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Research Article

On Some Matrix Trace Inequalities

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We first present an inequality for the Frobenius norm of the Hadamard product of two any square matrices and positive semidefinite matrices. Then, we obtain a trace inequality for products of two positive semidefinite block matrices by using 2×2 block matrices.

1. Introduction and Preliminaries

Let $M_{m,n}$ denote the space of $m \times n$ complex matrices and write $M_n \equiv M_{n,n}$. The identity matrix in M_n is denoted I_n. As usual, $A^* = (\overline{A})^T$ denotes the conjugate transpose of matrix A. A matrix $A \in M_n$ is Hermitian if $A^* = A$. A Hermitian matrix A is said to be positive semidefinite or nonnegative definite, written as $A \geq 0$, if

$$x^* A x \geq 0, \quad \forall x \in \mathbb{C}^n. \tag{1.1}$$

A is further called positive definite, symbolized $A > 0$, if the strict inequality in (1.1) holds for all nonzero $x \in \mathbb{C}^n$. An equivalent condition for $A \in M_n$ to be positive definite is that A is Hermitian and all eigenvalues of A are positive real numbers. Given a positive semidefinite matrix A and $p > 0$, A^p denotes the unique positive semidefinite pth power of A.

Let A and B be two Hermitian matrices of the same size. If $A - B$ is positive semidefinite, we write

$$A \geq B \quad \text{or} \quad B \leq A. \tag{1.2}$$

Denote $\lambda_1(A), \dots, \lambda_n(A)$ and $s_1(A), \dots, s_n(A)$ eigenvalues and singular values of matrix A, respectively. Since A is Hermitian matrix, its eigenvalues are arranged in decreasing order, that is, $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and if A is any matrix, its singular values are arranged in decreasing order, that is, $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) > 0$. The trace of a square matrix A

(the sum of its main diagonal entries, or, equivalently, the sum of its eigenvalues) is denoted by $\operatorname{tr} A$.

Let A be any $m \times n$ matrix. The Frobenius (Euclidean) norm of matrix A is

$$\|A\|_F = \left[\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right]^{1/2}. \tag{1.3}$$

It is also equal to the square root of the matrix trace of AA^*, that is,

$$\|A\|_F = \sqrt{\operatorname{tr}(AA^*)}. \tag{1.4}$$

A norm $\|\cdot\|$ on $M_{m,n}$ is called unitarily invariant $\|UAV\| = \|A\|$ for all $A \in M_{m,n}$ and all unitary $U \in M_m, V \in M_n$.

Given two real vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in decreasing order, we say that x is weakly log majorized by y, denoted $x \prec_{w\log} y$, if $\prod_{i=1}^{k} x_i \le \prod_{i=1}^{k} y_i,\ k = 1, 2, \ldots, n$, and we say that x is weakly majorized by y, denoted $x \prec_w y$, if $\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i,\ k = 1, 2, \ldots, n$. We say x is majorized by y denoted by $x \prec y$, if

$$x \prec_w y, \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{1.5}$$

As is well known, $x \prec_{w\log} y$ yields $x \prec_w y$ (see, e.g., [1, pages 17–19]).

Let A be a square complex matrix partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{1.6}$$

where A_{11} is a square submatrix of A. If A_{11} is nonsingular, we call

$$\widetilde{A}_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12} \tag{1.7}$$

the Schur complement of A_{11} in A (see, e.g., [2, page 175]). If A is a positive definite matrix, then A_{11} is nonsingular and

$$A_{22} \ge \widetilde{A}_{11} \ge 0. \tag{1.8}$$

Recently, Yang [3] proved two matrix trace inequalities for positive semidefinite matrices $A \in M_n$ and $B \in M_n$,

$$\begin{gathered} 0 \le \operatorname{tr}(AB)^{2n} \le (\operatorname{tr} A)^2\left(\operatorname{tr} A^2\right)^{n-1}\left(\operatorname{tr} B^2\right)^{n}, \\ 0 \le \operatorname{tr}(AB)^{2n+1} \le (\operatorname{tr} A)(\operatorname{tr} B)\left(\operatorname{tr} A^2\right)^{n}\left(\operatorname{tr} B^2\right)^{n}, \end{gathered} \tag{1.9}$$

for $n = 1, 2, \ldots$.

Also, authors in [4] proved the matrix trace inequality for positive semidefinite matrices A and B,

$$\operatorname{tr}(AB)^m \le \left\{\operatorname{tr}(A)^{2m}\operatorname{tr}(B)^{2m}\right\}^{1/2}, \tag{1.10}$$

where m is a positive integer.

Furthermore, one of the results given in [5] is

$$n(\det A\cdot\det B)^{m/n} \le \operatorname{tr}(A^m B^m) \tag{1.11}$$

for A and B positive definite matrices, where m is any positive integer.

2. Lemmas

Lemma 2.1 (see, e.g., [6]). *For any A and $B\in M_n$, $\sigma(A\circ B)\prec_w\sigma(A)\circ\sigma(B)$.*

Lemma 2.2 (see, e.g., [7]). *Let $A,B\in M_{m,n}$, then*

$$\begin{aligned}\sum_{i=1}^{t}\left|\delta_i\left((AB)^{2m}\right)\right| &\le \sum_{i=1}^{t}\lambda_i\left((A^*ABB^*)^m\right)\\ &\le \sum_{i=1}^{t}\lambda_i\left((A^*A)^m(BB^*)^m\right),\quad 1\le t\le n,\ m\in\mathbb{N}.\end{aligned} \tag{2.1}$$

Lemma 2.3 (Cauchy-Schwarz inequality). *Let $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ be real numbers. Then,*

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2 \le \left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right),\quad \forall a_i,b_i\in\mathbb{R}. \tag{2.2}$$

Lemma 2.4 (see, e.g., [8, page 269]). *If A and B are poitive semidefinite matrices, then,*

$$0\le\operatorname{tr}(AB)\le\operatorname{tr}A\operatorname{tr}B. \tag{2.3}$$

Lemma 2.5 (see, e.g., [9, page 177]). *Let A and B are $n\times n$ matrices. Then,*

$$\sum_{i=1}^{k}s_i(AB)\le\sum_{i=1}^{k}s_i(A)s_i(B)\quad(1\le k\le n). \tag{2.4}$$

Lemma 2.6 (see, e.g., [10]). *Let F and G are positive semidefinite matrices. Then,*

$$\sum_{i=1}^{t}\lambda_i^m(FG)\le\sum_{i=1}^{t}\lambda_i(F^mG^m),\quad 1\le t\le n, \tag{2.5}$$

where m is a positive integer.

3. Main Results

Horn and Mathias [11] show that for any unitarily invariant norm $\|\cdot\|$ on M_n

$$\begin{aligned}\|A^*B\|^2 &\leq \|A^*A\|\|B^*B\| \quad \forall A, B \in M_{m,n},\\ \|A\circ B\|^2 &\leq \|A^*A\|\|B^*B\| \quad \forall A, B \in M_n.\end{aligned} \tag{3.1}$$

Also, the authors in [12] show that for positive semidefinite matrix $A = \begin{pmatrix} L & X \\ X^* & M\end{pmatrix}$, where $X \in M_{m,n}$

$$\left\| |X|^p \right\|^2 \leq \|L^p\|\|M^p\| \tag{3.2}$$

for all $p > 0$ and all unitarily invariant norms $\|\cdot\|$.

By the following theorem, we present an inequality for Frobenius norm of the power of Hadamard product of two matrices.

Theorem 3.1. *Let A and B be n-square complex matrices. Then*

$$\left\|(A\circ B)^m\right\|_F^2 \leq \left\|(A^*A)^m\right\|_F \left\|(B^*B)^m\right\|_F, \tag{3.3}$$

where m is a positive integer. In particular, if A and B are positive semidefinite matrices, then

$$\left\|(A\circ B)^m\right\|_F^2 \leq \left\|A^{2m}\right\|_F \left\|B^{2m}\right\|_F. \tag{3.4}$$

Proof. From definition of Frobenius norm, we write

$$\left\|(A\circ B)^m\right\|_F^2 = \operatorname{tr}\left[(A\circ B)^m (A\circ B)^{m*}\right]. \tag{3.5}$$

Also, for any A and B, it follows that (see, e.g., [13])

$$\begin{pmatrix} AA^*\circ BB^* & A\circ B \\ A^*\circ B^* & I\end{pmatrix} \geq 0, \tag{3.6}$$

$$(A\circ B)(A\circ B)^* \leq AA^*\circ BB^*. \tag{3.7}$$

Since $|\operatorname{tr} A^{2m}| \leq \operatorname{tr}[A^m (A^*)^m] \leq \operatorname{tr}[(AA^*)^m]$ for $A \in M_n$ and from inequality (3.7), we write

$$\begin{aligned}\left\|(A\circ B)^m\right\|_F^2 &= \operatorname{tr}\,(A\circ B)^m (A\circ B)^{m*}\\ &\leq \operatorname{tr}\left[\left((A\circ B)(A\circ B)^*\right)^m\right]\\ &\leq \operatorname{tr}\left[(AA^*\circ BB^*)^m\right].\end{aligned} \tag{3.8}$$

From Lemma 2.1 and Cauchy-Schwarz inequality, we write

$$\begin{aligned}\operatorname{tr}(A^m \circ B^m) &= \sum_{i=1}^{n} \lambda_i[(A^m \circ B^m)] \le \sum_{i=1}^{n} \lambda_i(A^m)\lambda_i(B^m) \\ &\le \left\{ \sum_{i=1}^{n} \lambda_i^2(A^m) \sum_{i=1}^{n} \lambda_i^2(B^m) \right\}^{1/2} \\ &= \left\{ \operatorname{tr} A^{2m} \operatorname{tr} B^{2m} \right\}^{1/2}.\end{aligned} \tag{3.9}$$

By combining inequalities (3.7), (3.8), and (3.9), we arrive at

$$\begin{aligned}\operatorname{tr}\left[(AA^* \circ BB^*)^m\right] &\le \left\{ \operatorname{tr}\,(AA^*(AA^*))^m \operatorname{tr}\,(BB^*(BB^*))^m \right\}^{1/2} \\ &\le \left\{ \operatorname{tr}\,(AA^*AA^*)^m \operatorname{tr}\,(BB^*BB^*)^m \right\}^{1/2} \\ &= \left\{ \operatorname{tr}\,(AA^*)^{2m} \right\}^{1/2} \left\{ \operatorname{tr}\,(BB^*)^{2m} \right\}^{1/2} \\ &= \|(A^*A)^m\|_F \|(B^*B)^m\|_F.\end{aligned} \tag{3.10}$$

Thus, the proof is completed. Let A and B be positive semidefinite matrices. Then

$$\|(A \circ B)^m\|_F^2 \le \left\|A^{2m}\right\|_F \left\|B^{2m}\right\|_F, \tag{3.11}$$

where $m > 0$. □

Theorem 3.2. *Let $A_i \in M_n$ $(i = 1, 2, \ldots, k)$ be positive semidefinite matrices. For positive real numbers s, m, t*

$$\left(\sum_{i=1}^{k} \left\| A_i^{((s+t)/2)m} \right\|_F^2 \right)^2 \le \left(\sum_{i=1}^{k} \|A_i^{sm}\|_F^2 \right) \left(\sum_{i=1}^{k} \|A_i^{tm}\|_F^2 \right). \tag{3.12}$$

Proof. Let

$$A = \begin{pmatrix} A_1^{S/2} & 0 & \cdots & 0 \\ 0 & A_2^{s/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k^{s/2} \end{pmatrix}, \qquad B = \begin{pmatrix} A_1^{t/2} & 0 & \cdots & 0 \\ 0 & A_2^{t/2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k^{t/2} \end{pmatrix}. \tag{3.13}$$

We know that $A, B \geq 0$, then by using the definition of Frobenius norm, we write

$$\left\| (A \circ B)^m \right\|_F^2 = \sum_{i=1}^{k} \left\| A_i^{((s+t)/2)m} \right\|_F^2,$$
$$\left\| A^{2m} \right\|_F = \sqrt{\sum_{i=1}^{k} \| A_i^{sm} \|_F^2}, \qquad \left\| B^{2m} \right\|_F = \sqrt{\sum_{i=1}^{k} \| A_i^{tm} \|_F^2}. \tag{3.14}$$

Thus, by using Theorem 3.1, the desired is obtained. □

Now, we give a trace inequality for positive semidefinite block matrices.

Theorem 3.3. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \geq 0, \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \geq 0, \tag{3.15}$$

then,

$$\operatorname{tr}\left[\left(\widetilde{A}_{22} \right)^{1/2} B_{11}^{1/2} \right]^{2m} + \operatorname{tr}\left[A_{22}^{1/2} \left(\widetilde{B}_{11} \right)^{1/2} \right]^{2m} \leq \operatorname{tr}(AB)^m \leq \operatorname{tr}(A^m B^m), \tag{3.16}$$

where m is an integer.

Proof. Let

$$M = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} \tag{3.17}$$

with $Z = A_{22}^{1/2}, Y = A_{22}^{-1/2} A_{21}, X = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{1/2}$. Then $A = M^* M$ (see, e.g., [14]). Let

$$K = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} \tag{3.18}$$

with $Z = (B_{22} - B_{21} B_{11}^{-1} B_{12})^{1/2}$, $Y = B_{21} B_{11}^{-1/2}$, $X = B_{11}^{1/2}$. Then $B = KK^*$ (see, e.g., [14]). We know that

$$M^k = \begin{pmatrix} X^k & 0 \\ * & Z^k \end{pmatrix},$$

$$M \cdot K = \begin{bmatrix} \left(\left(A_{11} - A_{12} A_{22}^{-1} A_{21} \right)^{1/2} \right) B_{11}^{1/2} & 0 \\ A_{22}^{-1/2} A_{21} B_{11}^{1/2} + A_{22}^{1/2} B_{21} B_{11}^{-1/2} & A_{22}^{1/2} \left(B_{22} - B_{21} B_{11}^{-1} B_{12} \right)^{1/2} \end{bmatrix},$$

$$(M\cdot K)^{2m}=\begin{bmatrix}\left[\left(\left(A_{11}-A_{12}A_{22}^{-1}A_{21}\right)^{1/2}\right)B_{11}^{1/2}\right]^{2m} & 0\\ * & \left[A_{22}^{1/2}\left(B_{22}-B_{21}B_{11}^{-1}B_{12}\right)^{1/2}\right]^{2m}\end{bmatrix}. \tag{3.19}$$

By using Lemma 2.2, it follows that

$$\begin{aligned}\left|\operatorname{tr}(MK)^{2m}\right| &\le \sum_{i=1}^{n}s_i\left((MK)^{2m}\right)\le\sum_{i=1}^{n}(s_i(MK))^{2m}\\ &=\sum_{i=1}^{n}\left(s_i^2(MK)\right)^{m}=\sum_{i=1}^{n}\lambda_i\left((M^*MKK^*)^m\right)\\ &=\sum_{i=1}^{n}\lambda_i\left((AB)^m\right)=\sum_{i=1}^{n}\operatorname{tr}(AB)^m\le\sum_{i=1}^{n}\lambda_i\left((M^*M)^m(KK^*)^m\right)\\ &=\sum_{i=1}^{n}\lambda_i\left[(A)^m(B)^m\right]=\sum_{i=1}^{n}\operatorname{tr}(A^mB^m).\end{aligned} \tag{3.20}$$

Therefore, we get

$$\begin{aligned}\left|\operatorname{tr}(MK)^{2m}\right| &= \operatorname{tr}\left[\left(\left(A_{11}-A_{12}A_{22}^{-1}A_{21}\right)^{1/2}\right)B_{11}^{1/2}\right]^{2m}+\operatorname{tr}\left[A_{22}^{1/2}\left(B_{22}-B_{21}B_{11}^{-1}B_{12}\right)^{1/2}\right]^{2m}\\ &\le \operatorname{tr}(AB)^m\le\operatorname{tr}(A^mB^m).\end{aligned} \tag{3.21}$$

As result, we write

$$\operatorname{tr}\left[\left(\widetilde{A}_{22}\right)^{1/2}B_{11}^{1/2}\right]^{2m}+\operatorname{tr}\left[A_{22}^{1/2}\left(\widetilde{B}_{11}\right)^{1/2}\right]^{2m}\le\operatorname{tr}(AB)^m\le\operatorname{tr}(A^mB^m). \tag{3.22}$$

□

Example 3.4. Let

$$A=\begin{pmatrix}4&1\\1&1\end{pmatrix}>0,\qquad B=\begin{pmatrix}5&2\\2&1\end{pmatrix}>0. \tag{3.23}$$

Then $\operatorname{tr}AB=25$, $\det A=3$, $\det B=1$. From inequality (1.11), for $m=1$, we get

$$n(\det A\det B)^{1/n}=2\sqrt{3}\cong 3.464. \tag{3.24}$$

Also, for $m = 1$, since $\operatorname{tr}(\widetilde{A_{22}}^{1/2}B_{11}^{1/2})^2 = 15$ and $\operatorname{tr}(A_{22}^{1/2}\widetilde{B_{11}}^{1/2})^2 = 0.2$, we get

$$\operatorname{tr}\left(\widetilde{A_{22}}^{1/2}B_{11}^{1/2}\right)^2 + \operatorname{tr}\left(A_{22}^{1/2}\widetilde{B_{11}}^{1/2}\right)^2 = 15.2. \tag{3.25}$$

Thus, according to this example from (3.24) and (3.25), we get

$$n(\det A \det B)^{1/n} \leq \operatorname{tr}\left(\widetilde{A_{22}}^{1/2}B_{11}^{1/2}\right)^2 + \operatorname{tr}\left(A_{22}^{1/2}\widetilde{B_{11}}^{1/2}\right)^2 \leq \operatorname{tr}(AB). \tag{3.26}$$

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