Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2009, Article ID 868423, 14 pages
doi:10.1155/2009/868423

Research Article

Quadratic-Quartic Functional Equations in RN-Spaces

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Received 20 July 2009; Accepted 3 November 2009

Recommended by Andrea Laforgia

We obtain the general solution and the stability result for the following functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary t-norms $f(2x+y)+f(2x-y)=4[f(x+y)+f(x-y)]+2[f(2x)-4f(x)]-6f(y)$.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1,\cdot)$ be a group and let $(G_2,*,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x\cdot y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$? In other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f:E\to E'$ be a mapping between Banach spaces such that

$$\left\|f(x+y)-f(x)-f(y)\right\|\leq\delta \tag{1.1}$$

for all $x,y\in E$ and some $\delta>0$. Then there exists a unique additive mapping $T:E\to E'$ such that

$$\left\|f(x)-T(x)\right\|\leq\delta \tag{1.2}$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear. In 1978, Rassias [3] provided a generalization of Hyers' theorem which allows the Cauchy difference to be unbounded. In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [5–12]). The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.3}$$

is related to a symmetric biadditive mapping. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic functional equation (1.3) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exits a unique symmetric biadditive mapping B such that $f(x) = B(x,x)$ for all x (see [5, 13]). The biadditive mapping B is given by

$$B(x,y) = \frac{1}{4}\left(f(x+y) - f(x-y)\right). \tag{1.4}$$

The Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.3) was proved by Skof for mappings $f : A \to B$, where A is a normed space and B is a Banach space (see [14]). Cholewa [15] noticed that the theorem of Skof is still true if relevant domain A is replaced an abelian group. In [16], Czerwik proved the Hyers-Ulam-Rassias stability of the functional equation (1.3). Grabiec [17] has generalized the results mentioned above.

In [18], Park and Bae considered the following quartic functional equation

$$f(x+2y) + f(x-2y) = 4\left[f(x+y) + f(x-y) + 6f(y)\right] - 6f(x). \tag{1.5}$$

In fact, they proved that a mapping f between two real vector spaces X and Y is a solution of (1.5) if and only if there exists a unique symmetric multiadditive mapping $M : X^4 \to Y$ such that $f(x) = M(x,x,x,x)$ for all x. It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.5), which is called a quartic functional equation (see also [19]). In addition, Kim [20] has obtained the Hyers-Ulam-Rassias stability for a mixed type of quartic and quadratic functional equation.

The Hyers-Ulam-Rassias stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [21–26]. It should be noticed that in all these papers the triangle inequality is expressed by using the strongest triangular norm T_M.

The aim of this paper is to investigate the stability of the additive-quadratic functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary continuous t-norms.

In this sequel, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces, as in [22, 23, 27–29]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. Also, D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases} \tag{1.6}$$

Definition 1.1 (see [28]). A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a,1) = a$ for all $a \in [0,1]$;

(d) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Lukasiewicz t-norm). Recall (see [30, 31]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, then $T_{i=1}^{n} x_i$ is defined recurrently by $T_{i=1}^{1} x_i = x_1$ and $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \ge 2$. $T_{i=n}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i-1}$. It is known [31] that for the Lukasiewicz t-norm, the following implication holds:

$$\lim_{n \to \infty} (T_L)_{i=1}^{\infty} x_{n+i-1} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty. \tag{1.7}$$

Definition 1.2 (see [29]). A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, $\alpha \ne 0$;

(RN3) $\mu_{x+y}(t+s) \ge T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \ge 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|} \tag{1.8}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 1.3. Let (X, μ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \ge N$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \ge m \ge N$.

(3) An RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4 (see [28]). *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty}\mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Recently, Gordji et al. establish the stability of cubic, quadratic and additive-quadratic functional equations in RN-spaces (see [32, 33]).

In this paper, we deal with the following functional equation:

$$f(2x+y) + f(2x-y) = 4[f(x+y) + f(x-y)] + 2[f(2x) - 4f(x)] - 6f(y) \tag{1.9}$$

on RN-spaces. It is easy to see that the function $f(x) = ax^4 + bx^2$ is a solution of (1.9).

In Section 2, we investigate the general solution of the functional equation (1.9) when f is a mapping between vector spaces and in Section 3, we establish the stability of the functional equation (1.9) in RN-spaces.

2. General Solution

We need the following lemma for solution of (1.9). Throughout this section, X and Y are vector spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies* (1.9) *for all $x, y \in X$, then f is quadratic-quartic.*

Proof. We show that the mappings $g : X \to Y$ defined by $g(x) := f(2x) - 16f(x)$ and $h : X \to Y$ defined by $h(x) := f(2x) - 4f(x)$ are quadratic and quartic, respectively.

Letting $x = y = 0$ in (1.9), we have $f(0) = 0$. Putting $x = 0$ in (1.9), we get $f(-y) = f(y)$. Thus the mapping f is even. Replacing y by $2y$ in (1.9), we get

$$f(2x+2y) + f(2x-2y) = 4[f(x+2y) + f(x-2y)] + 2[f(2x) - 4f(x)] - 6f(2y) \tag{2.1}$$

for all $x, y \in X$. Interchanging x with y in (1.9), we obtain

$$f(2y+x) + f(2y-x) = 4[f(y+x) + f(y-x)] + 2[f(2y) - 4f(y)] - 6f(x) \tag{2.2}$$

for all $x, y \in X$. Since f is even, by (2.2), one gets

$$f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y)] + 2[f(2y) - 4f(y)] - 6f(x) \tag{2.3}$$

for all $x, y \in X$. It follows from (2.1) and (2.3) that

$$\begin{aligned} &[f(2(x+y)) - 16f(x+y)] + [f(2(x-y)) - 16f(x-y)] \\ &\quad = 2[f(2x) - 16f(x)] + 2[f(2y) - 16f(y)] \end{aligned} \tag{2.4}$$

for all $x, y \in X$. This means that

$$g(x+y) + g(x-y) = 2g(x) + 2g(y) \tag{2.5}$$

for all $x, y \in X$. Therefore, the mapping $g : X \to Y$ is quadratic.

To prove that $h : X \to Y$ is quartic, we have to show that

$$h(x+2y) + h(x-2y) = 4[h(x+y) + h(x-y) + 6h(y)] - 6h(x) \tag{2.6}$$

for all $x, y \in X$. Since f is even, the mapping h is even. Now if we interchange x with y in the last equation, we get

$$h(2x+y) + h(2x-y) = 4[h(x+y) + h(x-y) + 6h(x)] - 6h(y) \tag{2.7}$$

for all $x, y \in X$. Thus, it is enough to prove that h satisfies (2.7). Replacing x and y by $2x$ and $2y$ in (1.9), respectively, we obtain

$$f(2(2x+y)) + f(2(2x-y)) = 4[f(2(x+y)) + f(2(x-y))] + 2[f(4x) - 4f(2x)] - 6f(2y) \tag{2.8}$$

for all $x, y \in X$. Since $g(2x) = 4g(x)$ for all $x \in X$,

$$f(4x) = 20f(2x) - 64f(x) \tag{2.9}$$

for all $x \in X$. By (2.8) and (2.9), we get

$$f(2(2x+y)) + f(2(2x-y)) = 4[f(2(x+y)) + f(2(x-y))] + 32[f(2x) - 4f(x)] - 6f(2y) \tag{2.10}$$

for all $x, y \in X$. By multiplying both sides of (1.9) by 4, we get

$$4[f(2x+y) + f(2x-y)] = 16[f(x+y) + f(x-y)] + 8[f(2x) - 4f(x)] - 24f(y) \tag{2.11}$$

for all $x, y \in X$. If we subtract the last equation from (2.10), we obtain

$$\begin{aligned} h(2x+y) + h(2x-y) &= [f(2(2x+y)) - 4f(2x+y)] + [f(2(2x-y)) - 4f(2x-y)] \\ &= 4[f(2(x+y)) - 4f(x+y)] + 4[f(2(x-y)) - 4f(x-y)] \\ &\quad + 24[f(2x) - 4f(x)] - 6[f(2y) - 4f(y)] \\ &= 4[h(x+y) + h(x-y) + 6h(x)] - 6h(y) \end{aligned} \tag{2.12}$$

for all $x, y \in X$.

Therefore, the mapping $h : X \to Y$ is quartic. This completes the proof of the lemma. □

Theorem 2.2. *A mapping $f : X \to Y$ satisfies* (1.9) *for all $x, y \in X$ if and only if there exist a unique symmetric multiadditive mapping $M : X^4 \to Y$ and a unique symmetric bi-additive mapping $B : X \times X \to Y$ such that*

$$f(x) = M(x, x, x, x) + B(x, x) \tag{2.13}$$

for all $x \in X$.

Proof. Let f satisfy (1.9) and assume that $g, h : X \to Y$ are mappings defined by

$$g(x) := f(2x) - 16f(x), \qquad h(x) := f(2x) - 4f(x) \tag{2.14}$$

for all $x \in X$. By Lemma 2.1, we obtain that the mappings g and h are quadratic and quartic, respectively, and

$$f(x) = \frac{1}{12}h(x) - \frac{1}{12}g(x) \tag{2.15}$$

for all $x \in X$.

Therefore, there exist a unique symmetric multiadditive mapping $M : X^4 \to Y$ and a unique symmetric bi-additive mapping $B : X \times X \to Y$ such that $(1/12)h(x) = M(x, x, x, x)$ and $(-1/12)g(x) = B(x, x)$ for all $x \in X$ [5, 18]. So

$$f(x) = M(x, x, x, x) + B(x, x) \tag{2.16}$$

for all $x \in X$. The proof of the converse is obvious. □

3. Stability

Throughout this section, assume that X is a real linear space and (Y, μ, T) is a complete RN-space.

Theorem 3.1. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x,y}(t) \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{gathered}\lim_{n\to\infty} T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{2n+i+1}t\right)\right), T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{2n+i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{2n+i}t}{3}\right)\right)\right) = 1,\\ \lim_{n\to\infty}\rho_{2^n x,2^n y}\left(2^{2n}t\right) = 1\end{gathered} \tag{3.2}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quadratic mapping $Q_1 : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-Q_1(x)}(t) \geq T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(2^{i+1}t\right),\ T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^i t}{4}\right),\ \rho_{0,2^{i-1}x}\left(\frac{2^i t}{3}\right)\right)\right)\right) \tag{3.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = x$ in (3.1), we obtain

$$\mu_{f(3x)-6f(2x)+15f(x)}(t) \geq \rho_{x,x}(t) \tag{3.4}$$

for all $x \in X$ and all $t > 0$. Letting $y = 2x$ in (3.1), we get

$$\mu_{f(4x)-4f(3x)+4f(2x)+8f(x)-4f(-x)}(t) \geq \rho_{x,2x}(t) \tag{3.5}$$

for all $x \in X$ and all $t > 0$. Putting $x = 0$ in (3.1), we obtain

$$\mu_{3f(y)-3f(-y)}(t) \geq \rho_{0,y}(t) \tag{3.6}$$

for all $y \in X$ and all $t > 0$. Replacing y by x in (3.6), we see that

$$\mu_{3f(x)-3f(-x)}(t) \geq \rho_{0,x}(t) \tag{3.7}$$

for all $x \in X$ and all $t > 0$. It follows from (3.5) and (3.7) that

$$\mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t) \geq T\left(\rho_{x,2x}\left(\frac{t}{2}\right), \rho_{0,x}\left(\frac{2t}{3}\right)\right) \tag{3.8}$$

for all $x \in X$ and all $t > 0$. If we add (3.4) to (3.8), then we have

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq T\left(\rho_{x,x}(2t), T\left(\rho_{x,2x}\left(\frac{t}{4}\right), \rho_{0,x}\left(\frac{t}{3}\right)\right)\right). \tag{3.9}$$

Let

$$\varphi_{x,x}(t) = T\left(\rho_{x,x}(2t), T\left(\rho_{x,2x}\left(\frac{t}{4}\right), \rho_{0,x}\left(\frac{t}{3}\right)\right)\right) \tag{3.10}$$

for all $x \in X$ and all $t > 0$. Then we get

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \varphi_{x,x}(t) \tag{3.11}$$

for all $x \in X$ and all $t > 0$. Let $g : X \to Y$ be a mapping defined by $g(x) := f(2x) - 16f(x)$. Then we conclude that

$$\mu_{g(2x)-4g(x)}(t) \geq \psi_{x,x}(t) \tag{3.12}$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{g(2x)/2^2-g(x)}(t) \geq \psi_{x,x}\left(2^2 t\right) \tag{3.13}$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{g(2^{k+1}x)/2^{2(k+1)}-g(2^k x)/2^{2k}}(t) \geq \psi_{2^k x,2^k x}\left(2^{2(k+1)} t\right) \tag{3.14}$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. This means that

$$\mu_{g(2^{k+1}x)/2^{2(k+1)}-g(2^k x)/2^{2k}}\left(\frac{t}{2^{k+1}}\right) \geq \psi_{2^k x,2^k x}\left(2^{k+1} t\right) \tag{3.15}$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. By the triangle inequality, from $1 > 1/2 + 1/2^2 + \cdots + 1/2^n$, it follows that

$$\mu_{g(2^n x)/2^{2n}-g(x)}(t) \geq T_{k=1}^{n}\left(\mu_{g(2^k x)/2^{2k}-g(2^{k-1}x)/2^{2(k-1)}}\left(\frac{t}{2^k}\right)\right) \geq T_{i=1}^{n}\left(\psi_{2^{i-1}x,2^{i-1}x}\left(2^i t\right)\right) \tag{3.16}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{g(2^n x)/2^{2n}\}$, we replace x with $2^m x$ in (3.16) to obtain that

$$\mu_{g(2^{n+m}x)/2^{2(n+m)}-g(2^m x)/2^{2m}}(t) \geq T_{i=1}^{n}\left(\psi_{2^{i+m-1}x,2^{i+m-1}x}\left(2^{i+2m} t\right)\right). \tag{3.17}$$

Since the right-hand side of the inequality (3.17) tends to 1 as m and n tend to infinity, the sequence $\{g(2^n x)/2^{2n}\}$ is a Cauchy sequence. Thus we may define $Q_1(x) = \lim_{n\to\infty}(g(2^n x)/2^{2n})$ for all $x \in X$.

Now we show that Q_1 is a quadratic mapping. Replacing x, y with $2^n x$ and $2^n y$ in (3.1), respectively, we get

$$\begin{aligned}&\mu_{((g(2^n(2x+y))+g(2^n(2x-y))-4g(2^n(x+y))-4g(2^n(x-y))-2g(2^{n+1}x)+8g(2^n x)+6g(2^n y))/4^n)}(t)\\ &\quad\geq \rho_{(2^n x,2^n y)}\left(2^{2n} t\right).\end{aligned} \tag{3.18}$$

Taking the limit as $n \to \infty$, we find that Q_1 satisfies (1.9) for all $x, y \in X$. By Lemma 2.1, the mapping $Q_1 : X \to Y$ is quadratic.

Letting the limit as $n \to \infty$ in (3.16), we get (3.3) by (3.10).

Finally, to prove the uniqueness of the quadratic mapping Q_1 subject to (3.3), let us assume that there exists another quadratic mapping Q_1' which satisfies (3.3). Since $Q_1(2^n x) = 2^{2n}Q_1(x), Q_1'(2^n x) = 2^{2n}Q_1'(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, from (3.3), it follows that

$$\begin{aligned}
&\mu_{Q_1(x)-Q_1'(x)}(2t) \\
&\quad = \mu_{Q_1(2^n x)-Q_1'(2^n x)}\left(2^{2n+1}t\right) \\
&\quad \geq T\left(\mu_{Q_1(2^n x)-g(2^n x)}\left(2^{2n}t\right), \mu_{g(2^n x)-Q_1'(2^n x)}\left(2^{2n}t\right)\right) \\
&\quad \geq T\left(T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{2n+i+1}t\right), T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{2n+i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{2n+i}t}{3}\right)\right)\right)\right),\right. \\
&\qquad\quad \left. T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{2n+i+1}t\right), T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(\frac{2^{2n+i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{2n+i}t}{3}\right)\right)\right)\right)\right)
\end{aligned} \tag{3.19}$$

for all $x \in X$ and all $t > 0$. Letting $n \to \infty$ in (3.19), we conclude that $Q_1 = Q_1'$, as desired. □

Theorem 3.2. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x,y}(t) \tag{3.20}$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{aligned}
&\lim_{n\to\infty} T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{4n+3i+1}t\right),\right.\right. \\
&\qquad\qquad \left.\left. T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right) = 1, \\
&\lim_{n\to\infty} \rho_{2^n x,2^n y}\left(2^{4n}t\right) = 1
\end{aligned} \tag{3.21}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q_2 : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q_2(x)}(t) \geq T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(2^{3i+1}t\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^{3i}t}{4}\right), \rho_{0,2^{i-1}x}\left(\frac{2^{3i}t}{3}\right)\right)\right)\right) \tag{3.22}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = x$ in (3.20), we obtain

$$\mu_{f(3x)-6f(2x)+15f(x)}(t) \geq \rho_{x,x}(t) \tag{3.23}$$

for all $x \in X$ and all $t > 0$. Letting $y = 2x$ in (3.20), we get

$$\mu_{f(4x)-4f(3x)+4f(2x)+8f(x)-4f(-x)}(t) \geq \rho_{x,2x}(t) \tag{3.24}$$

for all $x \in X$ and all $t > 0$. Putting $x = 0$ in (3.20), we obtain

$$\mu_{3f(y)-3f(-y)}(t) \geq \rho_{0,y}(t) \tag{3.25}$$

for all $y \in X$ and all $t > 0$. Replacing y by x in (3.25), we get

$$\mu_{3f(x)-3f(-x)}(t) \geq \rho_{0,x}(t) \tag{3.26}$$

for all $x \in X$ and all $t > 0$. It follows from (3.5) and (3.26) that

$$\mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t) \geq T\left(\rho_{x,2x}\left(\frac{t}{2}\right), \rho_{0,x}\left(\frac{2t}{3}\right)\right) \tag{3.27}$$

for all $x \in X$ and all $t > 0$. If we add (3.23) to (3.27), then we have

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq T\left(\rho_{x,x}(2t), T\left(\rho_{x,2x}\left(\frac{t}{4}\right), \rho_{0,x}\left(\frac{t}{3}\right)\right)\right). \tag{3.28}$$

Let

$$\psi_{x,x}(t) = T\left(\rho_{x,x}(2t), T\left(\rho_{x,2x}\left(\frac{t}{4}\right), \rho_{0,x}\left(\frac{t}{3}\right)\right)\right) \tag{3.29}$$

for all $x \in X$ and all $t > 0$. Then we get

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \psi_{x,x}(t) \tag{3.30}$$

for all $x \in X$ and all $t > 0$. Let $h : X \to Y$ be a mapping defined by $h(x) := f(2x) - 4f(x)$. Then we conclude that

$$\mu_{h(2x)-16h(x)}(t) \geq \psi_{x,x}(t) \tag{3.31}$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{h(2x)/2^4-h(x)}(t) \geq \psi_{x,x}\left(2^4 t\right) \tag{3.32}$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{h(2^{k+1}x)/2^{4(k+1)}-h(2^kx)/2^{4k}}(t) \geq \varphi_{2^kx,2^kx}\left(2^{4(k+1)}t\right) \tag{3.33}$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. This means that

$$\mu_{h(2^{k+1}x)/2^{4(k+1)}-h(2^kx)/2^{4k}}\left(\frac{t}{2^{k+1}}\right) \geq \varphi_{2^kx,2^kx}\left(2^{3(k+1)}t\right) \tag{3.34}$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. By the triangle inequality, from $1 > 1/2 + 1/2^2 + \cdots + 1/2^n$, it follows that

$$\begin{aligned}\mu_{h(2^nx)/2^{4n}-h(x)}(t) &\geq T_{k=1}^{n}\left(\mu_{h(2^kx)/2^{4k}-h(2^{k-1}x)/2^{4(k-1)}}\left(\frac{t}{2^k}\right)\right)\\ &\geq T_{i=1}^{n}\left(\varphi_{2^{i-1}x,2^{i-1}x}\left(2^{3i}t\right)\right)\end{aligned} \tag{3.35}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{h(2^nx)/2^{4n}\}$, we replace x with 2^mx in (3.35) to obtain that

$$\mu_{h(2^{n+m}x)/2^{4(n+m)}-h(2^mx)/2^{4m}}(t) \geq T_{i=1}^{n}\left(\varphi_{2^{i+m-1}x,2^{i+m-1}x}\left(2^{3i+4m}t\right)\right). \tag{3.36}$$

Since the right-hand side of (3.36) tends to 1 as m and n tend to infinity, the sequence $\{h(2^nx)/2^{4n}\}$ is a Cauchy sequence. Thus we may define $Q_2(x) = \lim_{n\to\infty}(h(2^nx)/2^{4n})$ for all $x \in X$.

Now we show that Q_2 is a quartic mapping. Replacing x, y with 2^nx and 2^ny in (3.20), respectively, we get

$$\begin{aligned}&\mu_{(h(2^n(2x+y))+h(2^n(2x-y))-4h(2^n(x+y))-4h(2^n(x-y))-2h(2^{n+1}x)+8h(2^nx)+6h(2^ny))/16^n}(t)\\ &\quad\geq \rho_{2^nx,2^ny}\left(2^{4n}t\right).\end{aligned} \tag{3.37}$$

Taking the limit as $n \to \infty$, we find that Q_2 satisfies (1.9) for all $x, y \in X$. By Lemma 2.1 we get that the mapping $Q_2 : X \to Y$ is quartic.

Letting the limit as $n \to \infty$ in (3.35), we get (3.22) by (3.29).

Finally, to prove the uniqueness of the quartic mapping Q_2 subject to (3.24), let us assume that there exists a quartic mapping Q_2' which satisfies (3.22). Since $Q_2(2^n x) = 2^{4n}Q_2(x)$ and $Q_2'(2^n x) = 2^{4n}Q_2'(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, from (3.22), it follows that

$$\begin{aligned}
&\mu_{Q_2(x)-Q_2'(x)}(2t) \\
&\quad = \mu_{Q_2(2^n x)-Q_2'(2^n x)}\left(2^{4n+1}t\right) \\
&\quad \ge T\left(\mu_{Q_2(2^n x)-h(2^n x)}\left(2^{4n}t\right), \mu_{h(2^n x)-Q_2'(2^n x)}\left(2^{4n}t\right)\right), \\
&\quad \ge T\left(T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{4n+3i+1}t\right), T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right),\right. \\
&\qquad\quad \left. T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{4n+3i+1}t\right)T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right)\right)
\end{aligned} \tag{3.38}$$

for all $x \in X$ and all $t > 0$. Letting $n \to \infty$ in (3.38), we get that $Q_2 = Q_2'$, as desired. □

Theorem 3.3. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ($\rho(x,y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \ge \rho_{x,y}(t) \tag{3.39}$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{aligned}
&\lim_{n\to\infty} T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{4n+3i+1}t\right),\right.\right. \\
&\qquad\qquad \left.\left. T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right) = 1, \\
&\lim_{n\to\infty} \rho_{2^n x,2^n y}\left(2^{2n}t\right) = 1
\end{aligned} \tag{3.40}$$

for all $x, y \in X$ and all $t > 0$, then there exist a unique quadratic mapping $Q_1 : X \to Y$ and a unique quartic mapping $Q_2 : X \to Y$ such that

$$\begin{aligned}
&\mu_{f(x)-Q_1(x)-Q_2(x)}(t) \\
&\quad \ge T\left(T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(\frac{2^i t}{12}\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^i t}{4\cdot 24}\right), \rho_{0,2^{i-1}x}\left(\frac{2^i t}{3\cdot 24}\right)\right)\right)\right),\right. \\
&\qquad\quad \left. T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(\frac{2^{3i}t}{24}\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^{3i}t}{4\cdot 24}\right), \rho_{0,2^{i-1}x}\left(\frac{2^{3i}t}{3\cdot 24}\right)\right)\right)\right)\right)
\end{aligned} \tag{3.41}$$

for all $x \in X$ and all $t > 0$.

Proof. By Theorems 3.1 and 3.2, there exist a quadratic mapping $Q_1' : X \to Y$ and a quartic mapping $Q_2' : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-Q_1'(x)}(t) \geq T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(2^{i+1}t\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^i t}{4}\right), \rho_{0,2^{i-1}x}\left(\frac{2^i t}{3}\right)\right)\right)\right),$$
$$\mu_{f(2x)-4f(x)-Q_2'(x)}(t) \geq T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(2^{3i+1}t\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^{3i} t}{4}\right), \rho_{0,2^{i-1}x}\left(\frac{2^{3i} t}{3}\right)\right)\right)\right) \tag{3.42}$$

for all $x \in X$ and all $t > 0$. It follows from the last inequalities that

$$\begin{aligned}
&\mu_{f(x)+(1/12)Q_1'(x)-(1/12)Q_2'(x)}(t) \\
&\quad \geq T\left(\mu_{f(2x)-16f(x)-Q_1'(x)}\left(\frac{t}{24}\right), \mu_{f(2x)-4f(x)-Q_2'(x)}\left(\frac{t}{24}\right)\right) \\
&\quad \geq T\left(T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(\frac{2^i t}{12}\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^i t}{4\cdot 24}\right), \rho_{0,2^{i-1}x}\left(\frac{2^i t}{3\cdot 24}\right)\right)\right)\right),\right. \\
&\qquad \left. T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(\frac{2^{3i} t}{24}\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^{3i} t}{4\cdot 24}\right), \rho_{0,2^{i-1}x}\left(\frac{2^{3i} t}{3\cdot 24}\right)\right)\right)\right)\right)
\end{aligned} \tag{3.43}$$

for all $x \in X$ and all $t > 0$. Hence we obtain (3.41) by letting $Q_1(x) = -(1/12)Q_1'(x)$ and $Q_2(x) = (1/12)Q_2'(x)$ for all $x \in X$. The uniqueness property of Q_1 and Q_2 is trivial. □

Acknowledgment

C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2009-0070788).

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