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Research Article

Admissible Estimators in the General Multivariate Linear Model with Respect to Inequality Restricted Parameter Set

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By using the methods of linear algebra and matrix inequality theory, we obtain the characterization of admissible estimators in the general multivariate linear model with respect to inequality restricted parameter set. In the classes of homogeneous and general linear estimators, the necessary and suffcient conditions that the estimators of regression coeffcient function are admissible are established.

1. Introduction

Throughout this paper, $R_{m\times n}, R_m^s$, and $R_m^{\geq}$ denote the set of $m \times n$ real matrices, the subset of $R_{m\times m}$ consisting of symmetric matrices, and the subset of R_m^s consisting of nonnegative definite matrices, respectively. The symbols $A', \mu(A), A^+, A^-$, and $\operatorname{tr}(A)$ stand for the transpose, the range, Moore-Penrose inverse, generalized inverse, and trace of $A \in R_{m\times n}$, respectively. For any $A, B \in R_m^s$, $A \geq B$ means $A - B \geq 0$.

Consider the general multivariate linear model with respect to inequality restricted parameter set:

$$\begin{gathered} Y = XB + \varepsilon, \\ \overrightarrow{\varepsilon} \sim (0, \Sigma \otimes V), \\ (B - B_0)' X' N X (B - B_0) \leq \Sigma, \end{gathered} \tag{1.1}$$

where $Y \in R_{n\times q}$ is an observable random matrix, $X \in R_{n\times p}$, $B_0 \in R_{p\times q}$, $V \in R_n^{\geq}$, and $N \in R_n^{\geq}$ are known matrices, respectively. $B \in R_{p\times q}$, $\Sigma \in R_q^{\geq}$ $(\Sigma \neq 0)$ are unknown matrices. ε is the error matrix. $\overrightarrow{\varepsilon}$ denotes the vector made of the columns of ε and $\otimes$ denotes the Kronecker product.

Let $H(N, B_0) = \{(B, \Sigma) : (B - B_0)' X' N X (B - B_0) \leq \Sigma\}$. For the linear function $KB(K \in R_{k\times p})$, we use the following matrix loss function:

$$L(d(Y), KB) = (d(Y) - KB)(d(Y) - KB)', \tag{1.2}$$

where $d(Y)$ is a linear estimator of KB. The risk function is the expected value of loss function:

$$R(d, B, \Sigma) := R(d(Y), KB) = E(d(Y) - KB)(d(Y) - KB)'. \tag{1.3}$$

Suppose $d_1(Y)$ and $d_2(Y)$ are two estimators of KB, if for any (B, Σ), we have

$$R(d_1, B, \Sigma) \leq R(d_2, B, \Sigma), \tag{1.4}$$

and there exists (B_*, Σ_*), such that $R(d_2, B_*, \Sigma_*) - R(d_1, B_*, \Sigma_*) \neq 0$, then $d_1(Y)$ is said to be better than $d_2(Y)$. If there does not exist any estimator in set Ω that is better than $d(Y)$, where parameters $(B, \Sigma) \in H(N, B_0)$, then $d(Y)$ is called the admissible estimator of KB in the set Ω. We denote it by $d(Y) \overset{\Omega}{\sim} KB[H(N, B_0)]$.

In the case of $X' N X = 0$, model (1.1) degenerates to the general multivariate linear model without restrictions. Under the quadratic loss function, many articles discussed the admissibility of linear estimators, such as Cohen [1], Rao [2], LaMotte [3], etc. Under the matrix loss function, Zhu and Lu [4] and Baksalary and Markiewicz [5] studied the admissibility of linear estimators when $q = 1$ respectively. Deng et al. [6] discussed the admissibility under the matrix loss in multivariate model. Markiewicz [7] discussed the admissibility in the general multivariate linear model. Marquardt [8] and Perlman [9] pointed out that the least square estimator is not still the admissible estimator if the parameters are restricted. Further, Groß and Markiewicz [10] pointed out that the admissible linear estimator has the form of ridge estimator if the parameters have no restrictions. Therefore, it is useful and important to discuss the admissibility of linear estimators when the parameters have some restrictions.

Zhu and Zhang [11], Lu [12], Deng and Chen [13] studied the admissibility of linear estimators under the quadratic loss and matrix loss when $q = 1$. Qin et al. [14] studied the admissibility of the estimators of estimable function under the loss function $(d(Y) - KB)'(d(Y) - KB)$ in multivariate linear model with respect to restricted parameter set when $B_0 = 0$. In their case, whether an estimator is better than another or not does not depend on the regression parameters. It is easy to generalize the conclusions from univariate linear model to multivariate linear model. However under the matrix loss (1.2), it is more complicated. In this case, whether an estimator is better than another depends on the regression parameters.

In this paper, using the methods of linear algebra and matrix theory, we discuss the admissibility of linear estimators in model (1.1) under the matrix loss (1.2). We prove that the admissibility of the estimators of estimable function under univariate linear model and multivariate linear model are equivalent in the class of homogeneous linear estimators, and some sufficient and necessary conditions that the estimators in the general multivariate linear model with respect to restricted parameter set are admissible are obtained whether the function of parameter is estimable or not, which enriches the theory of admissibility in multivariate linear model.

2. Main Results

Let $HL = \{DY : D \in R_{k\times n}\}$ denote the class of homogeneous linear estimators, and let $L = \{DY + C : D \in R_{k\times n}, C \in R_{k\times q}\}$ denote the class of general linear estimators.

Lemma 2.1. *Under model* (1.1) *with the loss function* (1.2)*, suppose $DY \in HL$ is an estimator of KB, one has*

$$R(DY, B, \Sigma) \geq R(DP_X Y, B, \Sigma). \tag{2.1}$$

The equality holds if and only if

$$DV = DP_X V, \tag{2.2}$$

where $P_X = X(X'E^+X)^- X'E^+,\ E = V + XX'$.

Proof. Since

$$\begin{aligned} R(DY, B, \Sigma) &= E(DY - KB)(DY - KB)' \\ &= \operatorname{tr}(\Sigma)DVD' + (DX - K)BB'(DX - K)', \end{aligned} \tag{2.3}$$

It is easy to verify that (2.1) holds, and the equality holds if and only if

$$E(DY - DP_X Y)(DY - DP_X Y)' = 0. \tag{2.4}$$

Expanding it, we have

$$\operatorname{tr}(\Sigma)DVD' - \operatorname{tr}(\Sigma)DP_X VD' = 0. \tag{2.5}$$

Thus $DVD' = DP_X VD' = DP_X VP_X' D'$, that is $DV = DP_X V$. □

Lemma 2.2. *Under model* (1.1) *with the loss function* (1.2)*, if $B_0 = 0$, suppose $D_1 Y, DY \in HL$ are estimators of $K\beta$, then $D_1 Y$ is better than DY if and only if*

$$D_1 V D_1' \leq DVD', \tag{2.6}$$

$$\begin{aligned} &\forall B \in R_{p\times q}, \quad \operatorname{tr}\left(B'X'NXB\right)\left(D_1 V D_1' - DVD'\right) \\ &\quad \leq (DX - K)BB'(DX - K)' - (D_1 X - K)BB'(D_1 X - K)', \end{aligned} \tag{2.7}$$

and the two equalities above cannot hold simultaneously.

Proof. Since $B_0 = 0$, $B'X'NXB \leq \Sigma$, $\operatorname{tr}(B'X'NXB) \leq \operatorname{tr}(\Sigma)$, (2.3) implies the sufficiency is true. Suppose D_1Y is better than DY, then for any $(B,\Sigma) \in H(N,0)$, we have

$$\begin{aligned} R(D_1Y,B,\Sigma) &= \operatorname{tr}(\Sigma)D_1VD_1' + (D_1X-K)BB'(D_1X-K)' \\ &\leq \operatorname{tr}(\Sigma)DVD' + (DX-K)BB'(DX-K)' \\ &= R(DY,B,\Sigma), \end{aligned} \tag{2.8}$$

and there exists some (B_*,Σ_*) such that the equality in (2.8) cannot hold. Taking $B = 0$ in (2.8), (2.6) follows. Let $\Sigma = B'X'NXB + mI_q$, $m > 0$, I is the identity matrix, then for any $B \in R_{p\times q}$, $(B,\Sigma) \in H(N,0)$, by (2.8), we have

$$\begin{aligned} \lim_{m\to 0} R(D_1Y,B,\Sigma) &= \operatorname{tr}(B'X'NXB)D_1VD_1' + (D_1X-K)BB'(D_1X-K)' \\ &\leq \lim_{m\to 0} R(DY,B,\Sigma) = \operatorname{tr}\big(B'X'NXB\big)DVD' \\ &\quad + (DX-K)BB'(DX-K)'. \end{aligned} \tag{2.9}$$

Therefore, (2.7) holds. It is obvious that the two equalities in (2.6) and (2.7) cannot hold simultaneously.

Consider univariate linear model with respect to restricted parameter set:

$$\begin{gathered} y = X\beta + e, \\ e \sim \left(0,\sigma^2 V\right), \\ \beta'X'NX\beta \leq \sigma^2, \end{gathered} \tag{2.10}$$

and the loss function

$$\big(d(y)-K\beta\big)\big(d(y)-K\beta\big)', \tag{2.11}$$

where X, V, N and K are as defined in (1.1) and (1.2), $\beta \in R_{p\times 1}$ and σ^2 are unknown parameters. Set $H_1(N) = \{(\beta,\sigma^2) : \beta'X'NX\beta \leq \sigma^2\}$. If $d(y)$ is an admissible estimator of $K\beta$, we denote it by $d(y) \sim K\beta[H_1(N)]$. □

Similarly to Lemma 2.2, we have the following lemma.

Lemma 2.3. *Under model* (2.10) *with the loss function* (2.11), *suppose D_1y and Dy are estimators of $K\beta$, then D_1y is better than Dy if and only if*

$$D_1VD_1' \leq DVD', \tag{2.12}$$

$$\begin{aligned} \forall\beta \in R_{p\times 1}, \quad & \beta'X'NX\beta\big(D_1VD_1' - DVD'\big) \\ & \leq (DX-K)\beta\beta'(DX-K)' - (D_1X-K)\beta\beta'(D_1X-K)', \end{aligned} \tag{2.13}$$

and the two equalities above cannot hold simultaneously.

Theorem 2.4. *Consider the model* (1.1) *with the loss function* (1.2), $DY \overset{HL}{\sim} KB[H(N,0)]$ *if and only if* $Dy \sim K\beta[H_1(N)]$ *in model* (2.10) *with the loss function* (2.11).

Proof. From Lemmas 2.2 and 2.3, we need only to prove the equivalence of (2.7) and (2.13).

Suppose (2.7) is true, we can take $B = (\beta, 0, \dots, 0)$, $\beta \in R_{p\times 1}$, and plug it into (2.7). Then (2.13) follows.

For the inverse part, suppose (2.13) is true, let $B = (b_1, b_2, \dots, b_q)$, $b_i \in R_{p\times 1}$, we have

$$\begin{aligned}\operatorname{tr}(B'X'NXB)(D_1VD_1' - DVD') &= \sum_{i=1}^{q} b_i'X'NXb_i(D_1VD_1' - DVD') \\ &\le \sum_{i=1}^{q}\left[(DX-K)b_ib_i'(DX-K)' - (D_1X-K)b_ib_i'(D_1X-K)'\right] \\ &= (DX-K)BB'(DX-K)' - (D_1X-K)BB'(D_1X-K)'.\end{aligned} \tag{2.14}$$

The claim follows. □

Remark 2.5. From this Theorem, we can easily generalize the result under univariate linear model to the case under multivariate linear model in the class of homogeneous linear estimators.

Theorem 2.6. *Consider the model* (1.1) *with the loss function* (1.2), *if* KB *is estimable, then* $DY \overset{HL}{\sim} KB[H(N,0)]$ *if and only if:*

(1) $DV = DP_XV$,

(2) *if there exists* $\lambda > 0$, *such that*

$$2DVD' + 2DVNVD' - DXWK' - KWX'D' \ge \lambda(DX-K)W(DX-K)', \tag{2.15}$$

then $DX = K$, $DVN = 0$, *where* $W = (X'E^{+}X)^{-} - I_p$.

Proof. From the corresponding theorem in article Deng and Chen [13], under the model (2.10) with the loss function (2.11), if $K\beta$ is estimable, then $DY \sim K\beta[H_1(N)]$ if and only if (1) and (2) in Theorem 2.6 are satisfied. Now Theorem 2.6 follows from Theorem 2.4. □

Lemma 2.7. *Consider the model* (1.1) *with the loss function* (1.2), *suppose* $DY + C \in L$ *is an estimator of* KB. *One has*

$$R(DY + C, B, \Sigma) \ge R(DP_XY + C, B, \Sigma), \tag{2.16}$$

and the equality holds if and only if $DV = DP_XV$.

Proof. The proof follows from the following equalities:

$$\begin{aligned} R(DY+C,B,\Sigma) &= E(DY+C-KB)(DY+C-KB)' \\ &= \operatorname{tr}(\Sigma)DVD' + [(DX-K)B+C][(DX-K)B+C]', \\ R(DP_XY+C,B,\Sigma) &= \operatorname{tr}(\Sigma)DP_XVP_X'D' + [(DP_XX-K)B+C][(DP_XX-K)B+C]' \\ &= \operatorname{tr}(\Sigma)DP_XVP_X'D' + [(DX-K)B+C][(DX-K)B+C]'. \end{aligned} \tag{2.17}$$

□

Lemma 2.8. *Assume $A, B \in R_n^s$, one has*

(1) *if $A \geq 0$ and $\mu(B) \subset \mu(A)$, then there exists $t \geq 0$, for every $|r| \leq t$, $A - rB \geq 0$ and* $\operatorname{rank}(A - rB) = \operatorname{rank}(A)$.

(2) *$\mu(B) \subset \mu(A)$ if and only if for any vector $\alpha \in R_{n\times 1}$, $\alpha' A = 0$ implies $\alpha' B = 0$.*

Proof. (1) If $A = 0$, the claim is trivial. If $A \geq 0$, $A = P\operatorname{diag}\{a_1,\dots,a_k,0,\dots,0\}P'$, where P is an orthogonal matrix, $a_1 \geq a_2 \geq \cdots \geq a_k > 0$, $k = \operatorname{rank}(A)$. From $\mu(B) \subset \mu(A)$, we have $\mu(P'BP) \subset \mu(P'AP)$, notice that $B' = B$, we get $P'BP = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $B_1 \in R_k^s$. Clearly, there exists $r_2 > 0 > r_1$, such that $r_2 I_k \geq B_1 \geq r_1 I_k$. Let $t = \min_k\{-a_k/r_1, a_k/r_2\}$, then $t > 0$, and for every $|r| < t$, $\operatorname{diag}\{a_1,\dots,a_k,0,\dots,0\} > rB_1$, thus $A - rB \geq 0$ and $\operatorname{rank}(A-rB) = \operatorname{rank}(A)$.

(2) The claim is easy to verify. □

Theorem 2.9. *Consider the model* (1.1) *with the loss function* (1.2), *if KB is estimable, then $DY+C \overset{L}{\sim} KB[H(N,0)]$ if and only if:*

(1) *$DV = DP_XV$,*

(2) *if there exists $\lambda > 0$ such that*

$$2DVD' + 2DVNVD' - DXWK' - KWX'D' \geq \lambda(DX-K)W(DX-K)', \tag{2.18}$$

then $DX = K$, $DVN = 0$ and $C = 0$, where $W = (X'E^+X)^- - I_p$.

Proof. If $DX = K$, by (2.17) we obtain $R(DY+C,B,\Sigma) = \operatorname{tr}(\Sigma)DVD' + CC'$. Then $DY + C \overset{L}{\sim} KB[H(N,0)]$ implies $C = 0$. The claim is true by Theorem 2.6. Now we assume $DX \neq K$.

Necessity

Assume $DY + C \overset{L}{\sim} KB[H(N,0)]$, by Lemma 2.7, (1) is true. Now we will prove (2). Denote $F = K(X'E^+X)^- X'E^+$, $FX = K$. Since $DV = DP_XV$, rewrite (2.18) as the following

$$2DVD' + 2DVNVD' - DVF' - FVD' \geq \lambda(D-F)V(D-F)'. \tag{2.19}$$

If there exists $\lambda > 0$ such that (2.19) holds, for sufficient small $\eta > 0$, take $M = (1-\eta)D - 2\eta DVN + \eta F$. Since

$$MXB + (1-\eta)C - KB = (1-\eta)DXB + (1-\eta)C - (1-\eta)FXB - 2\eta DVNXB. \tag{2.20}$$

Thus

$$
\begin{aligned}
&R(DY+C,B,\Sigma)-R(MY+(1-\eta)C,B,\Sigma)\\
&\quad=\operatorname{tr}(\Sigma)DVD'-\operatorname{tr}(\Sigma)MVM'+(DXB+C-FXB)(DXB+C-FXB)'\\
&\qquad-\left[(1-\eta)(DXB+C-FXB)-2\eta DVNXB\right]\left[(1-\eta)(DXB+C-FXB)-2\eta DVNXB\right]'\\
&\quad=\operatorname{tr}(\Sigma)DVD'-\operatorname{tr}(\Sigma)MVM'+\left(2\eta-\eta^2\right)(DXB+C-FXB)(DXB+C-FXB)'\\
&\qquad+2\eta(1-\eta)(DXB+C-FXB)B'X'NVD'+2\eta(1-\eta)DVNXB(DXB+C-FXB)'\\
&\qquad-4\eta^2DVNXBB'X'N'VD'\\
&\quad=\operatorname{tr}(\Sigma)DVD'-\operatorname{tr}(\Sigma)MVM'\\
&\qquad+\eta(2-\eta)\left(DXB+C-FXB+\frac{2-2\eta}{2-\eta}DVNXB\right)\\
&\qquad\times\left(DXB+C-FXB+\frac{2-2\eta}{2-\eta}DVNXB\right)'\\
&\qquad-\frac{\eta(2-2\eta)^2}{2-\eta}DVNXBB'X'N'VD'-4\eta^2DVNXBB'X'N'VD'
\end{aligned}
\tag{2.21}
$$

$$
\geq\operatorname{tr}(\Sigma)DVD'-\operatorname{tr}(\Sigma)MVM'-\left(\frac{\eta(2-2\eta)^2}{2-\eta}+4\eta^2\right)DVNXBB'X'N'VD' \tag{2.22}
$$

$$
\geq\operatorname{tr}(\Sigma)DVD'-\operatorname{tr}(\Sigma)MVM'-\left(2\eta+4\eta^2\right)DVNXBB'X'N'VD' \tag{2.23}
$$

$$
\geq\operatorname{tr}(\Sigma)DVD'-\operatorname{tr}(\Sigma)MVM'-\left(2\eta+4\eta^2\right)\operatorname{tr}(\Sigma)DVNVD'. \tag{2.24}
$$

In the above, η is sufficiently small, $(2-2\eta)^2/(2-\eta)<2$, thus (2.23) follows. $B'X'NXB\leq\Sigma$, $\operatorname{tr}(B'X'NXB)=\operatorname{tr}(N^{1/2}B'X'NXBN^{1/2})\leq\operatorname{tr}(\Sigma)$, $N^{1/2}B'X'NXBN^{1/2}\leq\operatorname{tr}(\Sigma)N$, thus (2.24) follows

$$
\begin{aligned}
&\frac{1}{\eta\operatorname{tr}(\Sigma)}\left\{\operatorname{tr}(\Sigma)DVD'-\operatorname{tr}(\Sigma)MVM'-\left(2\eta+4\eta^2\right)\operatorname{tr}(\Sigma)DVNVD'\right\}\\
&\quad=2DVD'+4DVNVD'-DVF'-FVD'\\
&\qquad-\eta\left[DVD'+4DVNVD'+FVF'-DVF'-FVD'+4DVNVD'\right.\\
&\qquad\qquad\left.-2DVNVF'-2FVNVD'\right]-2DVNVD'-4\eta DVNVD'\\
&\quad=2DVD'+2DVNVD'-DVF'-FVD'\\
&\qquad-\eta\left[DVD'+4DVNVD'+FVF'-DVF'-FVD'+8DVNVD'\right.\\
&\qquad\qquad\left.-2DVNVF'-2FVNVD'\right].
\end{aligned}
\tag{2.25}
$$

For any compatible vector α, assume

$$\alpha'\left[2DVD' + 2DVNVD' - DVF' - FVD'\right] = 0. \tag{2.26}$$

By (2.19) we obtain $\alpha'(D-F)V(D-F)'\alpha = 0$, that is, $\alpha' DV = \alpha' FV$, plug it into (2.26), then $\alpha' DVNV'D'\alpha = 0$, $\alpha' DVN = 0$, thus

$$\begin{aligned}
&\alpha'\left[DVD' + 4DVNVD' + FVF' - DVF' - FVD' + 8DVNVD' - 2DVNVF' - 2FVNVD'\right] \\
&\quad = \alpha'\left[DVD' + FVF' - DVF' - FVD' - 2FVNVD'\right] \\
&\quad = \alpha'\left[DVD' + DVF' - DVF' - DVD' - 2DVNVN'\right] \\
&\quad = 0.
\end{aligned} \tag{2.27}$$

From Lemma 2.8, we have

$$\begin{aligned}
&\mu\left[DVD' + 4DVNVD' + FVF' - DVF' - FVD' + 8DVNVD' - 2DVNVF' - 2FVNVD'\right] \\
&\quad \subset \mu\left[2DVD' + 2DVNV'D' - DVF' - FVD'\right].
\end{aligned} \tag{2.28}$$

Therefore, there exists $t > 0$, for $0 < \eta < t$, the right side of (2.25) is nonnegative definite and its rank is $\operatorname{rank}(2DVD' + 2DVNV'D' - DVF' - FVD')$. If η is small enough, for every $(B,\Sigma) \in H(N,0)$, we have $R(DY + C, B, \Sigma) \geq R(MY + (1-\eta)C, B, \Sigma)$, and the equality cannot always hold if (2) does not hold. It contradicts $DY + C \overset{L}{\sim} KB[H(N,0)]$.

Sufficiency

Assume (1) and (2) are true. Since $DX \neq K$, by Theorem 2.6, $DY \overset{HL}{\sim} KB[H(N,0)]$. If there exists an estimator $D_1Y + C_1$ that is better than $DY + C$, then for every $(B,\Sigma) \in H(N,0)$,

$$\begin{aligned}
&\operatorname{tr}(\Sigma)DVD' + (DXB + C - KB)(DXB + C - KB)' \\
&\quad \geq \operatorname{tr}(\Sigma)D_1VD_1' + (D_1XB + C_1 - KB)(D_1XB + C_1 - KB)'.
\end{aligned} \tag{2.29}$$

Note that for any $r > 0$, if $(B,\Sigma) \in H(N,0)$, then $(rB, r^2\Sigma) \in H(N,0)$. Replace B and Σ in (2.29) with rB and $r^2\Sigma$, respectively, divide by r^2 on both sides, and let $r \to +\infty$, we get

$$\begin{aligned}
&\operatorname{tr}(\Sigma)DVD' + (DXB - KB)(DXB - KB)' \\
&\quad \geq \operatorname{tr}(\Sigma)D_1VD_1' + (D_1XB - KB)(D_1XB - KB)'.
\end{aligned} \tag{2.30}$$

Since $DY \overset{HL}{\sim} KB[H(N,0)]$, we have $DVD' = D_1VD_1'$ and $DX = D_1X$ (otherwise, $(1/2)(D + D_1)Y$ is better than DY). Plug them into (2.29), for every $B \in R_{p\times q}$,

$$CC' - C_1C_1' + (C - C_1)B'(DX - K)' + (DX - K)B(C - C_1)' \geq 0. \tag{2.31}$$

Thus $CC' - C_1C_1' \geq 0$ and $(C - C_1)B'(DX - K)' = 0$. $DX \neq K$ implies $C = C_1$, and the equality in (2.29) holds always. It contradicts that $D_1Y + C_1$ is better than $DY + C$. □

Theorem 2.10. *Under model* (1.1) *and the loss function* (1.2), *if* KB *is estimable, then* $DY + C \overset{L}{\sim} KB[H(N, B_0)]$ *if and only if* $DY + C \overset{L}{\sim} KB[H(N,0)]$.

Proof. Denote $Z = Y - XB_0$, $G = B - B_0$, model (1.1) is transformed into

$$\begin{gathered} Z = XG + \varepsilon, \\ \overrightarrow{\varepsilon} \sim (0, \Sigma \otimes V), \\ G'X'NXG \leq \Sigma, \end{gathered} \tag{2.32}$$

Since

$$\begin{aligned} &E(DY + C - KB)(DY + C - KB)' \\ &\quad = E\left[(DZ + C + (DX - K)B_0 - KG)(DZ + C + (DX - K)B_0 - KG)'\right], \end{aligned} \tag{2.33}$$

then (2.33) implies that $DY + C \sim KB[H(N, B_0)] \Leftrightarrow DZ + [C + (DX - K)B_0] \sim KG[H(N,0)]$, which combining Theorem 2.4 and the fact that "if $DX = K$, then $C = 0 \Leftrightarrow C+(DX-K)B_0 = 0$" yields $DY + C \sim KB[H(N, B_0)] \Leftrightarrow DY + C \sim KB[H(N,0)]$. □

Corollary 2.11. *Under model* (1.1) *and the loss function* (1.2), *if* KB *is estimable, then* $DY \overset{HL}{\sim} KB[H(N, B_0)]$ *if and only if* $DY \overset{HL}{\sim} KB[H(N,0)]$.

Lemma 2.12. *Consider model* (1.1) *with the loss function* (1.2), *suppose* $D_1Y, DY \in HL$, *if* $D_1X = DX$, *then*

$$D_1VD_1' \geq DP_XVP_X'D'. \tag{2.34}$$

Proof.

$$\begin{aligned} DP_XVP_X'D' &= DX(X'E^+X)^-X'E^+(E - XX')E^+X(X'E^+X)^-X'D' \\ &= D_1X(X'E^+X)^-X'D_1' - D_1XX'D_1' \\ &= D_1E^{1/2}Q_{E^{+1/2}X}E^{1/2}D_1' - D_1XX'D_1' \\ &\leq D_1E^{1/2}E^{1/2}D_1' - D_1XX'D_1' = D_1VD_1', \end{aligned} \tag{2.35}$$

where $Q_A = A(A'A)^-A'$ refers to the orthogonal projection onto $\mu(A)$. □

Lemma 2.13. *Suppose S and G are $t \times q$ and $k \times t$ real matrices, respectively, there exists a $q \times k$ matrix $B \neq 0$ such that $H \equiv SBG + G'B'S' \neq 0$ if and only if $S \neq 0$ and $G \neq 0$.*

Proof (necessity is obvious). For the proof of sufficiency, we need only to prove that there exists a $B_1 \neq 0$ such that SB_1G is not an inverse symmetric matrix.

Since $S_{t\times q} = (s_1, \ldots, s_t)' \neq 0$, $G_{k\times t} = (g_1, \ldots, g_t)' \neq 0$.

(1) If there is $i \in \{1, \ldots, t\}$ such that $s_i \neq 0, g_i \neq 0$, take $B_1 = s_i \cdot g_i' \neq 0$, then

$$e_i' SB_1Ge_i = e_i'S \cdot \left(s_i g_i'\right) \cdot Ge_i = s_i's_i \cdot g_i'g_i \neq 0, \tag{2.36}$$

where e_i is the column vector whose only nonzero entry is a 1 in the ith position.

(2) If there does not exist i such that $s_i \neq 0, g_i \neq 0$, then there must exist $i \neq j$ such that $s_i \neq 0$, $g_j \neq 0$ and $s_j = 0$, $g_i = 0$, take $B_1 = s_i \cdot g_j' \neq 0$, then

$$\begin{aligned} e_i'SB_1Ge_j &= e_i'S \cdot \left(s_i g_i'\right) \cdot Ge_j = s_i's_i \cdot g_j'g_j \neq 0, \\ e_j'SB_1Ge_i &= e_j'S \cdot \left(s_i g_i'\right) \cdot Ge_i = s_j's_i \cdot g_j'g_i = 0. \end{aligned} \tag{2.37}$$

That is, $e_i'SB_1Ge_j \neq -e_j'SB_1Ge_i$.

The proof is complete. □

Theorem 2.14. *Consider the model* (1.1) *with the loss function* (1.2), *if KB is inestimable, then $DY \overset{HL}{\sim} KB[H(N, B_0)]$ if and only if $DV = DP_XV$.*

Proof. Lemma 2.1 implies the necessity. For the proof of the inverse part, assume there exists $D_1Y \in HL$, for any $(B, \Sigma) \in H(N, B_0)$, we have

$$R(D_1Y, B, \Sigma) \leq R(DY, B, \Sigma). \tag{2.38}$$

Since

$$\begin{aligned} R(DY, B, \Sigma) = {} & \operatorname{tr}(\Sigma)DVD' + DXB(DXB - KTB)' - KTB(DXB)' \\ & - K(I - T)B(DXB)' - DXB[K(I - T)B]' + KBB'K', \end{aligned} \tag{2.39}$$

where $T = X^+X$, thus

$$\begin{aligned} R(DY, B, \Sigma) - R(D_1Y, B, \Sigma) = {} & G(XB, \Sigma) + K(I - T)B(D_1XB - DXB)' \\ & + (D_1XB - DXB)[K(I - T)B]' \geq 0, \end{aligned} \tag{2.40}$$

where $G(XB, \Sigma)$ is a known function. If there exists $(B_1, \Sigma_1) \in H(N, B_0)$ such that

$$D_1XB_1 - DXB_1 \neq 0, \tag{2.41}$$

note that KB is inestimable, then $K(I-T) \neq 0$, by Lemma 2.13, there exists $B_2 \neq 0$ such that

$$K(I-T)B_2(D_1XB_1 - DXB_1)' + (D_1XB_1 - DXB_1)[K(I-T)B_2]' \neq 0. \tag{2.42}$$

Take $B = m(I-T)B_2 + TB_1$, $m \in R$, since $XB = XB_1$, so $(B, \Sigma_1) \in H(N, B_0)$.

According to (2.40), we have for any real m,

$$G(XB_1, \Sigma_1) + m\{K(I-T)B_2(D_1XB - DXB)' + (D_1XB_1 - DXB_1)[K(I-T)B_2]'\} \geq 0. \tag{2.43}$$

It is a contradiction. Therefore $D_1X = DX$. Since $DV = DP_XV$, by Lemma 2.12, we obtain

$$D_1VD_1' \geq DVD'. \tag{2.44}$$

Take $B = 0$ in (2.38), we have

$$D_1VD_1' \leq DVD'. \tag{2.45}$$

Thus $D_1VD_1' = DVD'$, $R(D_1Y, B, \Sigma) \equiv R(DY, B, \Sigma)$. There is no estimator that is better than DY in HL. □

Similarly to Theorem 2.14, we have the following theorem.

Theorem 2.15. *Under model* (1.1) *and the loss function* (1.2), *if* KB *is inestimable, then* $DY + C \overset{HL}{\sim} KB[H(N, B_0)]$ *if and only if* $DV = DP_XV$.

Remark 2.16. This theorem indicates that if KB is inestimable, then the admissibility of $DY+C$ has no relation with the choice of C owing to $DX - K \neq 0$.

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