leq [u(m)]^{\varepsilon/q}\left\{\sum_{n=n_1}^{\infty}K(u(m),v(n))\frac{[u(m)]^{\lambda/r}v'(n)}{[v(n)]^{1-\lambda/s}}+\sum_{n=n_0}^{n_1-1}K(u(m),v(n))\frac{[u(m)]^{\lambda/r}v'(n)}{[v(n)]^{1-\lambda/s+\varepsilon/q}}\right\}\\ &= [u(m)]^{\varepsilon/q}\left\{\omega_\lambda(m,s)-\sum_{n=n_0}^{n_1-1}K(u(m),v(n))\frac{[u(m)]^{\lambda/r}v'(n)}{[v(n)]^{1-\lambda/s}}\right.\\ &\qquad\left.+\sum_{n=n_0}^{n_1-1}K(u(m),v(n))\frac{[u(m)]^{\lambda/r}v'(n)}{[v(n)]^{1-\lambda/s+\varepsilon/q}}\right\}.\end{aligned} \tag{3.15}$$

The series is uniformly convergent for $\varepsilon \geq 0$, so it has

$$\lim_{\varepsilon\to 0^+}\sum_{n=n_0}^{\infty}K(u(m),v(n))\frac{[u(m)]^{\lambda/r+\varepsilon/q}}{[v(n)]^{1-\lambda/s+\varepsilon/q}}v'(n) = \omega_\lambda(m,s) \tag{3.16}$$

and for $m > n_0$, there exists $\varepsilon_0 > 0$, when $0 < \varepsilon < \varepsilon_0$, it has

$$\sum_{n=n_0}^{\infty}K(u(m),v(n))\frac{[u(m)]^{\lambda/r+\varepsilon/q}}{[v(n)]^{1-\lambda/s+\varepsilon/q}}v'(n) > \omega_\lambda(m,s)-\frac{1}{u(m)}. \tag{3.17}$$

By (3.14) and (3.8), when $0 < \varepsilon < \varepsilon_0$, it has

$$\begin{aligned}
\left(T\tilde{a},\tilde{b}\right) &\geq \sum_{m=n_0}^{\infty} \frac{u'(m)}{[u(m)]^{1+\varepsilon}} \left(\omega_\lambda(m,s) - \frac{1}{u(m)}\right) \\
&\geq k_\lambda \sum_{m=n_0}^{\infty} \frac{u'(m)}{[u(m)]^{1+\varepsilon}} \left\{ \left[1 - O\left(\frac{1}{[u(m)]^{\rho}}\right)\right] - \frac{1}{k_\lambda u(m)} \right\} \\
&= k_\lambda \sum_{m=n_0}^{\infty} \frac{u'(m)}{[u(m)]^{1+\varepsilon}} \left\{ 1 - \left(\sum_{m=n_0}^{\infty} \frac{u'(m)}{[u(m)]^{1+\varepsilon}} \right)^{-1} \right. \\
&\qquad \left. \times \left(\sum_{m=n_0}^{\infty} \frac{u'(m)}{[u(m)]^{1+\varepsilon}} \left[O\left(\frac{1}{[u(m)]^{\rho}}\right) + \frac{1}{k_\lambda u(m)} \right] \right) \right\}.
\end{aligned} \tag{3.18}$$

In view of (3.13) and (3.18), letting $\varepsilon \to 0^+$, it has $k_\lambda \leq K$. This means that $K = k_\lambda$; that is, $\|T\|_{p,\varphi} = k_\lambda$. Theorem 3.1 is proved. □

Theorem 3.2. *Suppose that (p,q) and (r,s) are two pairs of conjugate exponents, $r > 1$, $p > 1$, $\lambda \in R$. Let*

$$\begin{aligned}
f(y) &:= K_\lambda(u(m),v(y)) \frac{u^{\lambda/r}(m)}{v^{1-\lambda/s}(y)} v'(y), \\
g(y) &:= K_\lambda(u(y),v(n)) \frac{v^{\lambda/s}(n)}{u^{1-\lambda/r}(y)} u'(y).
\end{aligned} \tag{3.19}$$

Here, $u(y)$, $v(y)$ satisfy the conditions as in Theorem 3.1. Set

$$R(m,s) := \int_0^{v(n_0)/u(m)} K_\lambda(1,u) u^{\lambda/s-1} du - \frac{1}{2} f(n_0) + \frac{1}{12} f'(n_0), \tag{3.20}$$

$$\tilde{R}(n,r) := \int_0^{u(n_0)/v(n)} K_\lambda(\mu,1) \mu^{\lambda/r-1} d\mu - \frac{1}{2} g(n_0) + \frac{1}{12} g'(n_0), \tag{3.21}$$

$$\eta(m,s) := \int_0^{v(n_0)/u(m)} K_\lambda(1,u) u^{\lambda/s-1} du - \frac{1}{2} f(n_0). \tag{3.22}$$

If (a) $K_\lambda(x,y) \geq 0$ is a homogeneous measurable kernel function of "λ" degree in R_+^2, such that

$$0 < k_\lambda(s) := \int_0^{\infty} K_\lambda(1,u) u^{\lambda/s-1} du < \infty, \tag{3.23}$$

(b) functions $f(y)$, $g(y)$ satisfy the conditions of (1.6); that is,

$$(-1)^i F^{(i)}(y) > 0 (y > n_0), \quad F^{(i)}(\infty) = 0 \quad (F = f, g,\ i = 0,1,2,3,4), \tag{3.24}$$

(c) there exists $\rho > 0$, such that

$$R(m,s) > 0, \quad \widetilde{R}(n,r) > 0, \quad 0 < \eta(m,s) = O\left(\frac{1}{u^{\rho}(m)}\right) \quad (m \longrightarrow \infty), \tag{3.25}$$

then it has

(1) *if $a \in \ell_{\phi}^{p}$, $b \in \ell_{\varphi}^{q}$, and $\|a\|_{p,\phi} > 0$, $\|b\|_{q,\varphi} > 0$, then*

$$(Ta,b) = \sum_{n=n_0}^{\infty} \sum_{m=n_0}^{\infty} K_{\lambda}(u(m),v(n)) a_m b_n < k_{\lambda}(s) \|a\|_{p,\phi} \|b\|_{q,\varphi}, \tag{3.26}$$

(2) *if $a \in \ell_{\phi}^{p}$ and $\|a\|_{p,\phi} > 0$, then*

$$\|Ta\|_{p,\varphi} = \left\{ \sum_{n=n_0}^{\infty} [v(n)]^{p\lambda/s-1} v'(n) \left[\sum_{m=n_0}^{\infty} K(u(m),v(n)) a_m \right]^{p} \right\}^{1/p} < k_{\lambda}(s) \|a\|_{p,\phi}, \tag{3.27}$$

where inequality (3.27) *is equivalent to* (3.26) *and the constant factor $k_{\lambda}(s) = \widetilde{k}_{\lambda}(r) := \int_0^{\infty} K_{\lambda}(u,1) u^{\lambda/r-1} du$ is the best possible.*

Proof. By (3.24), (1.6), it has

$$\begin{aligned} \int_{n_0}^{\infty} f(y)dy - \frac{1}{2} f(n_0) < \omega_{\lambda}(m,s) &= \sum_{n=n_0}^{\infty} K_{\lambda}(u(m),v(n)) \frac{[u(m)]^{\lambda/r}}{[v(n)]^{1-\lambda/s}} v'(n) \\ &= \sum_{n=n_0}^{\infty} f(n) < \int_{n_0}^{\infty} f(y)dy - \frac{1}{2} f(n_0) + \frac{1}{12} f'(n_0), \end{aligned} \tag{3.28}$$

$$\begin{aligned} 0 < \vartheta_{\lambda}(n,r) &= \sum_{m=n_0}^{\infty} K_{\lambda}(u(m),v(n)) \frac{[v(n)]^{\lambda/s}}{[u(m)]^{1-\lambda/r}} u'(m) \\ &= \sum_{m=n_0}^{\infty} g(m) < \int_{n_0}^{\infty} g(y)dy - \frac{1}{2} g(n_0) + \frac{1}{12} g'(n_0). \end{aligned} \tag{3.29}$$

Letting $\nu = v(y)/u(m)$ and $\mu = u(y)/v(n)$ in the integral of (3.28) and (3.29), respectively, by (3.23), it has

$$\begin{aligned}
\int_{n_0}^{\infty} f(y)dy &= \int_{n_0}^{\infty} K_\lambda(u(m), v(y)) \frac{u^{\lambda/r}(m)}{v^{1-\lambda/s}(y)} v'(y)dy = \int_{v(n_0)/u(m)}^{\infty} K_\lambda(1,\nu)\nu^{\lambda/s-1}d\nu \\
&= \int_0^{\infty} K_\lambda(1,\nu)\nu^{\lambda/s-1}d\nu - \int_0^{v(n_0)/u(m)} K_\lambda(1,\nu)\nu^{\lambda/s-1}d\nu = k_\lambda(s) \\
&\quad - \int_0^{v(n_0)/u(m)} K_\lambda(1,\nu)\nu^{\lambda/s-1}d\nu,
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
\int_{n_0}^{\infty} g(y)dy &= \int_{n_0}^{\infty} K_\lambda(u(y), v(n)) \frac{v^{\lambda/s}(n)}{u^{1-\lambda/r}(y)} u'(y)dy = \int_{u(n_0)/v(n)}^{\infty} K_\lambda(\mu,1)\mu^{\lambda/r-1}d\mu \\
&= \int_0^{\infty} K_\lambda(\mu,1)\mu^{\lambda/r-1}d\mu - \int_0^{u(n_0)/v(n)} K_\lambda(\mu,1)\mu^{\lambda/r-1}d\mu = k_\lambda(s) \\
&\quad - \int_0^{u(n_0)/v(n)} K_\lambda(\mu,1)\mu^{\lambda/r-1}d\mu,
\end{aligned} \tag{3.31}$$

(where, letting $t = 1/u$, it has $\widetilde{k}_\lambda(r) = \int_0^\infty K_\lambda(u,1)u^{\lambda/r-1}du = \int_0^\infty K_\lambda(1,t)t^{\lambda/s-1}dt = k_\lambda(s)$). In view of (3.28), (3.30), (3.20), (3.22), and with (3.25), it has

$$\begin{gathered}
0 < \omega_\lambda(m,s) < k_\lambda(s) - R_\lambda(m,s) < k_\lambda(s), \\
\omega_\lambda(m,s) > k_\lambda(s) - \eta_\lambda(m,s) = k_\lambda(s)\left[1 - O_1\left(\frac{1}{u^\rho(m)}\right)\right] \quad (\rho > 0, m \longrightarrow \infty).
\end{gathered} \tag{3.32}$$

Similarly, with (3.29), (3.31), (3.21), and (3.25), it has

$$0 < \vartheta_\lambda(n,r) < k_\lambda(s) \tag{3.33}$$

also. By Theorem 3.1, it has

$$\|Ta\|_{p,\psi} < k_\lambda(s)\|a\|_{p,\phi}, \tag{3.34}$$

and (3.27) holds. In view of

$$(Ta,b) \le \|Ta\|_{p,\psi}\|b\|_{q,\varphi}, \tag{3.35}$$

(3.26) holds also.

If (3.26) holds, from (3.26) and $\|a\|_{p,\phi} > 0$, there exists $n_1 > n_0$, such that $\sum_{m=n_0}^{H} \phi(m) a_m^p > 0$ and $b_n(H) := \varphi(n)[\sum_{m=n_0}^{H} [K_\lambda(u(m), v(n)) a_m]^{p-1} > 0$ when $H > n_1$. For $\widetilde{b} := \{b_n(H)\}_{n=n_0}^{H}$, it has

$$\begin{aligned}
0 &< \sum_{n=n_0}^{H} \varphi(n) b_n^q(H) \\
&= \sum_{n=n_0}^{H} \psi(n) \left[\sum_{m=n_0}^{H} K_\lambda(u(m), v(n)) a_m \right]^p = \sum_{n=n_0}^{H} \sum_{m=n_0}^{H} K_\lambda(u(m), v(n)) a_m b_n(H) \\
&< k_\lambda(s) \left[\sum_{n=n_0}^{H} \phi(n) a_n^p \right]^{1/p} \left[\sum_{n=n_0}^{H} \varphi(n) b_n^q(H) \right]^{1/q} < \infty.
\end{aligned} \tag{3.36}$$

By $p > 1$ and $q > 1$, it follows that

$$0 < \sum_{n=n_0}^{H} \varphi(n) b_n^q(H) < k_\lambda^p(s) \sum_{n=n_0}^{\infty} \phi(n) a_n^p < \infty. \tag{3.37}$$

Letting $H \to \infty$ in (3.37), it has $0 < \sum_{n=n_0}^{\infty} \varphi(n) b_n^q(\infty) < \infty$, and it means that $b = \{b_n(\infty)\}_{n=n_0}^{\infty} \in \ell_\varphi^q$ and $\|b\|_{q,\varphi} > 0$. Therefore, the inequality (3.36) keeps the form of the strict inequality when $H \to \infty$. In view of $\sum_{n=n_0}^{\infty} \varphi(n) b_n^q(\infty) = \|Ta\|_{p,\psi}^p$, inequality (3.27) holds and (3.27) is equivalent to (3.26). By $\|T\|_{p,\psi} = k_\lambda(s)$, it is obvious that the constant factor $k_\lambda(s) = k_\lambda(r)$ is the best possible. This completes the proof of Theorem 3.2. □

4. Applications

Example 4.1. Set (p, q), (r, s) be two pairs of conjugate exponents and $p > 1$, $s > 1$, $\beta \geq \alpha \geq e^{7/12}$, $0 < \lambda \leq 1$. Then it has the following.

(1) If $0 < \sum_{n=1}^{\infty} ([\ln \alpha n]^{p(1-\lambda/r)-1} a_n^p / n^{p-1}) < \infty$, and $0 < \sum_{n=1}^{\infty} ([\ln \beta n]^{q(1-\lambda/s)-1} \ b_n^q / n^{q-1}) < \infty$, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{\ln^\lambda \alpha m + \ln^\lambda \beta n} < \frac{\pi}{\lambda \sin(\pi/s)} \left\{ \sum_{n=1}^{\infty} \frac{[\ln \alpha n]^{p(1-\lambda/r)-1} a_n^p}{n^{p-1}} \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \frac{[\ln \beta n]^{q(1-\lambda/s)-1} b_n^q}{n^{q-1}} \right\}^{1/q}. \tag{4.1}$$

(2) If $0 < \sum_{n=1}^{\infty} ([\ln \alpha n]^{p(1-\lambda/r)-1} a_n^p / n^{p-1}) < \infty$, then

$$\sum_{n=1}^{\infty} \frac{[\ln \beta n]^{p\lambda/s-1}}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln^\lambda \alpha m + \ln^\lambda \beta n} \right)^p < \left(\frac{\pi}{\lambda \sin(\pi/s)} \right)^p \sum_{n=1}^{\infty} \frac{[\ln \alpha n]^{p(1-\lambda/r)-1} a_n^p}{n^{p-1}}, \tag{4.2}$$

where the constant factors $k_\lambda(s) = k_\lambda(r) := \pi/\lambda \sin(\pi/s)$ and $(\pi/\lambda \sin(\pi/s))^p$ are both the best possible. Inequality (4.2) is equivalent to (4.1).

Proof. Setting $K_\lambda(x,y) = 1/(x^\lambda + y^\lambda)$, $((x,y) \in R_+^2)$, it is a homogeneous measurable kernel function of "λ" degree. Letting $t = u^\lambda$, it has

$$0 < k_\lambda(s) := \int_0^\infty K_\lambda(1,u)u^{\lambda/s-1}du = \int_0^\infty \frac{u^{\lambda/s-1}}{1+u^\lambda}du = \frac{1}{\lambda}\int_0^\infty \frac{t^{1/s-1}}{1+t}dt = \frac{1}{\lambda}B\left(\frac{1}{s},\frac{1}{r}\right) = k_\lambda(r) < \infty. \tag{4.3}$$

Setting $u(x) = \ln \alpha x$, $v(x) = \ln \beta x$, then both $u(x)$ and $v(x)$ are strictly monotonic increasing differentiable functions in $[1,\infty)$ and satisfy

$$U(1) > 0, \qquad U(\infty) = \infty \quad (U = u, v),$$
$$0 < \sum_{n=1}^\infty \frac{v'(n)}{[v(n)]^{1+\varepsilon}} = \sum_{n=1}^\infty \frac{1}{n[\ln \beta n]^{1+\varepsilon}} \le \sum_{n=1}^\infty \frac{u'(n)}{[u(n)]^{1+\varepsilon}} = \sum_{n=1}^\infty \frac{1}{n[\ln \alpha n]^{1+\varepsilon}} < \infty \tag{4.4}$$

for $\varepsilon > 0$. As $\beta \ge \alpha \ge e^{7/12}$, $0 < \lambda \le 1$, $s > 1$, and $n_0 = 1$, letting

$$f(y) = K_\lambda(u(m), v(y))\frac{u^{\lambda/r}(m)}{v^{1-\lambda/s}(y)}v'(y) = \frac{1}{\ln^\lambda \alpha m + \ln^\lambda \beta y}\frac{\ln^{\lambda/r}\alpha m}{(\ln^{1-\lambda/s}\beta y)y},$$
$$g(y) := K_\lambda(u(y), v(n))\frac{v^{\lambda/s}(n)}{u^{1-\lambda/r}(y)}u'(y) = \frac{1}{\ln^\lambda \alpha y + \ln^\lambda \beta n}\frac{\ln^{\lambda/s}\beta n}{(\ln^{1-\lambda/r}\alpha y)y}, \tag{4.5}$$
$$y \in [1,\infty),\ n, m \in N,$$

with (2.1)~(2.8), it has

$$R_\lambda(m,s) = \int_0^{v(1)/u(m)} K_\lambda(1,u)u^{\lambda/s-1}du - \frac{1}{2}f(1) + \frac{1}{12}f'(1) > 0,$$
$$0 < \eta_\lambda(m,s) = O\left(\left[\frac{1}{u(m)}\right]^\rho\right) \quad (\rho > 0, m \longrightarrow \infty), \tag{4.6}$$
$$\widetilde{R}_\lambda(n,r) := \int_0^{u(1)/v(n)} K_\lambda(\mu,1)\mu^{\lambda/r-1}d\mu - \frac{1}{2}g(1) + \frac{1}{12}g'(1) > 0.$$

When $0 < \sum_{n=1}^\infty([\ln \alpha n]^{p(1-\lambda/r)-1}a_n^p/n^{p-1}) < \infty$ and $0 < \sum_{n=1}^\infty [\ln \beta n]^{q(1-\lambda/s)-1}b_n^q/n^{q-1} < \infty$; that is, $a \in \ell_\phi^p$, $b \in \ell_\varphi^q$ and $\|a\|_{p,\phi} > 0$, $\|b\|_{q,\varphi} > 0$, by Theorem 3.2, inequality (4.1) holds, so does (4.2). And (4.2) is equivalent to (4.1), and the constant factors $k_\lambda(s) = k_\lambda(r) := \pi/\lambda \sin(\pi/s)$ and $(\pi/\lambda \sin(\pi/s))^p$ are both the best possible. □

Example 4.2. Set (p,q), (r,s) be two pairs of conjugate exponents and $p > 1$, $s > 1$, $\beta \ge \alpha \ge 1/2$, $0 < \lambda \le 1$. Then it has the following.

(1) If $0 < \sum_{n=0}^{\infty} (n+\alpha)^{p(1-\lambda/r)-1} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} (n+\beta)^{q(1-\lambda/s)-1} b_n^q < \infty$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\beta))a_m b_n}{(m+\alpha)^{\lambda}-(n+\beta)^{\lambda}} < \left[\frac{\pi}{\lambda\sin(\pi/s)}\right]^2 \left\{\sum_{n=0}^{\infty}(n+\alpha)^{p(1-\lambda/r)-1}a_n^p\right\}^{1/p}\left\{\sum_{n=0}^{\infty}(n+\beta)^{q(1-\lambda/s)-1}b_n^q\right\}^{1/q}. \tag{4.7}$$

(2) If $0 < \sum_{n=0}^{\infty} (n+\alpha)^{p(1-\lambda/r)-1} a_n^p < \infty$, then

$$\sum_{n=0}^{\infty}(n+\beta)^{p\lambda/s-1}\left(\sum_{m=0}^{\infty}\frac{\ln((m+\alpha)/(n+\beta))a_m}{(m+\alpha)^{\lambda}-(n+\beta)^{\lambda}}\right)^p < \left[\frac{\pi}{\lambda\sin(\pi/s)}\right]^{2p}\sum_{n=0}^{\infty}(n+\alpha)^{p(1-\lambda/r)-1}a_n^p, \tag{4.8}$$

where inequality (4.8) is equivalent to (4.7) and the constant factors $k_\lambda(s) = k_\lambda(r) := [\pi/\lambda\sin(\pi/s)]^2$ and $[\pi/\lambda\sin(\pi/s)]^{2p}$ are both the best possible.

Proof. Setting $K_\lambda(x,y) = \ln(x/y)/(x^\lambda - y^\lambda)((x,y) \in R_+^2)$, it is a homogeneous measurable kernel function of "λ" degree. Letting $t = u^\lambda$, it has [2]

$$\begin{aligned} 0 < k_\lambda(s) &:= \int_0^\infty K_\lambda(1,u)u^{\lambda/s-1}du = \frac{1}{\lambda}\int_0^\infty \frac{-\ln u^\lambda}{1-u^\lambda}u^{\lambda/s-1}du \\ &= \frac{1}{\lambda^2}\int_0^\infty \frac{\ln t}{t-1}t^{1/s-1}dt = \left[\frac{1}{\lambda}B\left(\frac{1}{s},\frac{1}{r}\right)\right]^2 = \left[\frac{\pi}{\lambda\sin(\pi/s)}\right]^2 = k_\lambda(r) < \infty. \end{aligned} \tag{4.9}$$

Setting $u(x) = x+\alpha$, $v(x) = x+\beta$, then both $u(x)$ and $v(x)$ are strictly monotonic increasing differentiable functions in $[0,\infty)$ and satisfy

$$\begin{gathered} U(0) > 0, \quad U(\infty) = \infty \quad (U = u, v), \\ 0 < \sum_{n=0}^{\infty}\frac{v'(n)}{[v(n)]^{1+\varepsilon}} = \sum_{n=0}^{\infty}\frac{1}{(n+\beta)^{1+\varepsilon}} \le \sum_{n=0}^{\infty}\frac{u'(n)}{[u(n)]^{1+\varepsilon}} = \sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^{1+\varepsilon}} < \infty \end{gathered} \tag{4.10}$$

for $\varepsilon > 0$. As $\beta \geq \alpha \geq 1/2$, $0 < \lambda \leq 1$, $s > 1$, and $n_0 = 0$, letting

$$\begin{aligned}
f_1(y) &= K_\lambda(u(m), v(y))\frac{u^{\lambda/r}(m)}{v^{1-\lambda/s}(y)}v'(y) = \frac{\ln((m+\alpha)/(y+\beta))}{(m+\alpha)^\lambda - (y+\beta)^\lambda}\frac{(m+\alpha)^{\lambda/r}}{(y+\beta)^{1-\lambda/s}} \\
&= \frac{1}{\lambda(m+\alpha)}\frac{\ln((y+\beta)/(m+\alpha))^{\lambda}}{(y+\beta/m+\alpha)^\lambda - 1}\left(\frac{y+\beta}{m+\alpha}\right)^{\lambda/s-1}, \\
g_1(y) &:= K_\lambda(u(y), v(n))\frac{v^{\lambda/s}(n)}{u^{1-\lambda/r}(y)}u'(y) = \frac{\ln((y+\alpha)/(n+\beta))}{(y+\alpha)^\lambda - (n+\beta)^\lambda}\frac{(n+\beta)^{\lambda/s}}{(y+\alpha)^{1-\lambda/r}} \\
&= \frac{1}{\lambda(n+\beta)}\frac{\ln((y+\alpha)/(n+\beta))^{\lambda}}{((y+\alpha)/(n+\beta))^\lambda - 1}\left(\frac{y+\alpha}{n+\beta}\right)^{\lambda/r-1}, \quad y \in [0,\infty),\ n, m \in N,
\end{aligned} \tag{4.11}$$

with (2.18)~(2.21), it has

$$\begin{aligned}
R_\lambda(m,s) &:= \int_0^{v(0)/u(m)} K_\lambda(1,u)u^{\lambda/s-1}du - \frac{1}{2}f_1(0) + \frac{1}{12}f_1'(0) \\
&= \frac{1}{\lambda^2}\int_0^{(\beta/(m+\alpha))^\lambda}\frac{\ln u}{u-1}u^{1/s-1}du - \frac{1}{2}f_1(0) + \frac{1}{12}f_1'(0) > 0, \\
\widetilde{R}_\lambda(n,r) &:= \int_0^{u(0)/v(n)} K_\lambda(\mu,1)\mu^{\lambda/r-1}d\mu - \frac{1}{2}g_1(0) + \frac{1}{12}g_1'(0) > 0, \\
0 < \eta_\lambda(m,s) &= O\left(\left[\frac{1}{u(m)}\right]^\rho\right) \quad (\rho > 0,\ m \longrightarrow \infty).
\end{aligned} \tag{4.12}$$

When $0 < \sum_{n=0}^\infty (n+\alpha)^{p(1-\lambda/r)-1}a_n^p < \infty$ and $0 < \sum_{n=0}^\infty (n+\beta)^{q(1-\lambda/s)-1}b_n^q < \infty$; that is, $a \in \ell_\phi^p$, $b \in \ell_\varphi^q$ and $\|a\|_{p,\phi} > 0$, $\|b\|_{q,\varphi} > 0$, by Theorem 3.2, inequality (4.7) holds, so does (4.8). And (4.8) is equivalent to (4.7), and the constant factors $k_\lambda(s) = k_\lambda(r) := [\pi/\lambda\sin(\pi/s)]^2$ and $[\pi/\lambda\sin(\pi/s)]^{2p}$ are both the best possible. □

Remark 4.3. It can be proved similarly that, if the conditions "$\beta \geq \alpha \geq e^{7/12}$" in Lemma 2.1 and "$\beta \geq \alpha \geq 1/2$" in Lemma 2.2 are changed into "$\alpha \geq \beta \geq e^{7/12}$" and "$\alpha \geq \beta \geq 1/2$", respectively, Lemmas 2.1 and 2.2 are also valid. So the conditions "$\beta \geq \alpha \geq e^{7/12}$" in Example 4.1 and "$\beta \geq \alpha \geq 1/2$" in Example 4.2 can be replaced by "$\beta \geq e^{7/12}$, $\alpha \geq e^{7/12}$" and "$\beta \geq 1/2$, $\alpha \geq 1/2$", respectively.

Acknowledgment

This paper is supported by *the National Natural Science Foundation of China (no. 10871073)*. The author would like to thank the anonymous referee for his or her suggestions and corrections.

References

[1] H. Weyl, *Singulare Integralgleichungen mit besonderer Beriicksichtigung des Fourierschen Integraltheorems*, Inaugeral dissertation, University of Göttingen, Göttingen, Germany, 1908.

[2] G. Hardy, J. Littiewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, UK, 1934.

[3] B. C. Yang, "On a more accurate Hardy-Hilbert-type inequality and its applications," *Acta Mathematica Sinica*, vol. 49, no. 2, pp. 363–368, 2006 (Chinese).

[4] B. Yang, "On a more accurate Hilbert's type inequality," *International Mathematical Forum*, vol. 2, no. 37–40, pp. 1831–1837, 2007.

[5] W. Zhong, "A Hilbert-type linear operator with the norm and its applications," *Journal of Inequalities and Applications*, vol. 2009, Article ID 494257, 18 pages, 2009.

[6] B. Yang, *The Norm of Operator and Hilbert-Type Inequalities*, Science Press, Beijing, China, 2009.

[7] J. Kuang, *Applied Inequalities*, Shangdong Science and Technology Press, Jinan, China, 2004.