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Research Article

A Note on Essential Components and Essential Weakly Efficient Solutions for Multiobjective Optimization Problems

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The concept of essential component of weakly efficient solution set is introduced first. Then we obtain some sufficient conditions for the existence of an essential component or an essential weakly efficient solution in the weakly efficient solution sets for multiobjective optimization problems.

1. Introduction

In [1–7] and the references therein, the authors have studied the existence and the stability of (vector-valued, set-valued, semi-infinite vector) optimization and multiobjective optimization problems, while the author in [2] shows that most of the weakly efficient solution sets of multiobjective optimization problems (in the sense of Baire category) are stable. By the fact that there are still quite a few weakly efficient solution sets of multiobjective optimization problems which are not stable, in this paper, we discusses the stability of solution set for multiobjective optimization problems from the perspective of essential components.

2. Definitions and Lemmas

Let X be a nonempty compact subset of a Banach space E, $R = (-\infty, +\infty)$

$$f : X \to R^n, \quad \text{where } f = (f_1, f_2, \dots, f_n), \qquad f_i : X \to R \quad \forall i \in \{1, 2, \dots, n\}. \tag{2.1}$$

Consider the following multiobjective optimization problem:

$$\min f(x), \quad \text{s.t. } x \in X. \tag{VMP}$$

Definition 2.1. (1) A point $x^* \in X$ is called a weakly efficient solution to (VMP), if there is no $x \in X$ such that

$$f_i(x) < f_i(x^*), \quad \forall i = 1, 2, \ldots, n. \tag{2.2}$$

(2) A point $x^* \in X$ is called an efficient solution to (VMP), if there is no $x \in X$ $(x \neq x^*)$ such that

$$f_i(x) \leq f_i(x^*), \quad \forall i = 1, 2, \ldots, n. \tag{2.3}$$

$\mathrm{WE}(f, X)$ or $\mathrm{WE}(f)$ is used to denote the weakly efficient solution set of (VMP), and $E(f, X)$ or $E(f)$ is used to denote the efficient solution set of (VMP).

Clearly, $E(f, X) \subset \mathrm{WE}(f, X)$, but the reverse containment may not hold.

Example 2.2 (see [8]). Let $X = [0, 2]$, define

$$\begin{gathered} f_i(x) = x, \quad i \in \{1, 2, \ldots, n-1\}, \\ f_n(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 2 - x, & 1 \leq x \leq 2. \end{cases} \end{gathered} \tag{2.4}$$

Then $E(f, X) = \{0\} \cup (1, 2]$, $\mathrm{WE}(f, X) = [0, 2]$.

It is easy to see that $n = 1$, and (VMP) is just a scalar optimization problem, in this case, we still denote by $\mathrm{WE}(f, X)$ or $\mathrm{WE}(f)$ the set of optimal solution.

Remark 2.3. Let $f = (f_1, f_2, \ldots, f_n) : X \to R^n$, for all $i \in \{1, 2, \ldots, n\}$, one has $\mathrm{WE}(f_i, X) \subset \mathrm{WE}(f, X)$.

Denote

$$Y = \{f \mid f : X \to R^n \text{ continuous}, X \text{ be a nonempty compact subset of a Banach space}\}. \tag{2.5}$$

For any $f = (f_1, f_2, \ldots, f_n)$, $g = (g_1, g_2, \ldots, g_n) \in Y$, define

$$\rho(f, g) = \sum_{i=1}^{n} \sup_{x \in X} |f_i(x) - g_i(x)|. \tag{2.6}$$

Clearly, (Y, ρ) is a complete metric space.

For any $f = (f_1, f_2, \ldots, f_n) \in Y$, by [2], it has been shown that $\mathrm{WE}(f, X) \subset X$ is a nonempty compact subset, and the following Lemma 2.4 is due to [2].

Lemma 2.4. *The mapping $WE : Y \to 2^X$ is upper semicontinuous with nonempty compact values.*

For any $f \in Y$, the component of a point $x \in WE(f,X)$ is the union of all connected subsets of $WE(f,X)$ which contain the point x. From [9, page 356], one knows that components are connected closed subsets of $WE(f,X)$ and thus they are also compact as $WE(f,X)$ is compact. It is easy to see that the components of two distinct points of $WE(f,X)$ either coincide or are disjoint, so that all components constitute a decomposition of $WE(f,X)$ into connected pairwise disjoint compact subsets, that is,

$$WE(f,X) = \bigcup_{\alpha\in\Lambda} C_\alpha(f), \tag{2.7}$$

where Λ is an index set, for any $\alpha \in \Lambda$, $C_\alpha(f)$ is a nonempty connected compact and for any α, $\beta \in \Lambda\ (\alpha \neq \beta)$, $C_\alpha(f) \cap C_\beta(f) = \emptyset$.

Definition 2.5. For $f \in Y$, $C_\alpha(f)$ is called an essential component of $\mathrm{WE}(f,X)$ if, for any open set O containing $C_\alpha(f)$, there exists $\delta > 0$ such that for all $g \in Y$ with $\rho(f,g) < \delta$, $\mathrm{WE}(g,X) \cap O \neq \emptyset$.

The following Definition 2.6 is from [2].

Definition 2.6. For $f \in Y$, $x \in \mathrm{WE}(f,X)$ is said to be an essential weakly efficient solution to (VMP) if, for any open neighborhood $N(x)$ of x in X, there exists an open neighborhood $O(f)$ at f in Y such that $\mathrm{WE}(g,X) \cap N(x) \neq \emptyset$ for all $g \in O(f)$.

Remark 2.7. For $f \in Y$, if $x \in \mathrm{WE}(f,X)$ is an essential weakly efficient solution to (VMP), then the component which contains the point x is an essential component.

Remark 2.8. For $f \in Y$, maybe, there is no essential component in $\mathrm{WE}(f,X)$, and no essential weakly efficient solution in $\mathrm{WE}(f,X)$.

Example 2.9. Let $X = [0,6]$, define

$$f_1(x) = f_2(x) = \cdots = f_n(x) = \begin{cases} 1-x, & x \in [0,1], \\ 0, & x \in (1,2], \\ x-2, & x \in (2,3], \\ 4-x, & x \in (3,4], \\ 0, & x \in (4,5], \\ x-5, & x \in (5,6]. \end{cases} \tag{2.8}$$

Then, $\mathrm{WE}(f,X) = [1,2] \cup [4,5]$, and for all $i \in \{1,2,\ldots,n\}$, $\mathrm{WE}(f,X) = \mathrm{WE}(f_i,X)$. By [3, Theorem 2.3], $\mathrm{WE}(f,X) = \mathrm{WE}(f_i,X)$ has no essential component.

Example 2.10. Let $X = [0,2]$, define

$$f_1(x) = f_2(x) = \cdots = f_n(x) = \begin{cases} x, & x \in [0,1], \\ 2-x, & x \in (1,2]. \end{cases} \tag{2.9}$$

Then, $\mathrm{WE}(f,X) = \{0\} \cup \{2\}$, and for all $i \in \{1,2,\ldots,n\}$, $\mathrm{WE}(f,X) = \mathrm{WE}(f_i,X)$. By [3, Theorem 3.2], $\mathrm{WE}(f,X) = \mathrm{WE}(f_i,X)$ contains no essential weakly efficient solution.

Definition 2.11. Let X be a nonempty convex subset of a Banach space, and $f_i : X \to R$, the function f_i is said to be strongly quasiconvex on X, if

$$f_i(tx_1 + (1-t)x_2) < \max\{f_i(x_1), f_i(x_2)\} \tag{2.10}$$

for all $x_1, x_2 \in X$, $x_1 \neq x_2$, $t \in (0,1)$.

3. Essential Component and Essential Weakly Efficient Solution

Theorem 3.1. *For $f \in Y$, $x^* \in WE(f,X)$, if there is $i \in \{1,2,\ldots,n\}$ such that $WE(f_i,X) = \{x^*\}$, then x^* is an essential weakly efficient solution of* (VMP)*. Hence, the component that contains the point x^* is an essential component.*

Proof. Suppose that $f \in Y$, $x^* \in \mathrm{WE}(f)$, and there exists $i \in \{1,2,\ldots,n\}$ such that $\mathrm{WE}(f_i,X) = \{x^*\}$. For any open neighborhood $N(x^*)$ of x^* in X, there is an open neighborhood $M(x^*)$ of x^* in X such that $\overline{M(x^*)} \subset N(x^*)$, where $\overline{M(x^*)}$ denotes the closure of $M(x^*)$.

Since $f_i(x^*) = \min_{x\in X} f_i(x)$, and $\mathrm{WE}(f_i,X) = \{x^*\}$, then

$$\inf_{x\in X\setminus\overline{M(x^*)}} f_i(x) - f_i(x^*) > 0. \tag{3.1}$$

Take $\delta > 0$ such that

$$\inf_{x\in X\setminus\overline{M(x^*)}} f_i(x) - \delta > f_i(x^*) + \delta. \tag{3.2}$$

Then for any $g = (g_1, g_2, \ldots, g_n) \in Y$ with $\rho(f,g) < \delta$, one has

$$\left|f_i(x) - g_i(x)\right| < \delta, \quad \forall x \in X. \tag{3.3}$$

Then

$$f_i(x) - \delta < g_i(x) < f_i(x) + \delta, \quad \forall x \in X, \tag{3.4}$$

by (3.2) and (3.4), one has

$$\begin{gathered}\min_{x\in\overline{M(x^*)}} g_i(x) \le \min_{x\in\overline{M(x^*)}} f_i(x) + \delta = f_i(x^*) + \delta, \\ f_i(x^*) + \delta < \inf_{x\in X\setminus\overline{M(x^*)}} f_i(x) - \delta \le \inf_{x\in X\setminus\overline{M(x^*)}} g_i(x),\end{gathered} \tag{3.5}$$

therefore

$$\min_{x\in\overline{M(x^*)}} g_i(x) < \inf_{x\in X\setminus\overline{M(x^*)}} g_i(x). \tag{3.6}$$

Then, $\mathrm{WE}(g_i, X) \cap \overline{M(x^*)} \neq \emptyset$, and hence $\mathrm{WE}(g_i, X) \cap N(x^*) \neq \emptyset$, which implies $\mathrm{WE}(g, X) \cap N(x^*) \neq \emptyset$. By Definition 2.6, x^* is an essential weakly efficient solution to (VMP). Hence, the component that contains the point x^* is an essential component. □

Corollary 3.2 (see [2]). *When $n = 1$, for $f \in Y$, if $WE(f) = \{x\}$ is a singleton, then x is an essential optimum solution.*

Lemma 3.3. *Let X be a nonempty compact convex subset of a Banach space, the function $f_i : X \to R$ continuous and strongly quasiconvex, then $WE(f_i, X)$ is a singleton.*

Proof. Suppose $x_1, x_2 \in \mathrm{WE}(f_i, X)$, and $x_1 \neq x_2$. By Definition 2.11,

$$f_i(tx_1 + (1-t)x_2) < \max\{f_i(x_1), f_i(x_2)\} = \min_{x\in X} f_i(x), \quad t \in (0,1), \tag{3.7}$$

which is a contradiction, then $\mathrm{WE}(f_i, X)$ is a singleton. □

By Theorem 3.1 and Lemma 3.3, we have the following Theorem 3.4.

Theorem 3.4. *Let X be a nonempty compact convex subset of a Banach space, for $f = (f_1, f_2, \ldots, f_n) \in Y$, if there exists $i \in \{1, 2, \ldots, n\}$ such that f_i is strongly quasiconvex, then $WE(f, X)$ has an essential weakly efficient solution, consequently, $WE(f, X)$ has an essential component.*

Theorem 3.5. *For $f \in Y$, if $WE(f, X)$ has only one component $C(f)$, then $C(f)$ is an essential component.*

Proof. By Lemma 2.4, the mapping $\mathrm{WE} : Y \to 2^X$ is upper semicontinuous at $f \in Y$, hence, for any open set O with $O \supset \mathrm{WE}(f, X) = C(f)$, there exists $\delta > 0$ such that for all $g \in Y$ with $\rho(f, g) < \delta$, one has $\mathrm{WE}(g, X) \subset O$, and hence, $\mathrm{WE}(g, X) \cap O \neq \emptyset$. By Definition 2.5, $\mathrm{WE}(f, X) = C(f)$ is an essential component. □

Remark 3.6. When $n = 1$, by [3], the optimum solution set $\mathrm{WE}(f, X) = \mathrm{WE}(f_1, X)$ (where $f = f_1$) has an essential component if and only if $\mathrm{WE}(f, X) = \mathrm{WE}(f_1, X)$ is connected. But, when $n \ge 2$, it is not true.

Example 3.7. Let $X = [0,3]$, define

$$f_1(x) = f_2(x) = \cdots = f_{n-1}(x) = x, \quad 0 \le x \le 3,$$
$$f_n(x) = \begin{cases} 1+x, & x \in [0,1], \\ 3-x, & x \in (1,3]. \end{cases} \tag{3.8}$$

$\mathrm{WE}(f,X) = \{0\} \cup [2,3]$ is disconnected; however, $x^* = 0 \in \mathrm{WE}(f,X)$ is an essential weakly efficient solution, $C_1(f) = \{0\}$ is an essential component, and $C_2(f) = [2,3]$ is an essential component.

Example 3.8. Let $X = [0,1]$, define

$$f_1(x) = \begin{cases} x, & x \in \left[0, \dfrac{1}{2}\right], \\ 1-x, & x \in \left(\dfrac{1}{2}, 1\right], \end{cases}$$
$$f_2(x) = \begin{cases} 1-x, & x \in \left[0, \dfrac{1}{2}\right], \\ x, & x \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{3.9}$$

Then $\mathrm{WE}(f,X) = [0,1]$, $\mathrm{WE}(f_1,X) = \{0,1\}$, $\mathrm{WE}(f_2,X) = \{1/2\}$. By Theorem 3.1, $x_1 = 1/2$ is an essential weakly efficient solution. But, $x_2 = 0$ and $x_3 = 1$ are not essential weakly efficient solutions. In fact, for $x_2 = 0$, take $N(0,1/4) = [0,1/4)$, $\overline{N(0,1/4)} = [0,1/4]$, for all $\varepsilon : 0 < \varepsilon < 1$, take $g = (g_1, g_2)$ as the following:

$$g_1(x) = \begin{cases} x + \dfrac{\varepsilon}{2}\left(\dfrac{1}{4} - x\right), & x \in \left[0, \dfrac{1}{4}\right], \\ f_1(x), & x \in \left(\dfrac{1}{4}, 1\right], \end{cases}$$
$$g_2(x) = \begin{cases} f_2(x), & x \in \left[0, \dfrac{3}{4}\right], \\ x - \dfrac{\varepsilon}{2}\left(x - \dfrac{3}{4}\right), & x \in \left(\dfrac{3}{4}, 1\right]. \end{cases} \tag{3.10}$$

Then $\mathrm{WE}(g,X) \neq \emptyset$, and

$$\rho(f,g) = \sup_{x\in X}\left|f_1(x) - g_1(x)\right| + \sup_{x\in X}\left|f_2(x) - g_2(x)\right| < \varepsilon. \tag{3.11}$$

But $\mathrm{WE}(g, X) \cap N(0, 1/4) = \emptyset$. In fact, for all $t \in N(0, 1/4)$, if $t = 0$, $g_1(0) = \varepsilon/8$, $g_2(0) = 1$, take $x^* = 1 \in X$, we have

$$g_1(x^*) = 0 < g_1(0), \qquad g_2(x^*) = 1 - \frac{\varepsilon}{8} < g_2(0). \tag{3.12}$$

If $t \in (0, 1/4)$, $g_1(t) = t + (\varepsilon/2)(1/4 - t)$, $g_2(t) = 1 - t$, take $x^* = 1 - t \in X$, we have

$$\begin{aligned} &g_1(x^*) = 1 - x^* = t < g_1(t), \\ &g_2(x^*) = x^* - \frac{\varepsilon}{2}\left(x^* - \frac{3}{4}\right) = 1 - t - \frac{\varepsilon}{2}\left(\frac{1}{4} - t\right) < g_2(t). \end{aligned} \tag{3.13}$$

Thus $\mathrm{WE}(g, X) \cap N(0, 1/4) = \emptyset$; therefore, $x_2 = 0$ is not an essential weakly efficient solution. Similarly, $x_3 = 1$ is not an essential weakly efficient solution.

4. Conclusions

When $n = 1$, by [3], for $f \in Y$, $\mathrm{WE}(f)$ has an essential component if and only if $\mathrm{WE}(f)$ is connected, and $\mathrm{WE}(f)$ has an essential optimum solution if and only if $\mathrm{WE}(f)$ is a singleton.

When $n \geq 2$, we obtain some sufficient conditions for the existence of an essential component or an essential weakly efficient solution in $\mathrm{WE}(f, X)$. Example 3.7 shows that $\mathrm{WE}(f, X)$ disconnected, but $\mathrm{WE}(f, X)$ has an essential component and $\mathrm{WE}(f, X)$ is not a singleton, but $\mathrm{WE}(f, X)$ has an essential weakly efficient solution. Example 3.8 shows that, if $\mathrm{WE}(f_i, X)$ is not a singleton, for some i, then for any $x \in \mathrm{WE}(f_i, X)$, x is not an essential weakly efficient solution.

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