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Research Article

Weighted Rellich Inequality on H-Type Groups and Nonisotropic Heisenberg Groups

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We prove a sharp weighted Rellich inequality associated with a class of Greiner-type vector fields on H-type groups. We also obtain some weighted Hardy- and Rellich-type inequalities on nonisotropic Heisenberg groups. As an application, we get a Rellich-Sobolev-type inequality on Heisenberg groups.

1. Introduction

The study of partial differential operators constructed from noncommutative vector fields satisfying the Hörmander condition [1] has had much development. We refer to [2, 3] and the references therein for a systematic account of the study. Recently there have been considerable interests in studying the sub-Laplacians as square sums of vector fields that are not invariant or do not satisfy the Hörmander condition. Among the examples of such sub-Laplacians are the Grushin operators, Greiner-type operators, and the sub-Laplacian constructed by Kohn [4]. Those noninvariant sub-Laplacians also appear naturally in complex analysis. In [5] Beals et al. considered the CR operators $\{Z_j, \overline{Z_j}\}_{j=1}^{n}$ on $\mathbb{R}^{2n+1}$ as boundary of the complex domain

$$\left\{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} > \left(\sum_{j=1}^{n} |z_j|^2 \right)^k \right\}, \tag{1.1}$$

where $Z_j = (1/2)X_j - iY_j$,

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial t}, \tag{1.2}$$

and k is a positive integer. The space $\mathbb{R}^{2n+1}$ has a natural structure of a Heisenberg group, but the vector fields are not left or right invariant. In [6] Zhang and Niu studied the Greiner vector fields on $\mathbb{R}^{2n+1}$ for general parameter $k \geq 1$ and got the corresponding Hardy-type inequality. Note that for nonintegral k these vector fields do not satisfy the Hörmander condition and are not smooth.

H-type groups were introduced by Kaplan [7] as direct generalizations of Heisenberg groups. In [8] we define a class of vector fields X (see (2.5)) on H-type groups generalizing the vector fields (1.2) considered in [5, 6] and find the fundamental solution of the corresponding p-Laplacian with singularity at the identity element. Also we prove a Hardy-type inequality associated to X.

The goal of the present paper is to continue our study on analysis associated with Greiner-type vector fields X introduced in [8]. We will throughout study the Rellich inequality which is a generalization of Hardy inequality to higher-order derivatives. They have various applications in the study of elliptic and parabolic PDEs. The classical Rellich inequality in $\mathbb{R}^n$ states that for $n \geq 5$ and $\phi \in C_0^\infty(R^n \setminus \{0\})$,

$$\int_{R^n} |\Delta\phi(x)|^2 dx \geq \left(\frac{n(n-4)}{4}\right)^2 \int_{R^n} \frac{|\phi(x)|^2}{|x|^4} dx. \tag{1.3}$$

The constant $n^2(n-4)^2/16$ is sharp and is never achieved. Davies and Hinz [9] generalized (1.3) to the L^p case and showed that for any $p \in (1, n/2)$ there holds

$$\int_{R^n} |\Delta\phi(x)|^p dx \geq \left(\frac{n(p-1)(n-2p)}{p^2}\right)^p \int_{R^n} \frac{|\phi(x)|^p}{|x|^{2p}} dx. \tag{1.4}$$

See also the works in [10–13].

In a recent paper [14] Yang obtains an L^2 version of Rellich inequality associated with the left-invariant vector fields in the setting of Heisenberg group, there is a similar L^2 Rellich inequality on the general carnot group in [15] with different approach. A natural question is to find an L^p version of Rellich inequality in this general setting. The main purpose of the present paper is to prove some weighted L^p-Rellich inequalities associated with Greiner-type vector fields on H-type groups. Our approach depends on the fundamental solution of the corresponding square operator and the weighted Hardy inequality proved in our earlier paper [8]. We prove also some weighted L^p Hardy and Rellich inequalities on nonisotropic Heisenberg groups by a different method caused by the absence of the explicit representation formula for fundamental solution. As an application, we get a Rellich-Sobolev-type inequality on Heisenberg groups.

The plan of the paper is as follows. In Section 2 we introduce a class of Greiner-type vector field and prove the corresponding weighted L^p-Rellich inequality on H-type groups; Section 3 is devoted to the proof of weighted Hardy- and Rellich-type inequalities on nonisotropic Heisenberg groups and a Rellich-Sobolev-type inequality on Heisenberg groups.

2. Rellich Inequality on H-Type Groups

We recall that a simply connected nilpotent group $\mathbb{G}$ is of Heisenberg type, or of H-type, if its Lie algebra $\mathfrak{n} = V \oplus \mathfrak{t}$ is of step two, $[V, V] \subset \mathfrak{t}$, $[\mathfrak{t}, V] = 0$ and if there is an inner product $\langle \cdot\, , \, \cdot \rangle$ on $\mathfrak{n}$ such that the linear map

$$J : \mathfrak{t} \longrightarrow \operatorname{End}(V), \tag{2.1}$$

defined by the condition

$$\langle J_t(u), v \rangle = \langle t, [u, v] \rangle \quad u, v \in V,\ z \in \mathfrak{t}, \tag{2.2}$$

satisfies

$$J_t^2 = -|t|^2 \mathbf{Id} \tag{2.3}$$

for all $t \in \mathfrak{t}$, where $|t|^2 = \langle t, t \rangle$.

Groups of H-type were introduced by Kaplan in [7] as direct generalizations of Heisenberg groups, and they have been studied quite extensively; see [16–19] and the references therein.

We identify $\mathbb{G}$ with its Lie algebra $\mathfrak{n}$ via the exponential map, $\exp : V \oplus \mathfrak{t} \to \mathbb{G}$. The Lie group product is given by

$$(u, t)(v, s) = \left(u + v, t + s + \frac{1}{2}[u, v] \right). \tag{2.4}$$

For $g \in \mathbb{G}$, we write $g = (z(g), t(g)) \in V \oplus \mathfrak{t}$.

In [8] the authors constructed a family of Greiner-type vector fields $X = \{X_1, \dots, X_m\}$ on $\mathbb{G}$:

$$X_j = \partial_j + \frac{1}{2} k |z|^{2k-2} \partial_{[z, e_j]}, \quad j = 1, 2 \dots, m, \tag{2.5}$$

where $\partial_j = \partial_{e_j}$, $\partial_{[z, e_j]}$ are the directional derivatives, $\{e_j\}_{j=1,\dots,m}$ is an orthonormal basis of V, and $k \geq 1$ is a fixed parameter. If $k = 1$, X_j are the left-invariant vector fields defined by the orthonormal basis $\{e_j\}_{j=1}^m$ on V. The corresponding degenerate p-sub-Laplacian is

$$L_{p,k} u = \operatorname{div}_X \left(|\nabla_X u|^{p-2} \nabla_X u \right), \tag{2.6}$$

where

$$\nabla_X u = (X_1 u, \ldots, X_m u), \qquad \mathrm{div}_X(u_1, \ldots, u_m) = \sum_{j=1}^{m} X_j u_j. \tag{2.7}$$

We denote

$$\Delta_X = L_{2,k} = \sum_{i=1}^{m} X_i^2. \tag{2.8}$$

There is a one-parameter group of dilations $\{\delta_\lambda : \lambda > 0\}$ on $\mathbb{G}$:

$$\delta_\lambda : (z,t) \longmapsto (w,s) = \left(\lambda z, \lambda^{2k} t\right). \tag{2.9}$$

The volume element is transformed by δ via

$$dw\, ds = \lambda^Q dz\, dt, \tag{2.10}$$

where $Q := m+2kq$ with $m = \dim V$ and $q = \dim \mathfrak{t}$. Q will be called the degree of homogeneity and is the homogeneous dimension if $k = 1$. We define a corresponding *homogeneous norm* by

$$d = d(z,t) := \left(|z|^{4k} + 16|t|^2\right)^{1/4k}. \tag{2.11}$$

Remark 2.1. Note that when $p = 2$ and $k = 1$, $L_{p,k}$ becomes the sub-Laplacian $\Delta_{\mathbb{G}}$ on the H-type group $\mathbb{G}$. If $p = 2$, $q = 1$, and $k = 2, 3, \ldots$, $L_{p,k}$ is a Greiner operator (see [5, 20]). Also we note that vector fields $\{X_j\}$ are neither left nor right invariant and they do not satisfy Hörmander's condition for $k > 1$, $k \notin Z$.

The main results in [8] are the following.

Theorem 2.2. *Let $\mathbb{G}$ be an H-type group, $k \geq 1$, and $Q = m + 2kq$. Then for $1 < p < \infty$,*

$$\Gamma_p = \begin{cases} C_p\, d^{(p-Q)/(p-1)}, & p \neq Q, \\ C_Q \log \dfrac{1}{d}, & p = Q \end{cases} \tag{2.12}$$

is a fundamental solution of $L_{p,k}$ with singularity at the identity element $0 \in \mathbb{G}$. Here $d(z,t)$ is defined in (2.11),

$$C_p = \frac{p-1}{p-Q}(\sigma_p)^{-1/(p-1)}, \qquad C_Q = -(\sigma_Q)^{-1/(Q-1)},$$
$$\sigma_p = \left(\frac{1}{4}\right)^{q-1/2} \frac{\pi^{(q+m)/2}\Gamma\left(((2k-1)p+m)/4k\right)}{\Gamma(m/2)\Gamma\left(((2k-1)p+Q)/4k\right)}. \tag{2.13}$$

Theorem 2.3. *Let $\mathbb{G}$ be an H-type group, $k \geq 1$, and $Q = m + 2kq$. Suppose that $1 < p < Q + \alpha$. Then the following Hardy-type inequality holds for $\Phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$:*

$$\int_{\mathbb{G}} d^\alpha |\nabla_X \Phi|^p \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_{\mathbb{G}} d^\alpha \left(\frac{|z|}{d}\right)^{(2k-1)p} \left|\frac{\Phi}{d}\right|^p, \tag{2.14}$$

where $\nabla_X f = (X_1 f, X_2 f, \ldots, X_m f)$ is the gradient defined by the vector fields (2.5). *Moreover, the constant $((Q+\alpha-p)/p)^p$ is sharp.*

Based on the above two theorems, we will prove the following L^p version of weighted Rellich inequality on H-type groups.

Theorem 2.4. *Let $\mathbb{G}$ be an H-type group, $k \geq 1$, and $Q = m + 2kq$. Suppose that $1 < p < +\infty$, $2 - Q < \alpha < \min\{(p-1)(Q-2), (Q-2)\}$. Then the following Rellich-type inequality holds for $\Phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$:*

$$\int_{\mathbb{G}} d^{\alpha+4k(p-1)} |z|^{(2-4k)(p-1)} |\Delta_X \phi|^p \geq \left(\frac{(Q+\alpha-2)[(p-1)(Q-2)-\alpha]}{p^2}\right)^p \int_{\mathbb{G}} d^{\alpha-4k} |z|^{4k-2} |\phi|^p. \tag{2.15}$$

Moreover, the constant $((Q+\alpha-2)[(p-1)(Q-2)-\alpha]/p^2)^p$ is sharp.

Remark 2.5. In the abelian case $\mathbb{G} = R^n$, the above result recovers the classical Rellich inequality (1.4) with $d(x) = |x|$, $\alpha = 2 - 2p$ under the condition $2p < n$.

Remark 2.6. When we take $p = 2$ and $k = q = 1$, our inequality (2.15) is just inequality (1.5) in [14] and inequality (5.2) in [15] for Heisenberg groups.

Now we prove Theorem 2.4.

Proof. We denote $u = d^{2-Q}$, where d is as in (2.11), then

$$\begin{aligned}\Delta_X(d^\alpha) &= \Delta_X\left(\left(d^{2-Q}\right)^{\alpha/(2-Q)}\right) = \Delta_X\left(u^{\alpha/(2-Q)}\right)\\ &= \frac{\alpha}{2-Q} u^{\alpha/(2-Q)-1} \Delta_X u + \frac{\alpha}{2-Q}\left(\frac{\alpha}{2-Q} - 1\right) u^{\alpha/(2-Q)-2} |\nabla_X u|^2,\end{aligned} \tag{2.16}$$

for $\alpha > 2 - Q$; we have

$$\begin{aligned}\int_{\mathbb{G}} |\phi|^p \Delta_X d^\alpha &= \int_{\mathbb{G}} \frac{\alpha}{2-Q} u^{\alpha/(2-Q)-1} \Delta_X u |\phi|^p + \int_{\mathbb{G}} \frac{\alpha}{2-Q}\left(\frac{\alpha}{2-Q} - 1\right) u^{\alpha/(2-Q)-2} |\nabla_X u|^2 |\phi|^p\\ &= \int_{\mathbb{G}} \frac{\alpha}{2-Q}\left(\frac{\alpha}{2-Q} - 1\right) u^{\alpha/(2-Q)-2} |\nabla_X u|^2 |\phi|^p\\ &= \alpha(\alpha - 2 + Q) \int_{\mathbb{G}} d^{\alpha-2} |\nabla_X d|^2 |\phi|^p\end{aligned} \tag{2.17}$$

since $C_2 \cdot u$ is the fundamental solution of Δ_X at the origin by Theorem 2.2; however the left-hand side is

$$\begin{aligned}\int_{\mathbb{G}} |\phi|^p \Delta_X d^\alpha &= -\int_{\mathbb{G}} p|\phi|^{p-2}\phi\nabla_X\phi\nabla_X(d^\alpha)\\ &= p\int_{\mathbb{G}} d^\alpha\Big((p-1)|\phi|^{p-2}|\nabla_X\phi|^2 + |\phi|^{p-2}\phi\Delta_X\phi\Big).\end{aligned} \tag{2.18}$$

Thus, by (2.17), (2.18), and the corresponding weighted Hardy inequality (Theorem 2.3), we have

$$\begin{aligned}&\alpha(\alpha-2+Q)\int_{\mathbb{G}} d^{\alpha-2}|\nabla_X d|^2|\phi|^p - p\int_{\mathbb{G}} d^\alpha|\phi|^{p-2}\phi\Delta_X\phi\\ &\quad= p(p-1)\int_{\mathbb{G}} d^\alpha|\nabla_X\phi|^2|\phi|^{p-2} = \frac{4p(p-1)}{p^2}\int_{\mathbb{G}} d^\alpha\left|\nabla_X\left(|\phi|^{p/2}\right)\right|^2\\ &\quad\geq \frac{4(p-1)}{p}\left(\frac{Q+\alpha-2}{2}\right)^2\int_{\mathbb{G}} d^\alpha|\phi|^p\frac{|\nabla_X d|^2}{d^2} = \frac{(p-1)}{p}(Q+\alpha-2)^2\int_{\mathbb{G}} d^{\alpha-2}|\nabla_X d|^2|\phi|^p,\end{aligned} \tag{2.19}$$

this implies that

$$\begin{aligned}-p\int_{\mathbb{G}} d^\alpha|\phi|^{p-2}\phi\Delta_X\phi &\geq \left[\frac{(p-1)}{p}(Q+\alpha-2)^2 - \alpha(\alpha-2+Q)\right]\int_{\mathbb{G}} d^{\alpha-2}|\nabla_X d|^2|\phi|^p\\ &= (Q+\alpha-2)\left[\frac{(p-1)}{p}(Q+\alpha-2)-\alpha\right]\int_{\mathbb{G}} d^{\alpha-2}|\nabla_X d|^2|\phi|^p\\ &= \frac{(Q+\alpha-2)\left[(p-1)(Q-2)-\alpha\right]}{p}\int_{\mathbb{G}} d^{\alpha-2}|\nabla_X d|^2|\phi|^p.\end{aligned} \tag{2.20}$$

Applying Hölder's inequality, we get

$$\begin{aligned}&\frac{(Q+\alpha-2)\left[(p-1)(Q-2)-\alpha\right]}{p^2}\int_{\mathbb{G}} d^{\alpha-2}|\nabla_X d|^2|\phi|^p\\ &\quad\leq \left(\int_{\mathbb{G}} d^{\alpha-2}|\nabla_X d|^2|\phi|^p\right)^{(p-1)/p}\left(\int_{\mathbb{G}} d^\alpha\left|\frac{\nabla_X d}{d}\right|^{2(1-p)}|\Delta_X\phi|^p\right)^{1/p}.\end{aligned} \tag{2.21}$$

Noticing that

$$|\nabla_X d| = \left(\frac{|z|}{d}\right)^{2k-1}, \tag{2.22}$$

we have thus proved (2.15).

It remains to show the sharpness of the constant $((Q+\alpha-p)\ /p)^p$. Let B be any constant satisfying the inequality

$$\int_{\mathbb{G}} d^{\alpha-2+2p}|\nabla_X d|^{2-2p}\left|\Delta_X \phi\right|^p \geq B\int_{\mathbb{G}} d^{\alpha-2}|\nabla_X d|^2\left|\phi\right|^p. \tag{2.23}$$

We will prove that $B \leq ((Q+\alpha-2)[(p-1)(Q-2)-\alpha]/p^2)^p$. The idea is to find functions $\{u_j\}_{j=1}^{\infty}$ so that the difference between the left- and right-hand sides approximates to 0. Given any positive integer j it is elementary that there exists ψ_j in $C_0^{\infty}(0,\infty)$ such that $\operatorname{supp}\psi_j = [2^{-j-1},2]$, $\psi_j(x)=1$ on $[2^{-j},1]$, and $|\psi_j'(x)| \leq C2^j$, $|\psi_j''(x)| \leq C2^{2j}$ on $[2^{-j-1},2^{-j}]$, where C is a constant independent of j. Let

$$u_j(z,t) = d(z,t)^{(2-Q-\alpha)/p-1/j}\psi_j(d(z,t)). \tag{2.24}$$

Clearly $u_j \in C^{\infty}(G\setminus\{0\})$ which is radial. Denoting C_j by $(2-Q-\alpha)/p-1/j$, it is easy to see that

$$\nabla_X u_j = \begin{cases} 0, & 0\leq d<2^{-j-1},\ \text{or}\ d>2\\ C_j d^{C_j-1}\nabla_X d, & 2^{-j}<d<1,\end{cases}$$
$$\Delta_X u_j = \begin{cases} 0, & 0\leq d<2^{-j-1},\ \text{or}\ \ d>2\\ C_j(C_j+Q-2)d^{C_j-2}|\nabla_X d|^2, & 2^{-j}<d<1.\end{cases} \tag{2.25}$$

Here we used the fact that

$$d\Delta_X d = (Q-1)|\nabla_X d|^2. \tag{2.26}$$

The left-hand side of the above inequality (2.23) is

$$\text{LHS} = \int_{\mathbb{G}} = \int_{2^{-j}<d<1} + \int_{2^{-j-1}<d\leq 2^{-j}} + \int_{1\leq d<2} = \int_{2^{-j}<d<1} + I + II. \tag{2.27}$$

The first integration is

$$\begin{aligned}&\int_{2^{-j}<d<1} d^{\alpha-2+2p}|\nabla_X d|^{2-2p}\left|\Delta_X\phi\right|^p\\ &\quad=\left(\frac{Q+\alpha-2}{p}+\frac{1}{j}\right)^p\left(\frac{(p-1)(Q-2)-\alpha}{p}-\frac{1}{j}\right)^p\int_{2^{-j}<d<1} d^{-Q-p/j}|\nabla_X d|^2,\end{aligned} \tag{2.28}$$

which can be evaluated as the last computations in the proof of Theorem 1 in [8], and is

$$\left(\frac{Q+\alpha-2}{p}+\frac{1}{j}\right)^p\left(\frac{(p-1)(Q-2)-\alpha}{p}-\frac{1}{j}\right)^p C_0(2^p-1)j, \tag{2.29}$$

where C_0 is a positive constant. Similarly,

$$\text{RHS} = B\int_{2^{-j}<d<1} + III + IV. \tag{2.30}$$

The first integration is precisely the same as above and is

$$B\int_{2^{-j}<d<1} = BC_0(2^p - 1)j. \tag{2.31}$$

It is easy to estimate the error terms which they are all bounded:

$$I, II, III, IV \leq C. \tag{2.32}$$

The inequality (2.23) now becomes

$$\begin{aligned}&\left(\frac{Q+\alpha-2}{p}+\frac{1}{j}\right)^p\left(\frac{(p-1)(Q-2)-\alpha}{p}-\frac{1}{j}\right)^p C_0(2^p-1)j + I + II\\&\quad\geq BC_0(2^p-1)j + III + IV.\end{aligned} \tag{2.33}$$

Dividing both sides by j and letting $j \to \infty$ prove our claim. □

The following is an immediate consequence of Theorem 2.4, which is an extension of the uncertainty principle inequality.

Corollary 2.7. *Let $\mathbb{G}$ be a Heisenberg-type group, $k \geq 1$, and $Q = m + 2kq$. Suppose that $1 < p < +\infty$, $1/p + 1/q = 1$, $2 - Q < \alpha < \min\{(p-1)(Q-2), (Q-2)\}$. Then for all $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$ the following inequality holds:*

$$\begin{aligned}&\left(\int_{\mathbb{G}} d^{\alpha+2(p-1)}|\nabla_X d|^{2(1-p)}|\Delta_X\phi|^p\right)^{1/p}\left(\int_{\mathbb{G}} d^{(2-\alpha)q/p}|\nabla_X d|^{2q/p}|\phi|^q\right)^{1/q}\\&\quad\geq \frac{(Q+\alpha-2)\left[(p-1)(Q-2)-\alpha\right]}{p^2}\int_{\mathbb{G}}|\phi|^2.\end{aligned} \tag{2.34}$$

Remark 2.8. We mention that when $p = q = 2$ and $k = 1$, our inequality (2.34) goes back to inequality (5.7) in [15] in the setting of H-type group.

We end this section with the following Rellich-type inequality on the polarizable group which can be proved by the same method if we noted Theorem 2.15 and Proposition 2.18 in [21] and the weighted Hardy inequality (Theorem 4.1) in [22].

Theorem 2.9. *Let $\mathbb{G}$ be a polarizable group with homogeneous dimension $Q > 4$. Suppose that $1 < p < +\infty$, $2 - Q < \alpha < \min\{(p-1)(Q-2), (Q-2)\}$. Then the following inequality holds for all $\phi \in C_0^\infty(\mathbb{G} \setminus \{0\})$:*

$$\int_{\mathbb{G}} N^\alpha \left|\frac{\nabla_{\mathbb{G}} N}{N}\right|^{2(1-p)} |\Delta_{\mathbb{G}}\phi|^p \geq \left(\frac{(Q+\alpha-2)[(p-1)(Q-2)-\alpha]}{p^2}\right)^p \int_{\mathbb{G}} N^\alpha \left|\frac{\nabla_{\mathbb{G}} N}{N}\right|^2 |\phi|^p. \tag{2.35}$$

Here $N = u^{1/(2-Q)}$ is the homogeneous norm associated with the fundamental solution u for the Kohn sub-Laplacian. Moreover, the constant $((Q+\alpha-2)[(p-1)(Q-2)-\alpha]/p^2)^p$ is sharp.

3. Rellich Inequality on Nonisotropic Heisenberg Groups

Let $a = (a_1, a_2, \ldots, a_n)$, $a_1, a_2, \ldots, a_n > 0$. Let $H = H(a) = \mathbb{R}^{2n} \oplus \mathbb{R}$ be the corresponding nonisotropic Heisenberg group with the product

$$(\zeta, t) \circ (\eta, s) = \left(\zeta + \eta, t + s + 2\sum_{j=1}^n a_j(\zeta_{j+n}\eta_j - \zeta_j\eta_{j+n})\right). \tag{3.1}$$

We consider the following nonisotropic Greiner-type vector fields:

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2k a_j y_j \left(\sum_{j=1}^n a_j |z_j|^2\right)^{k-1} \frac{\partial}{\partial t}, \\ Y_j &= \frac{\partial}{\partial y_j} - 2k a_j x_j \left(\sum_{j=1}^n a_j |z_j|^2\right)^{k-1} \frac{\partial}{\partial t}, \end{aligned} \quad j = 1, \ldots, n, \tag{3.2}$$

where $(x, y, t) \in \mathbb{R}^{2n} \oplus \mathbb{R}$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$. Denote

$$\begin{aligned} \nabla_H u &= (X_1 u, \ldots, X_n u, Y_1 u, \ldots, Y_n u), \\ \operatorname{div}_H(u_1, \ldots, u_{2n}) &= \sum_{j=1}^n (X_j u_j + Y_j u_{n+j}), \\ d &= \left(\left(\sum_{j=1}^n a_j |z_j|^2\right)^{2k} + t^2\right)^{1/4k}, \\ Q &= 2\sum_{j=1}^n a_j + 2k. \end{aligned} \tag{3.3}$$

For further information on the nonisotropic Heisenberg group, see, for example, [23, 24].

In this section we prove firstly the L^p Hardy-type inequality associated with the Greiner-type vector fields (3.2) on the nonisotropic Heisenberg group H.

Theorem 3.1. *Let $H = H(a)$ be the anisotropic Heisenberg group with $a_j \le 1$, $j = 1, \dots, n$. Let $\alpha \in \mathbb{R}$, $2 \le p < Q + \alpha$ and $\Phi \in C_0^\infty(H \setminus \{0\})$. Then the following inequality is valid:*

$$\int_H d^\alpha |\nabla_L \Phi|^p \ge \left(\frac{Q+\alpha-p}{p}\right)^p \int_H d^{\alpha-2kp} \left(\sum_{j=1}^n a_j^2 |z_j|^2\right)^{p/2} \left(\sum_{j=1}^n a_j |z_j|^2\right)^{(k-1)p} |\Phi|^p, \qquad (3.4)$$

where $d = \left(\left(\sum_{j=1}^n a_j |z_j|^2\right)^{2k} + t^2\right)^{1/4k}$, $Q = 2\sum_{j=1}^n a_j + 2k$.

For the proof of the above inequality, we need the following lemma which can be proved by a similar method in [25].

Lemma 3.2. *Let $w \ge 0$ be a weight function in $\Omega \subset \mathbb{G}$ and $L_{p,k,w} u = \operatorname{div}_H(|\nabla_H u|^{p-2} w \nabla_H u)$. Suppose that for some $\lambda > 0$, there exists $v \in C^\infty(\Omega)$, $v > 0$ such that*

$$-L_{p,k,w} v \ge \lambda g v^{p-1} \qquad (3.5)$$

for some $g \ge 0$, in the sense of distribution acting on nonnegative test functions. Then for any $u \in HW_0^{1,p}(\Omega, w)$, it holds that

$$\int_\Omega |\nabla_H u|^p w \ge \lambda \int_\Omega g |u|^p, \qquad (3.6)$$

where $HW_0^{1,p}(\Omega, w)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\left(\int_\Omega |\nabla_H u|^p w\right)^{1/p}$.

We now prove Theorem 3.1.

Proof. Take $w = d^\alpha$ and $v = d^{(p-Q-\alpha)/p}$. Noting that

$$\begin{aligned}
L_{H,p,w} v &= \sum_{i=1}^n \left[X_i\left(|\nabla_L v|^{p-2} w X_i v\right) + Y_i\left(|\nabla_L v|^{p-2} w Y_i v\right)\right], \\
X_i d &= \frac{\left[\left(\sum_{j=1}^n a_j |z_j|^2\right)^{2k-1} a_i x_i + \left(\sum_{j=1}^n a_j |z_j|^2\right)^{k-1} a_i y_i t\right]}{d^{4k-1}}, \\
Y_i d &= \frac{\left[\left(\sum_{j=1}^n a_j |z_j|^2\right)^{2k-1} a_i y_i - \left(\sum_{j=1}^n a_j |z_j|^2\right)^{k-1} a_i x_i t\right]}{d^{4k-1}}, \\
|\nabla_L d|^2 &= \frac{\left(\sum_{j=1}^n a_j^2 |z_j|^2\right)\left(\sum_{j=1}^n a_j |z_j|^2\right)^{2k-2}}{d^{4k-2}},
\end{aligned} \qquad (3.7)$$

we get

$$-L_{H,p,w}v = -\sum_{i=1}^{n}\left[X_i\left(d^{\alpha}\left|\frac{Q+\alpha-p}{p}d^{-(Q+\alpha)/p}\right|^{p-2}|\nabla_L d|^{p-2}\left(\frac{p-Q-\alpha}{p}\right)d^{-(Q+\alpha)/p}X_i d\right)\right.$$
$$\left.+Y_i\left(d^{\alpha}\left|\frac{Q+\alpha-p}{p}d^{-(Q+\alpha)/p}\right|^{p-2}|\nabla_L d|^{p-2}\left(\frac{p-Q-\alpha}{p}\right)d^{-(Q+\alpha)/p}Y_i d\right)\right] \tag{3.8}$$
$$= \left(\frac{Q+\alpha-p}{p}\right)^{p-1}\{I_1+I_2+I_3+I_4\},$$

where

$$I_1 = \left(\frac{Q+\alpha-pQ}{p}+(2-p)(2k-1)\right)d^{(Q+\alpha-pQ-p)/p+(2-p)(2k-1)}$$
$$\times\left(\sum_{j=1}^{n}a_j^2|z_j|^2\right)^{(p-2)/2}\left(\sum_{j=1}^{n}a_j|z_j|^2\right)^{(p-2)(k-1)}|\nabla_L d|^2;$$
$$I_2 = (p-2)d^{(Q+\alpha-pQ)/p+(2-p)(2k-1)}\left(\sum_{j=1}^{n}a_j^2|z_j|^2\right)^{(p-4)/2}\left(\sum_{j=1}^{n}a_j|z_j|^2\right)^{(p-2)(k-1)}$$
$$\times\sum_{j=1}^{n}a_j^2(x_j\cdot X_j d+y_j\cdot Y_j d);$$
$$I_3 = 2(k-1)(p-2)d^{(Q+\alpha-pQ)/p+(2-p)(2k-1)}\left(\sum_{j=1}^{n}a_j^2|z_j|^2\right)^{(p-2)/2}\left(\sum_{j=1}^{n}a_j|z_j|^2\right)^{(p-2)(k-1)-1}$$
$$\times\sum_{j=1}^{n}a_j(x_j\cdot X_j d+y_j\cdot Y_j d);$$
$$I_4 = d^{(Q+\alpha-pQ)/p+(2-p)(2k-1)}\left(\sum_{j=1}^{n}a_j^2|z_j|^2\right)^{(p-2)/2}\left(\sum_{j=1}^{n}a_j|z_j|^2\right)^{(p-2)(k-1)}\sum_{j=1}^{n}\left(X_j^2 d+Y_j^2 d\right). \tag{3.9}$$

By direct computations we have

$$\sum_{j=1}^{n}a_j(x_j\cdot X_j d+y_j\cdot Y_j d) = \frac{\left(\sum_{j=1}^{n}a_j|z_j|^2\right)^{2k-1}\left(\sum_{j=1}^{n}a_j^2|z_j|^2\right)}{d^{4k-1}},$$
$$\sum_{j=1}^{n}a_j^2(x_j\cdot X_j d+y_j\cdot Y_j d) = \frac{\left(\sum_{j=1}^{n}a_j|z_j|^2\right)^{2k-1}\left(\sum_{j=1}^{n}a_j^3|z_j|^2\right)}{d^{4k-1}},$$

$$\sum_{j=1}^{n}\left(X_j^2 d + Y_j^2 d\right) = \left(\sum_{j=1}^{n} a_j |z_j|^2\right)^{2(k-1)} \frac{\left(2\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} a_j |z_j|^2\right) + \sum_{j=1}^{n} a_j^2 |z_j|^2}{d^{4k-1}}. \tag{3.10}$$

Then

$$-L_{H,p,w} v = \left(\frac{Q+\alpha-p}{p}\right)^{p-1} d^{(Q+\alpha)/p-1-Q+(2k-1)(2-p)} \left(\sum_{j=1}^{n} a_j^2 |z_j|^2\right)^{(p-4)/2} \left(\sum_{j=1}^{n} a_j |z_j|^2\right)^{(k-1)p} \cdot II, \tag{3.11}$$

where

$$\begin{aligned} II = \Bigg\{ &\left(\frac{Q+\alpha}{p} - Q + (2k-1)(2-p)\right)\left(\sum_{j=1}^{n} a_j^2 |z_j|^2\right)^2 + (p-2)\left(\sum_{j=1}^{n} a_j |z_j|^2\right)\left(\sum_{j=1}^{n} a_j^3 |z_j|^2\right) \\ &+ 2(k-1)(p-2)\left(\sum_{j=1}^{n} a_j^2 |z_j|^2\right)^2 + \left(\sum_{j=1}^{n} a_j^2 |z_j|^2\right) \\ &\times \left\{(2k-1)\left(\sum_{j=1}^{n} a_j^2 |z_j|^2\right) + \left(2\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} a_j |z_j|^2\right)\right\}\Bigg\}. \end{aligned} \tag{3.12}$$

Using Cauchy inequality

$$\left(\sum_{j=1}^{n} a_j |z_j|^2\right)\left(\sum_{j=1}^{n} a_j^3 |z_j|^2\right) \geq \left(\sum_{j=1}^{n} a_j^2 |z_j|^2\right)^2 \tag{3.13}$$

and that (since, by assumption, $a_j \leq 1,\ j = 1, \dots, n$)

$$\sum_{j=1}^{n} a_j |z_j|^2 \geq \sum_{j=1}^{n} a_j^2 |z_j|^2, \tag{3.14}$$

we find

$$II \geq \frac{Q+\alpha-p}{p}\left(\sum_{j=1}^{n} a_j^2 |z_j|^2\right)^2, \tag{3.15}$$

so for $p \geq 2$,

$$\begin{aligned} -L_{H,p,w}v &\geq \left(\frac{Q+\alpha-p}{p}\right)^{p}\left(d^{(p-Q-\alpha)/p}\right)^{p-1}d^{\alpha-2kp}\left(\sum_{j=1}^{n}a_j^2|z_j|^2\right)^{p/2}\left(\sum_{j=1}^{n}a_j|z_j|^2\right)^{(k-1)p} \\ &= \left(\frac{Q+\alpha-p}{p}\right)^{p}v^{p-1}d^{\alpha-p}|\nabla_L d|^p. \end{aligned} \tag{3.16}$$

Hence Theorem 3.1 follows from Lemma 3.2 with $\lambda = ((Q+\alpha-p)/p)^p$, $g = d^{\alpha-p}|\nabla_L d|^p$. □

Now it is time to prove the following Rellich inequality on nonisotropic Heisenberg group.

Theorem 3.3. *Let $H = H(a)$ be a nonisotropic Heisenberg group with $a_j \leq 1$, $j = 1,\dots,n$. Suppose that $\alpha \in \mathbb{R}$, $2-Q < \alpha \leq 0$ and $u \in C_0^\infty(H \setminus \{0\})$. Then the following inequality is valid:*

$$\begin{aligned} &\int_H \frac{d^{\alpha-4k+4kp}}{\left(\sum_{j=1}^{n}a_j^2|z_j|^2\right)^{(2k-1)(p-1)}}|\Delta_H u|^p \\ &\quad\geq \left(\frac{((p-1)(Q-2)-\alpha)(Q+\alpha-2)}{p^2}\right)^p\int_H d^{\alpha-4k}\left(\sum_{j=1}^{n}a_j^2|z_j|^2\right)^{2k-1}|u|^p, \end{aligned} \tag{3.17}$$

where $d = \left(\left(\sum_{j=1}^{n}a_j|z_j|^2\right)^{2k}+t^2\right)^{1/4k}$, $Q = 2\sum_{j=1}^{n}a_j + 2k$.

Remark 3.4. If $p = 2$, $\alpha = -2$, and $k = a_i = 1$ $(i = 1,\dots,n)$, then we get the Rellich inequality on the Heisenberg group $\mathbb{H}^n$ with homogeneous dimension $Q = 2n+2$:

$$\int_{\mathbb{H}^n}\frac{d^2}{|z|^2}|\Delta_{\mathbb{H}^n}u|^2 \geq \left(\frac{Q(Q-4)}{4}\right)^2\int_{\mathbb{H}^n}\frac{|z|^2}{d^6}|u|^2. \tag{3.18}$$

We also mention that to our knowledge, even in the special case $k = a_j = 1$, $j = 1,2,\dots,n$, our inequality (3.17) is new.

We now give the proof of Theorem 3.3.

Proof. We have

$$\begin{aligned} \int_H |u|^p\Delta_X d^\alpha &= \alpha\int_H |u|^p\left((\alpha-1)d^{\alpha-2}|\nabla_H d|^2 + d^{\alpha-1}\Delta_H d\right) \\ &= \alpha(\alpha-1)\int_H |u|^p d^{\alpha-2}|\nabla_H d|^2 + \alpha\int_H |u|^p d^{\alpha-1}\Delta_H d. \end{aligned} \tag{3.19}$$

By a direct computation we have

$$X_i d = \frac{\left[\left(\sum_{j=1}^n a_j|z_j|^2\right)^{2k-1} a_i x_i + \left(\sum_{j=1}^n a_j|z_j|^2\right)^{k-1} a_i y_i t\right]}{d^{4k-1}},$$

$$Y_i d = \frac{\left[\left(\sum_{j=1}^n a_j|z_j|^2\right)^{2k-1} a_i y_i - \left(\sum_{j=1}^n a_j|z_j|^2\right)^{k-1} a_i x_i t\right]}{d^{4k-1}},$$

$$|\nabla_H d|^2 = \frac{\left(\sum_{j=1}^n a_j^2|z_j|^2\right)\left(\sum_{j=1}^n a_j|z_j|^2\right)^{2k-2}}{d^{4k-2}},$$

$$\Delta_H d = \left(\sum_{j=1}^n a_j|z_j|^2\right)^{2(k-1)} \frac{\left(2\sum_{j=1}^n a_j\right)\left(\sum_{j=1}^n a_j|z_j|^2\right) + (2k-1)\sum_{j=1}^n a_j^2|z_j|^2}{d^{4k-1}}. \tag{3.20}$$

Noting that $a_i \le 1$ then

$$d\Delta_H d \ge (Q-1)|\nabla_H d|^2; \tag{3.21}$$

we take $\alpha < 0$ and thus deduce from (3.19) that

$$\int_H |u|^p \Delta_X d^\alpha \le \alpha(Q+\alpha-2)\int_H |u|^p d^{\alpha-2}|\nabla_H d|^2. \tag{3.22}$$

On the other hand,

$$\begin{aligned}\int_H |u|^p \Delta_H d^\alpha &= -p\int_H |u|^{p-2} u \nabla_H u \nabla_H (d^\alpha) \\ &= p\int_H d^\alpha \left((p-1)|u|^{p-2}|\nabla_H u|^2 + |u|^{p-2} u \Delta_H u\right).\end{aligned} \tag{3.23}$$

Combining with (3.22), (3.23), and the L^2 weighted Hardy inequality associated with the Greiner-type vector fields (3.2) on the nonisotropic Heisenberg group (3.4), then we only need to do the same steps as in Theorem 2.4 and thus finish the proof of Theorem 3.3. □

The following is also an uncertainty principle type inequality which is an immediate consequence of Theorem 3.3.

Corollary 3.5. *Let $H = H(a)$ be an anisotropic Heisenberg group with $a_j \leq 1,\ j = 1,\dots,n$. Then for $u \in C_0^\infty(H \setminus \{0\}), 1/p + 1/q = 1\ (1 < p < \infty)$, the following inequality holds:*

$$\left(\int_H \left(\frac{d^{4k}}{\left(\sum_{j=1}^n a_j^2 |z_j|^2\right)^{2k-1}}\right)^{p-1} |\Delta_H u|^p\right)^{1/p} \left(\int_H \left(\frac{d^{4k}}{\left(\sum_{j=1}^n a_j^2 |z_j|^2\right)^{2k-1}}\right)^{1/(p-1)} |u|^q\right)^{1/q}$$
$$\geq \frac{(p-1)(Q-2)^2}{p^2} \int_H |u|^2, \tag{3.24}$$

where $d = ((\sum_{j=1}^n a_j |z_j|^{2k})^2 + t^2)^{1/4k}$, $Q = 2\sum_{j=1}^n a_j + 2k$.

By a similar method, we can get the following inequality which does not contain the weight $|\nabla_H d|$, so we omit the proof.

Theorem 3.6. *Let $H = H(a)$ be a nonisotropic Heisenberg group with $a_j \leq 1,\ j = 1,\dots,n$. Let $\alpha \in \mathbb{R},\ 2 - Q < \alpha \leq 0$ and $u \in C_0^\infty(H \setminus \{0\})$. Then the following inequality is valid:*

$$\int_H |z|^\alpha |\Delta_H u|^p \geq \left(\frac{((p-1)(2\sum_{i=1}^n a_i) - \alpha)(2\sum_{i=1}^n a_i + \alpha - 2p)}{p^2}\right)^p \int_H |z|^{\alpha-2p} |u|^p. \tag{3.25}$$

With the help of Theorem 3.6, we can also obtain a Rellich-Sobolev-type inequality on the Heisenberg group.

Corollary 3.7. *Let $H = \mathbb{H}^n$ be the Heisenberg group with homogeneous dimension $Q = 2n + 2$. Suppose that $1 < p < (Q-2)/2,\ 0 \leq s \leq p,\ p_s = p(Q-2s)/(Q-2p)$. Then there exists a positive constant C such that for any $u \in C_0^\infty(H \setminus \{0\})$, the following inequality holds:*

$$\left(\int_H \frac{|u|^{p_s}}{|z|^{2s}}\right)^{1/p_s} \leq C\left(\int_H |\Delta_H u|^p\right)^{1/p}. \tag{3.26}$$

Proof. By Hölder inequality we have

$$\int_H \frac{|u|^{p_s}}{|z|^{2s}} = \int_H \frac{|u|^s}{|z|^{2s}} |u|^{Q(p-s)/(Q-2p)} \leq \left(\int_H \frac{|u|^p}{|z|^{2p}}\right)^{s/p} \left(\int_H |u|^{pQ/(Q-2p)}\right)^{(p-s)/p}. \tag{3.27}$$

Thanks to the Rellich inequality (3.25) with $\alpha = 0$, $a_i = 1$ ($i = 1, 2, \ldots, n$) and the Sobolev inequality on the nilpotent Lie group (Chapter IV Theorem 3.3.1 in [26]) we get

$$\int_H \frac{|u|^{p_s}}{|z|^{2s}} \leq C\left(\int_H |\triangle_H u|^p\right)^{s/p} \left(\int_H |\triangle_H u|^p\right)^{Q(p-s)/p(Q-2p)} = C\left(\int_H |\triangle_H u|^p\right)^{(Q-2s)/(Q-2p)}. \tag{3.28}$$

Thus we obtain the desired result. □

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References

[1] L. Hörmander, "Hypoelliptic second order differential equations," *Acta Mathematica*, vol. 119, pp. 147–171, 1967.

[2] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, vol. 28 of *Mathematical Notes*, Princeton University Press, Princeton, NJ, USA, 1982.

[3] A. Nagel, E. M. Stein, and S. Wainger, "Balls and metrics defined by vector fields. I. Basic properties," *Acta Mathematica*, vol. 155, no. 1-2, pp. 103–147, 1985.

[4] J. J. Kohn, "Hypoellipticity and loss of derivatives," *Annals of Mathematics. Second Series*, vol. 162, no. 2, pp. 943–986, 2005, With an appendix by Makhlouf Derridj and David S. Tartakoff.

[5] R. Beals, B. Gaveau, and P. Greiner, "Uniforms hypoelliptic Green's functions," *Journal de Mathématiques Pures et Appliqués*, vol. 77, no. 3, pp. 209–248, 1998.

[6] H. Zhang and P. Niu, "Hardy-type inequalities and Pohozaev-type identities for a class of p-degenerate subelliptic operators and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 54, no. 1, pp. 165–186, 2003.

[7] A. Kaplan, "Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms," *Transactions of the American Mathematical Society*, vol. 258, no. 1, pp. 147–153, 1980.

[8] Y. Jin and G. Zhang, "Fundamental solutions for a class of degenerate p-Laplacian operators and applications to Hardy type inequalities," to appear in *Canadian Journal of Mathematics*.

[9] E. B. Davies and A. M. Hinz, "Explicit constants for Rellich inequalities in $L_p(\Omega)$," *Mathematische Zeitschrift*, vol. 227, no. 3, pp. 511–523, 1998.

[10] Adimurthi, M. Grossi, and S. Santra, "Optimal Hardy-Rellich inequalities, maximum principle and related eigenvalue problem," *Journal of Functional Analysis*, vol. 240, no. 1, pp. 36–83, 2006.

[11] G. Barbatis, "Best constants for higher-order Rellich inequalities in $L^p(\Omega)$," *Mathematische Zeitschrift*, vol. 255, no. 4, pp. 877–896, 2007.

[12] G. Barbatis and A. Tertikas, "On a class of Rellich inequalities," *Journal of Computational and Applied Mathematics*, vol. 194, no. 1, pp. 156–172, 2006.

[13] A. Tertikas and N. B. Zographopoulos, "Best constants in the Hardy-Rellich inequalities and related improvements," *Advances in Mathematics*, vol. 209, no. 2, pp. 407–459, 2007.

[14] Q. Yang, "Best constants in the Hardy-Rellich type inequalities on the Heisenberg group," *Journal of Mathematical Analysis and Applications*, vol. 342, no. 1, pp. 423–431, 2008.

[15] I. Kombe, "Hardy, Rellich and uncertainty principle inequalities on Carnot groups," http://arxiv.org/abs/math/0611850.

[16] M. Cowling, A. Dooley, A. Korányi, and F. Ricci, "An approach to symmetric spaces of rank one via groups of Heisenberg type," *The Journal of Geometric Analysis*, vol. 8, no. 2, pp. 199–237, 1998.

[17] M. Cowling, A. H. Dooley, A. Korányi, and F. Ricci, "H-type groups and Iwasawa decompositions," *Advances in Mathematics*, vol. 87, no. 1, pp. 1–41, 1991.

[18] A. Kaplan, "Lie groups of Heisenberg type," *Rendiconti del Seminario Matematico. Università e Politecnico di Torino*, pp. 117–130, 1983, Conference on Differential Geometry on Homogeneous Spaces (Turin, 1983).

[19] A. Korányi, "Geometric properties of Heisenberg-type groups," *Advances in Mathematics*, vol. 56, no. 1, pp. 28–38, 1985.

[20] P. C. Greiner, "A fundamental solution for a nonelliptic partial differential operator," *Canadian Journal of Mathematics*, vol. 31, no. 5, pp. 1107–1120, 1979.

[21] Z. M. Balogh and J. T. Tyson, "Polar coordinates in Carnot groups," *Mathematische Zeitschrift*, vol. 241, no. 4, pp. 697–730, 2002.

[22] I. Kombe, "Sharp Hardy type inequalities on the Carnot group," http://arxiv.org/abs/math/0501522.

[23] D.-C. Chang and P. Greiner, "Harmonic analysis and sub-Riemannian geometry on Heisenberg groups," *Bulletin of the Institute of Mathematics. Academia Sinica*, vol. 30, no. 3, pp. 153–190, 2002.

[24] D. C. Chang and J. Z. Tie, "A note on Hermite and subelliptic operators," *Acta Mathematica Sinica*, vol. 21, no. 4, pp. 803–818, 2005.

[25] P. Niu, H. Zhang, and Y. Wang, "Hardy type and Rellich type inequalities on the Heisenberg group," *Proceedings of the American Mathematical Society*, vol. 129, no. 12, pp. 3623–3630, 2001.

[26] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and Geometry on Groups*, vol. 100 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, Mass, USA, 1992.