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Research Article

On Generalizations of Grüss Inequality in Inner Product Spaces and Applications

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Using a Kurepa's result for Gramians, we achieve refinements of well-known generalizations of Grüss inequality in inner product spaces. These results are further applied in $L_2(a,b)$ to derive improvements of some published trapezoid-Grüss and Ostrowki-Grüss type inequalities. Refinements of the discrete version of Grüss inequality as well as a reverse of the Schwarz inequality are also given.

1. Introduction

Let $\{e_1,\ldots,e_n\}$ be an orthonormal system of vectors in unitary space $(V,\langle\cdot,\cdot\rangle)$. It is well known that for all $x,y\in V$, the following inequality holds [1, page 333]:

$$\left|S_n(x,y)\right|^2 \le S_n(x,x)S_n(y,y), \tag{1.1}$$

where $S_n(x,y)$ is defined by

$$S_n(x,y) = \langle x,y\rangle - \sum_{k=1}^{n}\langle x,e_k\rangle\langle e_k,y\rangle. \tag{1.2}$$

The equality in (1.1) holds if and only if $\{x, y, e_1, \ldots, e_n\}$ is linearly dependent. Applying (1.1) on $L_2(a,b)$ for $n = 1$ by choosing $e_1 = 1/\sqrt{b-a}$, $x = (1/\sqrt{b-a})f$ and $y = (1/\sqrt{b-a})g$, we immediately obtain the Pre-Grüss inequality as follows:

$$(T(f,g))^2 \leq T(f,f)T(g,g), \tag{1.3}$$

where $f, g \in L_2(a,b)$ and $T(f,g)$ is the Chebyshev functional

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx. \tag{1.4}$$

Let γ, Γ, ϕ, and Φ be real numbers such that $\gamma \leq f(x) \leq \Gamma$ and $\phi \leq g(x) \leq \Phi$ for all $x \in (a,b)$. Combining (1.3) with the following well-known inequality:

$$T(f,f) \leq \frac{1}{4}(\Gamma-\gamma)^2, \tag{1.5}$$

we obtain a premature Grüss inequality,

$$|T(f,g)| \leq \frac{1}{2}(\Gamma-\gamma)\sqrt{T(g,g)}, \tag{1.6}$$

and the original Grüss inequality (see [2]),

$$|T(f,g)| \leq \frac{1}{4}(\Gamma-\gamma)(\Phi-\phi). \tag{1.7}$$

Note that the discrete version of inequality (1.7) has the following form:

$$\left|\frac{1}{n}\sum_{k=1}^n a_k b_k - \frac{1}{n}\sum_{k=1}^n a_k \cdot \frac{1}{n}\sum_{k=1}^n b_k\right| \leq \frac{1}{4}(A-a)(B-b), \tag{1.8}$$

where a_k, b_k are real numbers so that $a \leq a_k \leq A$, $b \leq b_k \leq B$ for all $k = 1, \ldots, n$, and $a < A$, $b < B$.

In [3, 4], Dragomir starting from inequality (1.2) proved the following Grüss type inequality in real or complex inner product spaces.

Let e be a unit vector in V. If ϕ, γ, Φ, and Γ are complex numbers and x, y are vectors in V that satisfy the conditions,

$$\operatorname{Re}\langle \Phi e - x, x - \phi e\rangle \geq 0, \qquad \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0, \tag{1.9}$$

then the following inequality holds:

$$|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{4}|\Phi-\phi||\Gamma-\gamma|. \tag{1.10}$$

The constant $1/4$ that appears at the right side of the inequality is optimal in the sense that it cannot be replaced by a smaller one.

Some generalizations and refinements of inequality (1.10) can be found in [1, 4–10]. In this paper, we achieve an improvement of inequality (1.1) in the sense of subtracting a nonnegative quantity from the right part of (1.1). In this way, every result that stems from (1.1), such as the inequality (1.10) and its generalizations and refinements, can also be improved. Furthermore, we apply our improvements of inequalities (1.3), (1.6), and (1.7) to achieve refinements of some well-known trapezoid-Grüss and Ostrowki-Grüss type inequalities. Some refinements of inequality (1.8) as well as an additive reverse of the Schwarz inequality are also given.

2. A Refinement of Inequality (1.1)

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{k}$. For our purpose, we need the following three lemmas.

Lemma 2.1 (see [2, page 599]). *For all $u_1,\dots,u_m, v_1,\dots,v_m \in V$, one has*

$$\left|\det\begin{bmatrix}\langle u_1,v_1\rangle & \cdots & \langle u_1,v_m\rangle \\ \vdots & & \vdots \\ \langle u_m,v_1\rangle & \cdots & \langle u_m,v_m\rangle\end{bmatrix}\right|^2 \le \Gamma(u_1,\dots,u_m)\Gamma(v_1,\dots,v_m), \tag{2.1}$$

where $\Gamma(u_1,\dots,u_m)$ is the Gramian of the vectors $u_1,\dots,u_m$. The equality in (2.1) holds if and only if $\operatorname{span}\{u_1,\dots,u_n\} = \operatorname{span}\{v_1,\dots,v_n\}$, or $\{u_1,\dots,u_m\}$ is linearly dependent, or $\{v_1,\dots,v_m\}$ is linearly dependent.

Lemma 2.2. *Let $\{e_1,\dots,e_n\}$ be an orthonormal system of vectors in V, then, for any vectors $x_1,\dots,x_m \in V$, one has that*

$$A := \left\{x_1 - \sum_{k=1}^{n}\langle x_1,e_k\rangle e_k,\dots,x_m - \sum_{k=1}^{n}\langle x_m,e_k\rangle e_k\right\} \tag{2.2}$$

is linearly dependent, if and only if $B := \{x_1,\dots,x_m,e_1,\dots,e_n\}$ is linearly dependent.

Proof. Let A be linearly dependent, then clearly B is also linearly dependent. Conversely, let B be linearly dependent, then since $\{e_1,\dots,e_n\}$ is linearly independent, there exist $c_1,\dots,c_n \in \mathbb{k}$ and $b_1,\dots,b_m \in \mathbb{k}$ not all zero, such that

$$\sum_{k=1}^{m} b_k x_k + \sum_{k=1}^{n} c_k e_k = 0, \tag{2.3}$$

which for some $d_k \in \mathbb{k}$ can be rewritten as follows:

$$\sum_{i=1}^{n} b_i\left(x_i - \sum_{j=1}^{n}\langle x_i,e_j\rangle e_j\right) + \sum_{k=1}^{n} d_k e_k = 0. \tag{2.4}$$

Hence, we get

$$\sum_{i=1}^{n} b_i \left\langle x_i - \sum_{j=1}^{n} \langle x_i, e_j \rangle e_j, e_k \right\rangle + d_k = 0, \quad 1 \le k \le n. \tag{2.5}$$

Further, for $i = 1, \ldots, m$, we have

$$\left\langle x_i - \sum_{j=1}^{n} \langle x_i, e_j \rangle e_j, e_k \right\rangle = 0. \tag{2.6}$$

Hence from (2.5), we get $d_k = 0$, $k = 1, \ldots, n$. Consequently, from (2.4) we obtain

$$\sum_{i=1}^{m} b_i \left(x_i - \sum_{j=1}^{n} \langle x_i, e_j \rangle e_j \right) = 0, \tag{2.7}$$

and because at least one of $b_1, \ldots, b_m$ is nonzero, it is derived that

$$\left\{ x_1 - \sum_{j=1}^{n} \langle x_1, e_j \rangle e_j, \ldots, x_m - \sum_{j=1}^{n} \langle x_m, e_j \rangle e_j \right\} \tag{2.8}$$

is linearly dependent. □

Lemma 2.3. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal system of vectors in V and $x_1, \ldots, x_m$ be any vectors in V, then,*

$$\operatorname{span}\{x_1, \ldots, x_m, e_1, \ldots, e_n\} = \operatorname{span}\{\widehat{x}_1, \ldots, \widehat{x}_m\} \oplus \operatorname{span}\{e_1, \ldots, e_n\}, \tag{2.9}$$

where

$$\widehat{u} := u - \sum_{k=1}^{n} \langle u, e_k \rangle e_k. \tag{2.10}$$

Proof. Define the linear mapping

$$T : \operatorname{span}\{x_1, \ldots, x_m, e_1, \ldots, e_n\} \longrightarrow \operatorname{span}\{x_1, \ldots, x_m, e_1, \ldots, e_n\} \tag{2.11}$$

given by $T(u) = \widehat{u}$. It is easy to verify that

$$\operatorname{Im} T = \operatorname{span}\{\widehat{x}_1, \ldots, \widehat{x}_m\}, \qquad \operatorname{Ker} T = \operatorname{span}\{e_1, \ldots, e_n\}, \tag{2.12}$$

which completes the proof. □

Theorem 2.4. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal system of vectors in V and let $\{l_1, l_2\} \subset V$ be linear independent, such that $\{l_1, l_2\} \perp \{e_1, \ldots, e_n\}$, then for all $x, y \in V$ the following inequality holds:*

$$\left|S_n(x,y)\right|^2 \leq S_n(x,x)S_n(y,y) - R(x,y;l_1,l_2), \tag{2.13}$$

where

$$R(x,y;l_1,l_2) := \frac{\left|\langle x,l_1\rangle\langle y,l_2\rangle - \langle x,l_2\rangle\langle y,l_1\rangle\right|^2}{\|l_1\|^2\|l_2\|^2 - |\langle l_1,l_2\rangle|^2} \geq 0, \tag{2.14}$$

and $S_n(x,y)$ is given as in (1.2). *The equality in* (2.13) *holds if and only if* $\operatorname{span}\{x, y, e_1, \ldots, e_n\} = \operatorname{span}\{l_1, l_2, e_1, \ldots, e_n\}$, *or $\{x, y, e_1, \ldots, e_n\}$ is linearly dependent.*

Proof. If we apply inequality (2.1) for $n = 2$ by choosing $u_1 = \widehat{x}$, $u_2 = \widehat{y}$, $v_1 = \widehat{l}_1$, and $v_2 = \widehat{l}_2$ and taking into consideration that for all $u, v \in V$, we have

$$\begin{aligned}
\langle \widehat{u}, \widehat{v}\rangle &= \left\langle u - \sum_{i=1}^{n}\langle u,e_i\rangle e_i,\, v - \sum_{j=1}^{n}\langle v,e_j\rangle e_j \right\rangle \\
&= \langle u,v\rangle - \sum_{j=1}^{n}\langle e_j,v\rangle\langle u,e_j\rangle - \sum_{i=1}^{n}\langle u,e_i\rangle\langle e_i,v\rangle + \sum_{k=1}^{n}\langle u,e_k\rangle\langle v,e_k\rangle \\
&= \langle u,v\rangle - \sum_{k=1}^{n}\langle u,e_k\rangle\langle e_k,v\rangle = S_n(u,v),
\end{aligned} \tag{2.15}$$

then we get

$$\left|\det\begin{bmatrix} S_n(x,l_1) & S_n(x,l_2) \\ S_n(y,l_1) & S_n(y,l_2)\end{bmatrix}\right|^2 \leq \begin{vmatrix} S_n(x,x) & S_n(x,y) \\ S_n(y,x) & S_n(y,y)\end{vmatrix} \begin{vmatrix} S_n(l_1,l_1) & S_n(l_1,l_2) \\ S_n(l_2,l_1) & S_n(l_2,l_2)\end{vmatrix}. \tag{2.16}$$

Now, from the condition $\{l_1, l_2\} \perp \{e_1, \ldots, e_n\}$, the following is derived:

$$S_n(x,l_i) = \langle x,l_i\rangle, \quad S_n(y,l_i) = \langle y,l_i\rangle, \quad i = 1,2, \tag{2.17}$$

$$S_n(l_i,l_j) = \langle l_i,l_j\rangle, \quad i,j = 1,2. \tag{2.18}$$

From the condition "$\{l_1, l_2\}$ is linear independent", it follows that $\|l_1\|^2\|l_2\|^2 - |\langle l_1,l_2\rangle|^2 \neq 0$. Now, if we set (2.17) and (2.18) in (2.16), we readily get the conclusion. Finally, according to Lemma 2.1 we have that the equality in (2.16) holds if and only if

$$\operatorname{span}\{\widehat{x}, \widehat{y}\} = \operatorname{span}\left\{\widehat{l}_1, \widehat{l}_2\right\}, \tag{2.19}$$

or $\{\hat{x},\hat{y}\}$ is linearly dependent, or $\{\hat{l}_1,\hat{l}_2\}$ is linearly dependent. Now, since $\hat{l}_1,\hat{l}_2,\hat{x},\hat{y}\perp\{e_1,\ldots,e_n\}$, (2.19) can be rewritten as follows:

$$\operatorname{span}\{\hat{x},\hat{y}\}\oplus\operatorname{span}\{e_1,\ldots,e_n\}=\operatorname{span}\left\{\hat{l}_1,\hat{l}_2\right\}\oplus\operatorname{span}\{e_1,\ldots,e_n\}. \tag{2.20}$$

Moreover, according to Lemma 2.3, we have that

$$\begin{aligned}\operatorname{span}\{\hat{x},\hat{y}\}\oplus\operatorname{span}\{e_1,\ldots,e_n\}&=\operatorname{span}\{x,y,e_1,\ldots,e_n\},\\ \operatorname{span}\left\{\hat{l}_1,\hat{l}_2\right\}\oplus\operatorname{span}\{e_1,\ldots,e_n\}&=\operatorname{span}\{l_1,l_2,e_1,\ldots,e_n\}.\end{aligned} \tag{2.21}$$

Combining (2.20) with (2.21), we conclude that (2.19) is equivalent to

$$\operatorname{span}\{x,y,e_1,\ldots,e_n\}=\operatorname{span}\{l_1,l_2,e_1,\ldots,e_n\}. \tag{2.22}$$

From the conditions of this theorem, we clearly have that $\{\hat{l}_1,\hat{l}_2\}=\{l_1,l_2\}$ is linearly independent and since $\{l_1,l_2\}\perp\{e_1,\ldots,e_n\}$ we obtain that $\{l_1,l_2,e_1,\ldots,e_n\}$ is also linear independent. Finally, according to Lemma 2.2, we have that $\{\hat{x},\hat{y}\}$ is linearly dependent if and only if $\{x,y,e_1,\ldots,e_n\}$ is linearly dependent, which completes the proof. □

Corollary 2.5. *Assuming that the conditions of Theorem 2.4 hold and $x,y\notin\operatorname{span}\{e_1,\ldots,e_n\}$, then one has*

$$|S_n(x,y)|\le\sqrt{S_n(x,x)}\sqrt{S_n(y,y)}-\frac{R(x,y;l_1,l_2)}{2\sqrt{S_n(x,x)}\sqrt{S_n(y,y)}}. \tag{2.23}$$

The equality in (2.23) *holds if and only if $\{x,y,e_1,\ldots,e_n\}$ is linearly dependent.*

Proof. From the condition $x,y\notin\operatorname{span}\{e_1,\ldots,e_n\}$, we obtain that $S_n(x,x),S_n(y,y)\neq 0$. Hence, (2.23) can be rewritten as follows:

$$|S_n(x,y)|\le\sqrt{S_n(x,x)}\sqrt{S_n(y,y)}-\sqrt{1-\frac{R(x,y;l_1,l_2)}{S_n(x,x)S_n(y,y)}}. \tag{2.24}$$

Furthermore, from (2.13) and (2.14), we have

$$0\le\frac{R(x,y;l_1,l_2)}{S_n(x,x)S_n(y,y)}\le 1. \tag{2.25}$$

Consequently, we can apply the elementary inequality

$$\sqrt{1-t}\le 1-\frac{1}{2}t,\quad t\in(-\infty,1], \tag{2.26}$$

for

$$t = \frac{R(x,y;l_1,l_2)}{S_n(x,x)S_n(y,y)}, \tag{2.27}$$

to obtain

$$\sqrt{1 - \frac{R(x,y;l_1,l_2)}{S_n(x,x)S_n(y,y)}} \leq 1 - \frac{R(x,y;l_1,l_2)}{2S_n(x,x)S_n(y,y)}. \tag{2.28}$$

Combining (2.24) with (2.28), we get the desired result.

Letting $\operatorname{span}\{x,e_1,\ldots,e_n\} = \operatorname{span}\{y,e_1,\ldots,e_n\}$, we easily derive that $|S_n(x,y)| = \sqrt{S_n(x,x)}\sqrt{S_n(y,y)}$ and $\langle x,l_1\rangle\langle y,l_2\rangle - \langle x,l_2\rangle\langle y,l_1\rangle = 0$, hence the equality in (2.23) holds. Conversely, it is clear that the equality in (2.23) holds if and only if the equalities in (2.24) and (2.28) hold. That is

$$\operatorname{span}\{x,y,e_1,\ldots,e_n\} = \operatorname{span}\{l_1,l_2,e_1,\ldots,e_n\}, \tag{2.29}$$

$$\langle x,l_1\rangle\langle y,l_2\rangle - \langle x,l_2\rangle\langle y,l_1\rangle = 0. \tag{2.30}$$

Now, from (2.29) it follows that for some $x_1,x_2,y_1,y_2 \in \mathbb{k}$,

$$x = x_1 l_1 + x_2 l_2 + \sum_{k=1}^{n}\langle x,e_k\rangle e_k, \tag{2.31}$$

$$y = y_1 l_1 + y_2 l_2 + \sum_{k=1}^{n}\langle y,e_k\rangle e_k. \tag{2.32}$$

Putting (2.31) and (2.32) in (2.30), we get after some algebraic calculations,

$$(x_1y_2 - x_2y_1)\left(\langle l_1,l_1\rangle\langle l_2,l_2\rangle - |\langle l_1,l_2\rangle|^2\right) = 0, \tag{2.33}$$

and since $\|l_1\|^2\|l_2\|^2 - |\langle l_1,l_2\rangle|^2 \neq 0$, we conclude that $x_1y_2 - x_2y_1 = 0$.

Therefore, there is $t \neq 0$ in $\mathbb{k}$ such that,

$$y_1 = tx_1, \qquad y_2 = tx_2. \tag{2.34}$$

Putting (2.34) in (2.32), dividing the result by t and finally subtracting this result from (2.31), we conclude that $\{x,y,e_1,\ldots,e_n\}$ is linearly dependent. □

Remark 2.6. The main result in [11] is an identity, which by setting $e = z/\|z\|$ can be equivalently written in the following form:

$$\begin{aligned}\left(\|x\|^2 - |\langle x,e\rangle|^2\right)\left(\|y\|^2 - |\langle y,e\rangle|^2\right) - |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle|^2 \\ = \left(\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2\right) \times \inf_{-\infty<s,t<+\infty} \|e - sx - ty\|^2.\end{aligned} \tag{2.35}$$

It is hard to find a positive lower bound of the term $\inf_{-\infty<s,t<+\infty}\|e - sx - ty\|^2$. Therefore, it is difficult to derive a refinement of inequality (1.1) like (2.13) for $n = 1$ from the above identity.

3. A Refinement of Dragomir's Inequality

The main result of this section is a refinement of Dragomir's inequality (1.10).

Theorem 3.1. *Let e be a unit vector in V. If ϕ, γ, Φ, and Γ are real or complex numbers with $\phi \neq \Phi$, $\gamma \neq \Gamma$, and x, y are vectors in V satisfying the following conditions:*

$$\operatorname{Re}\langle \Phi e - x, x - \phi e\rangle \geq 0, \qquad \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0, \tag{3.1}$$

then for all nonproportional vectors $l_1, l_2 \in V$ such that $\{e\} \perp \{l_1, l_2\}$, the following inequality holds

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \leq \frac{1}{4}|\Phi - \phi||\Gamma - \gamma| - \frac{2R(x,y;l_1,l_2)}{|\Phi - \phi||\Gamma - \gamma|}, \tag{3.2}$$

where $R(x,y;l_1,l_2)$ is as given in (2.14).

Proof. We distinguish two cases.

Case 1. Let either x or $y \in \operatorname{span}\{e\}$. Without loss of generality, let us assume that $y \in \operatorname{span}\{e\}$. Then from the condition $\{e\} \perp \{l_1, l_2\}$, we have that $\{y\} \perp \{l_1, l_2\}$. Thus, $\langle y, l_1\rangle = \langle y, l_2\rangle = 0$. So $R(x,y;l_1,l_2) = 0$. Consequently, (3.2) reduces to inequality (1.10).

Case 2. Let $x, y \notin \operatorname{span}\{e\}$, then we can apply inequality (2.23) by $n = 1$ to obtain

$$\begin{aligned}|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \leq \left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2}\left(\|y\|^2 - |\langle y,e\rangle|^2\right)^{1/2} \\ - \frac{R(x,y;l_1,l_2)}{2\left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2}\left(\|y\|^2 - |\langle y,e\rangle|^2\right)^{1/2}}.\end{aligned} \tag{3.3}$$

Furthermore, from the conditions (3.1), we have the following (see [3]):

$$\|x\|^2 - |\langle x,e\rangle|^2 \leq \frac{1}{4}|\Phi - \phi|^2, \qquad \|y\|^2 - |\langle y,e\rangle|^2 \leq \frac{1}{4}|\Gamma - \gamma|^2. \tag{3.4}$$

Finally, combining (3.3) with (3.4) leads to the asserted inequality (3.2). □

Now, working similar as above we can show the following result, which we will use in the next section, to obtain refinements of some known integral inequalities.

Theorem 3.2. *Let e be a unit vector in V. If ϕ, Φ are real or complex numbers with $\phi \neq \Phi$, and x, y are vectors in V satisfying the following condition:*

$$\operatorname{Re}\langle \Phi e - x, x - \phi e \rangle \geq 0, \tag{3.5}$$

then for all nonproportional vectors $l_1, l_2 \in V$ such that $\{e\} \perp \{l_1, l_2\}$, the following inequality holds:

$$|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}|\Phi - \phi|\left(\|y\|^2 - |\langle y, e\rangle|^2\right)^{1/2} - \frac{R(x, y; l_1, l_2)}{|\Phi - \phi|\left(\|y\|^2 - |\langle y, e\rangle|^2\right)^{1/2}}. \tag{3.6}$$

Remark 3.3. Based on inequality (2.23), one can improve, in a way similar to Theorem 3.1, all results related to Grüss inequality as in [1, 4–8]. For example, in [1, pages 333-334], [7, page 2751], and [4, page 90], Dragomir obtained the following inequality by using Arcel's inequality:

$$\begin{aligned}\left(\|x\|^2 - |\langle x, e\rangle|^2\right)^{1/2}&\left(\|y\|^2 - |\langle y, e\rangle|^2\right)^{1/2} \\ &\leq \frac{1}{4}|\Phi - \phi||\Gamma - \gamma| - \left|\frac{\Phi + \phi}{2} - |\langle x, e\rangle|\right|\left|\frac{\Gamma + \gamma}{2} - |\langle y, e\rangle|\right|,\end{aligned} \tag{3.7}$$

which is used to derive the following refinement of (1.10):

$$\begin{aligned}&|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| \\ &\quad\leq \frac{1}{4}|\Phi - \phi||\Gamma - \gamma| - \left|\frac{\Phi + \phi}{2} - |\langle x, e\rangle|\right|\left|\frac{\Gamma + \gamma}{2} - |\langle y, e\rangle|\right|.\end{aligned} \tag{3.8}$$

Now, if we combine the inequalities (3.3), (3.7), and (3.4), we get the following improvement of (3.8):

$$\begin{aligned}|\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| &\leq \frac{1}{4}|\Phi - \phi||\Gamma - \gamma| \\ &\quad - \left|\frac{\Phi + \phi}{2} - |\langle x, e\rangle|\right|\left|\frac{\Gamma + \gamma}{2} - |\langle y, e\rangle|\right| - \frac{2R(x, y; l_1, l_2)}{|\Phi - \phi||\Gamma - \gamma|}.\end{aligned} \tag{3.9}$$

4. A Refinement of Grüss Inequality and Applications

First we will use the results of Section 3 to obtain improvements of inequalities (1.3), (1.6), and (1.7).

Theorem 4.1. *Let $f, g \in L_2(a,b)$ be bounded on (a,b) and let $i, j \in L_2(a,b)$ be not proportional and so that $\int_a^b i(x)dx = \int_a^b j(x)dx = 0$. Then, provided that f, g are nonconstant functions, the following inequalities hold:*

$$|T(f,g)| \leq \sqrt{T(f,f)T(g,g)} - \frac{R(f,g;i,j)}{2\sqrt{T(f,f)T(g,g)}}, \tag{4.1}$$

$$|T(f,g)| \leq \frac{1}{2}(\Gamma - \gamma)\sqrt{T(g,g)} - \frac{R(f,g;i,j)}{(\Gamma-\gamma)\sqrt{T(g,g)}}, \tag{4.2}$$

$$|T(f,g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Phi - \phi) - \frac{2R(f,g;i,j)}{(\Gamma-\gamma)(\Phi-\phi)}, \tag{4.3}$$

where

$$R(f,g;i,j) := \frac{\left(\int_a^b f(x)i(x)dx \int_a^b g(x)j(x)dx - \int_a^b f(x)j(x)dx \int_a^b g(x)i(x)dx\right)^2}{(b-a)^2\left(\int_a^b i^2(x)dx \int_a^b j^2(x)dx - \left(\int_a^b i(x)j(x)dx\right)^2\right)},$$
$$\gamma = \inf_{x\in(a,b)} f(x), \qquad \Gamma = \sup_{x\in(a,b)} f(x), \qquad \phi = \inf_{x\in(a,b)} g(x), \qquad \Phi = \sup_{x\in(a,b)} g(x), \tag{4.4}$$

and $T(f,g)$ is the Chebyshev functional defined by (1.4).

Proof. Applying inequality (2.23) on $L_2(a,b)$ for $n = 1$ as well as the inequalities (3.6) and (3.2) by choosing $e = 1/\sqrt{b-a}$, $x = (1/\sqrt{b-a})f$, $y = (1/\sqrt{b-a})g$, $l_1 = i$, $l_2 = j$, we easily get the required inequalities. □

Note that all known trapezoid-Grüss, midpoint-Grüss, and Ostrowski-Grüss type inequalities, which are proved by applying the inequalities (1.3), (1.6), and (1.7), can be improved by using the inequalities of Theorem 4.1. For example, in [12, page 39] we can see the following trapezoid-Grüss type inequality:

$$\left|\frac{1}{b-a}\int_a^b h(x)dx - \frac{h(a)+h(b)}{2} + \frac{(b-a)}{12}(h'(b)-h'(a))\right| \leq \frac{1}{24\sqrt{5}}(b-a)^2(\Gamma-\gamma), \tag{4.5}$$

where h is a twice differentiable mapping on (a,b) such that $\gamma \leq h''(x) \leq \Gamma$, for all $x \in (a,b)$.

In the following, we apply inequality (4.2) to derive a refinement of inequality (4.5).

Proposition 4.2. *Let h, γ, and Γ be as above, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b h(x)dx - \frac{h(a)+h(b)}{2} + \frac{(b-a)}{12}(h'(b)-h'(a))\right| \\ &\quad\leq \frac{(b-a)^2(\Gamma-\gamma)}{24\sqrt{5}} - \frac{\sqrt{5}}{5(\Gamma-\gamma)}\left(\frac{h'(b)+h'(a)}{2} - \frac{h(b)-h(a)}{b-a}\right)^2.\end{aligned} \tag{4.6}$$

Proof. Let $f, g, i, j : [a,b] \to \mathbb{R}$ be functions defined by $f(x) = h''(x)$, $g(x) = (x-a)(b-x)$, $i(x) = (x-a)(b-x) - (1/6)(b-a)^2$, and $j(x) = x - (a+b)/2$, then it is easy to verify that

$$\begin{gathered}
T(f,g) = \frac{1}{6}\left(h'(b) - h'(a)\right)(b-a) - (h(b) - h(a)) + \frac{2}{b-a}\int_a^b h(x)dx, \\
T(g,g) = \frac{1}{180}(b-a)^4 , \\
\int g(x)j(x)dx = 0, \qquad \int_a^b g(x)i(x)dx = \frac{1}{180}(b-a)^5, \\
\int_a^b f(x)j(x)dx = \left(h'(b) + h'(a)\right)\frac{b-a}{2} - (h(b) - h(a)), \\
\int_a^b i^2(x)dx = \frac{1}{180}(b-a)^5, \qquad \int_a^b j^2(x)dx = \frac{1}{12}(b-a)^3, \qquad \int_a^b i(x)j(x)dx = 0.
\end{gathered} \tag{4.7}$$

Finally, since equalities $\int_a^b i(x)dx = 0$, $\int_a^b j(x)dx = 0$ hold, we can apply inequality (4.2), using the above relations to get the required inequality. □

In [13, page 167] we can see the following Ostrowski-Grüss type inequality for all $t \in (a,b)$:

$$|T(h,t)| \le \frac{1}{4\sqrt{3}}(\Gamma - \gamma)(b-a), \quad t \in (a,b), \tag{4.8}$$

where h is a differentiable function on (a,b) such that $\gamma \le h'(x) \le \Gamma$, for all $x \in (a,b)$, and $T(h,t)$ is defined by

$$T(h,t) = h(t) - \frac{1}{b-a}\int_a^b h(x)dx - \frac{h(b) - h(a)}{b-a}\left(t - \frac{a+b}{2}\right). \tag{4.9}$$

We propose improvement of this result as follows.

Proposition 4.3. *Let h be as above, then,*

$$\begin{aligned}
|T(h,t)| \le \frac{1}{4\sqrt{3}}(\Gamma - \gamma)(b-a) &- \frac{6\sqrt{3}(t-a)(b-t)(b-a)}{(\Gamma - \gamma)\left((b-a)^2 - 3(t-a)(b-t)\right)} \\
&\times \left(\left(\frac{h(t) - h(a)}{t-a} - \frac{h(b) - h(t)}{b-t}\right)\frac{b-a}{6} - T(h,t)\right)^2.
\end{aligned} \tag{4.10}$$

Proof. If we apply inequality (4.2) by choosing

$$f(x) = h'(x), \qquad g(x) = \begin{cases} x-a, & \text{if } a \le x < t, \\ x-b, & \text{if } t \le x < b, \end{cases} \tag{4.11}$$

$$i(x) = \begin{cases} \dfrac{1}{t-a}, & \text{if } a \le x < t, \\ -\dfrac{1}{b-t}, & \text{if } t \le x < b, \end{cases} \tag{4.12}$$

$$j(x) = g(x) - \left(t - \frac{a+b}{2}\right) \tag{4.13}$$

and take into account that, after some calculations, we obtain

$$T(f,g) = h(t) - \frac{1}{b-a}\int_a^b h(x)dx - \frac{h(b)-h(a)}{b-a}\left(t - \frac{a+b}{2}\right), \tag{4.14}$$

$$T(g,g) = \frac{1}{12}(b-a)^2, \tag{4.15}$$

$$\int_a^b f(x)i(x)dx = \frac{h(t)-h(a)}{t-a} - \frac{h(b)-h(t)}{b-t}, \qquad \int_a^b g(x)i(x)dx = \frac{b-a}{2}, \tag{4.16}$$

$$\int_a^b f(x)j(x)dx = (b-a)h(t) - \int_a^b h(x)dx - \left(t - \frac{a+b}{2}\right)(h(b)-h(a)) = T(h,t), \tag{4.17}$$

$$\int_a^b g(x)j(x)dx = \frac{1}{12}(b-a)^3, \tag{4.18}$$

$$\int_a^b i^2(x)dx = \frac{b-a}{(t-a)(b-t)}, \tag{4.19}$$

$$\int_a^b j^2(x)dx = \frac{1}{12}(b-a)^3, \qquad \int_a^b i(x)j(x)dx = \frac{b-a}{2}, \tag{4.20}$$

and we easily derive the asserted inequality. □

According to [13, page 168], the following inequality holds:

$$\left|\frac{1}{b-a}\int_a^b h(x)dx - h\left(\frac{a+b}{2}\right)\right| \le \frac{1}{4\sqrt{3}}(\Gamma-\gamma)(b-a). \tag{4.21}$$

If we apply inequality (4.10) for $t = (a+b)/2$, we get the following refinement of the previous inequality.

Corollary 4.4. *Let h be as in Proposition 4.3, then,*

$$\left|\frac{1}{b-a}\int_a^b h(x)dx - h\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{4\sqrt{3}}(\Gamma-\gamma)(b-a) - \frac{6\sqrt{3}}{(\Gamma-\gamma)(b-a)}\left(\frac{1}{b-a}\int_a^b h(x)dx - \frac{1}{3}\left(h(a)+h\left(\frac{a+b}{2}\right)+h(b)\right)\right)^2. \tag{4.22}$$

It can now be observed that (4.19) can be written as follows:

$$\left(\frac{1}{b-a}\int_a^b h(x)dx - \frac{1}{3}\left(h(a)+h\left(\frac{a+b}{2}\right)+h(b)\right)\right)^2 \leq \frac{1}{72}(\Gamma-\gamma)^2(b-a)^2 - \frac{(\Gamma-\gamma)(b-a)}{6\sqrt{3}}\left|\frac{1}{b-a}\int_a^b h(x)dx - h\left(\frac{a+b}{2}\right)\right|, \tag{4.23}$$

or equivalently

$$\left|\frac{1}{b-a}\int_a^b h(x)dx - \frac{1}{3}h(a)+h\left(\frac{a+b}{2}\right)+h(b)\right| \leq \frac{(\Gamma-\gamma)(b-a)}{6\sqrt{2}}\left[1-\frac{2\sqrt{3}}{(\Gamma-\gamma)(b-a)}\left|\frac{1}{b-a}\int_a^b h(x)dx - h\left(\frac{a+b}{2}\right)\right|\right]^{1/2}, \tag{4.24}$$

which, by using inequality (2.26), leads to the following result.

Corollary 4.5. *Let h be as above, then,*

$$\left|\frac{1}{b-a}\int_a^b h(x)dx - \frac{1}{3}\left(h(a)+h\left(\frac{a+b}{2}\right)+h(b)\right)\right| \leq \frac{(\Gamma-\gamma)(b-a)}{6\sqrt{2}} - \frac{1}{\sqrt{6}}\left|\frac{1}{b-a}\int_a^b h(x)dx - h\left(\frac{a+b}{2}\right)\right|. \tag{4.25}$$

Now, we will use again inequality (4.2) to give another improvement of inequality (4.8).

Proposition 4.6. *Let h be as in Proposition 4.3, then for all $t \in [a,b]$,*

$$|T(h,t)| \leq \frac{1}{4\sqrt{3}}(\Gamma-\gamma)(b-a) - \frac{2\sqrt{3}(a-t)^2(b-t)^2|a+b-2t|K(h,t)}{(\Gamma-\gamma)(b-a)^3}, \tag{4.26}$$

where

$$K(h,t)=\begin{cases}\left(\dfrac{h(2t-b)-h(a)}{2t-b-a}-\dfrac{h(b)-h(2t-b)}{2b-2t}\right)^2, & \text{if } \dfrac{a+b}{2}\le t\le b,\\[2ex] \left(\dfrac{h(2t-a)-h(a)}{2t-2a}-\dfrac{h(b)-h(2t-a)}{a+b-2t}\right)^2, & \text{if } a\le t<\dfrac{a+b}{2}.\end{cases} \tag{4.27}$$

Proof. Let us choose f, g, and i as given in (4.11) and (4.12). We distinguish the following two cases.

Case 1 ($(a+b)/2\le t\le b$). Then, by choosing

$$j(x):=\begin{cases}\dfrac{1}{2t-b-a}, & \text{if } a\le x<2t-b,\\[2ex] -\dfrac{1}{2b-2t}, & \text{if } 2t-b\le x<b,\end{cases} \tag{4.28}$$

it can be easily verified that

$$\int_a^b j(x)dx=0. \tag{4.29}$$

So, we can apply inequality (4.2), by using (4.12), (4.14), (4.15), and (4.16) as well as

$$\begin{gathered}\int_a^b g(x)j(x)dx=0,\\ \int_a^b f(x)j(x)dx=\frac{h(2t-b)-h(a)}{2t-b-a}-\frac{h(b)-h(2t-b)}{2b-2t},\\ \int_a^b i(x)j(x)dx=\frac{b-a}{2(t-a)(b-t)},\qquad \int_a^b j^2(x)dx=\frac{b-a}{(2t-b-a)(2b-2t)}\end{gathered} \tag{4.30}$$

to obtain the desired inequality (4.22).

Case 2 ($a\le t<(a+b)/2$). Then, by choosing

$$j(x):=\begin{cases}\dfrac{1}{2t-2a}, & \text{if } a\le x<2t-a,\\[2ex] -\dfrac{1}{a+b-2t}, & \text{if } 2t-a\le x<b,\end{cases}\quad t<\frac{a+b}{2} \tag{4.31}$$

we easily derive the following:

$$\int_a^b j(x)dx = 0, \qquad \int_a^b g(x)j(x)dx = 0,$$
$$\int_a^b f(x)j(x)dx = \frac{h(2t-a)-h(a)}{2t-2a} - \frac{h(b)-h(2t-a)}{a+b-2t}, \tag{4.32}$$
$$\int_a^b i(x)j(x)dx = \frac{b-a}{2(t-a)(b-t)}, \qquad \int_a^b j^2(x)dx = \frac{b-a}{(2t-2a)(a+b-2t)}.$$

Finally, application of inequality (4.2) using the above relations leads to the claimed result.

□

5. Refinements of Discrete Grüss Inequality

In this section, some refinements of the discrete version of Grüss inequality (1.8), as well as, an additive reverse of the Schwarz inequality are provided.

Theorem 5.1. *Let a_k, b_k, $k = 1, \ldots, n$ be real numbers so that $a \le a_k \le A$, $b \le b_k \le B$ for $k = 1, \ldots, n$, where $a < A$ and $b < B$. Let x_1, y_k be real numbers such that the vectors $(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in \mathbb{R}^n$ are not proportional and*

$$\sum_{k=1}^n x_k = \sum_{k=1}^n y_k = 0. \tag{5.1}$$

Then,

$$\begin{aligned} \left| \frac{1}{n}\sum_{k=1}^n a_k b_k - \frac{1}{n}\sum_{k=1}^n a_k \cdot \frac{1}{n}\sum_{k=1}^n b_k \right| &\le \frac{1}{4}(A-a)(B-b) \\ &- \frac{2}{(A-a)(B-b)n^2} \frac{\left(\sum_{k=1}^n x_k a_k \cdot \sum_{k=1}^n y_k b_k - \sum_{k=1}^n x_k b_k \cdot \sum_{k=1}^n y_k a_k\right)^2}{\sum_{k=1}^n x_k^2 \cdot \sum_{k=1}^n y_k^2 - \left(\sum_{k=1}^n x_k y_k\right)^2}. \end{aligned} \tag{5.2}$$

Proof. Define the vectors $x := (1/\sqrt{n})(a_1, \ldots, a_n)$, $y := (1/\sqrt{n})(b_1, \ldots, b_n)$, $e := (1/\sqrt{n})(1, \ldots, 1)$, $l_1 := (x_1, \ldots, x_n)$, $l_2 := (y_1, \ldots, y_n)$ in the Euclidian inner product space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. Then, we have

$$\begin{aligned} \langle Ae - x, x - ae \rangle &= \sum_{k=1}^n \frac{(A-a_k)(a_k-a)}{n} \ge 0, \\ \langle Be - y, y - be \rangle &= \sum_{k=1}^n \frac{(B-b_k)(b_k-b)}{n} \ge 0, \end{aligned} \tag{5.3}$$

and $\langle e, l_1 \rangle = \langle e, l_2 \rangle = 0$. Hence, we can apply inequality (3.2) of Theorem 3.1 for the vectors x, y, e, l_1, and l_2, as given above, to complete the proof. □

Corollary 5.2. *Let a_k, b_k be as is in Theorem 5.1, then for all $p, q, r, s \in \{1, \dots, n\}$,*

$$\left|\frac{1}{n}\sum_{k=1}^{n} a_k b_k - \frac{1}{n}\sum_{k=1}^{n} a_k \cdot \frac{1}{n}\sum_{k=1}^{n} b_k\right| \le \frac{1}{4}(A-a)(B-b) - \frac{2\left(\left(a_p - a_q\right)\left(b_r - b_s\right) - \left(b_p - b_q\right)\left(a_r - a_s\right)\right)^2}{\varepsilon(p,q,r,s)n^2(A-a)(B-b)}, \tag{5.4}$$

where

$$\varepsilon(p,q;r,s) = \begin{cases} 4, & \text{if } \{p,q\} \cap \{r,s\} = \emptyset, \\ 3, & \text{if } \{p,q\} \cap \{r,s\} \neq \emptyset. \end{cases} \tag{5.5}$$

Proof. If $p = q$ or $r = s$ or $\{p,q\} = \{r,s\}$, then inequality (5.4) reduces to (1.8). If $p \neq q$, $r \neq s$, and $\{p,q\} \neq \{r,s\}$, then by choosing the vectors $(x_1, \dots, x_n)$, $(y_1, \dots, x_n)$ such that

$$x_i = \begin{cases} 1, & \text{if } i = p, \\ -1, & \text{if } i = q, \\ 0, & \text{if } i \neq p, q, \end{cases} \qquad y_i = \begin{cases} 1, & \text{if } i = r, \\ -1, & \text{if } i = s, \\ 0, & \text{if } i \neq r, s, \end{cases} \tag{5.6}$$

it follows that these vectors satisfy the conditions of Theorem 5.1. Hence, we can apply Theorem 5.1 to obtain the desired result. □

Now, if we apply Corollary 5.2 by choosing $a_n = b_n = 0$ and $q = s = n$, we immediately obtain the following result.

Corollary 5.3. *Let a_k, b_k be real numbers so that $a \le a_k \le A$, $b \le b_k \le B$ for all $k = 1, \dots, n-1$, where $a < A$ and $b < B$. Then for all $p, r \in \{1, \dots, n-1\}$, one has the following inequality:*

$$\left|\frac{1}{n}\sum_{i=1}^{n-1} a_k b_k - \frac{1}{n}\sum_{k=1}^{n-1} a_i \cdot \frac{1}{n}\sum_{k=1}^{n-1} b_i\right| \le \frac{1}{4}(A-a)(B-b) - \frac{2\left(a_p b_r - b_p a_r\right)^2}{3n^2(A-a)(B-b)}. \tag{5.7}$$

Corollary 5.4. *Let $0 \le a_1 \le a_2 \le \cdots \le a_{n-1}$ and $b_1 \ge b_2 \ge \cdots \ge b_{n-1} \ge 0$, then*

$$\left|\frac{1}{n}\sum_{k=1}^{n-1} a_k b_k - \frac{1}{n}\sum_{k=1}^{n-1} a_k \cdot \frac{1}{n}\sum_{k=1}^{n-1} b_k\right| \le \frac{1}{4}(a_{n-1} - a_1)(b_1 - b_{n-1}) - \frac{2\left(a_{n-1} b_1 - a_1 b_{n-1}\right)^2}{3n^2(a_{n-1} - a_1)(b_1 - b_{n-1})}. \tag{5.8}$$

Proof. If we apply inequality (5.7) by choosing $a = a_1$, $A = a_{n-1}$, $b = b_{n-1}$, $B = b_1$ and $p = n-1$, $r = 1$, we directly get the desired inequality. □

Now, we will use inequality (5.7) to derive an additive reverse of the Schwarz inequality.

Applying $\binom{n-1}{2}$ times the inequality (5.7), namely for all pairs of integers (p, r) with $1 \le p < r \le n-1$, adding the resulting inequalities, using the Lagrange identity, and finally dividing the resulting inequality by $\binom{n-1}{2}$, then,

$$\left| \frac{1}{n}\sum_{k=1}^{n-1} a_k b_k - \frac{1}{n}\sum_{k=1}^{n-1} a_k \frac{1}{n}\sum_{k=1}^{n-1} b_k \right| \le \frac{1}{4}(A-a)(B-b) - \frac{4}{3(n-1)(n-2)n^2} \frac{\sum_{k=1}^{n-1} a_k^2 \sum_{k=1}^{n-1} b_k^2 - \left(\sum_{k=1}^{n-1} a_k b_k\right)^2}{(A-a)(B-b)}. \tag{5.9}$$

Now, if we solve inequality (5.9) with respect to $\sum_{k=1}^{n-1} a_k^2 \cdot \sum_{k=1}^{n-1} b_k^2 - (\sum_{k=1}^{n-1} a_k b_k)^2$ and then replace n by $n+1$, we obtain the following reverse of the Schwarz inequality.

Proposition 5.5. *Let a_k, b_k, $k = 1, \ldots, n$ be real numbers so that $a \le a_k \le A$, $b \le b_k \le B$ for all $k = 1, \ldots, n$, where $a < A$ and $b < B$. then,*

$$\begin{aligned} &\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k \right)^2 \\ &\quad \le \frac{3}{4} n(n-1)(n+1)^2 (A-a)(B-b) \\ &\quad\quad \times \left(\frac{1}{4}(A-a)(B-b) - \left| \frac{1}{n+1}\sum_{k=1}^{n} a_k b_k - \frac{1}{n+1}\sum_{k=1}^{n} a_k \frac{1}{n+1}\sum_{k=1}^{n} b_k \right| \right) \\ &\quad \le \frac{3}{16} n(n-1)(n+1)^2 (A-a)^2 (B-b)^2. \end{aligned} \tag{5.10}$$

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