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Research Article

A Recent Note on Quasi-Power Increasing Sequence for Generalized Absolute Summability

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We prove two theorems on $|A,\delta|_k$, $k \geq 1, 0 \leq \delta < 1/k$, summability factors for an infinite series by using quasi-power increasing sequences. We obtain sufficient conditions for $\sum a_n\lambda_n$ to be summable $|A,\delta|_k$, $k \geq 1$, $0 \leq \delta < 1/k$, by using quasi-f-increasing sequences.

1. Introduction

Quite recently, Savaş [1] obtained sufficient conditions for $\sum a_n\lambda_n$ to be summable $|A,\delta|_k$, $k \geq 1, 0 \leq \delta < 1/k$. The purpose of this paper is to obtain the corresponding result for quasi-f-increasing sequence. Our result includes and moderates the conditions of his theorem with the special case $\mu = 0$.

A sequence $\{\lambda_n\}$ is said to be of bounded variation (bv) if $\sum_n |\Delta\lambda_n| < \infty$. Let $bv_0 = bv \cap c_0$, where c_0 denotes the set of all null sequences.

The concept of absolute summability of order $k \geq 1$ was defined by Flett [2] as follows. Let $\sum a_n$ denote a series with partial sums $\{s_n\}$, and A a lower triangular matrix. Then $\sum a_n$ is said to be absolutely A-summable of order $k \geq 1$, written that $\sum a_n$ is summable $|A|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|T_{n-1} - T_n|^k < \infty, \tag{1.1}$$

where

$$T_n = \sum_{v=0}^{n} a_{nv}s_v. \tag{1.2}$$

In [3], Flett considered further extension of absolute summability in which he introduced a further parameter δ. The series $\sum a_n$ is said to be summable $|A,\delta|_k$, $k \geq 1$, $\delta \geq 0$, if

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |T_{n-1} - T_n|^k < \infty. \tag{1.3}$$

A positive sequence $\{b_n\}$ is said to be an almost increasing sequence if there exist an increasing sequence $\{c_n\}$ and positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [4]). Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = e^{(-1)^n} n$.

A positive sequence $\gamma := \{\gamma_n\}$ is said to be a quasi-β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$K n^{\beta} \gamma_n \geq m^{\beta} \gamma_m \tag{1.4}$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi-β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking an example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$ (see [5]). If (1.4) stays with $\beta = 0$, then γ is simply called a quasi-increasing sequence. It is clear that if $\{\gamma_n\}$ is quasi-β-power increasing, then $\{n^{\beta}\gamma_n\}$ is quasi-increasing.

A positive sequence $\gamma = \{\gamma_n\}$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(\gamma, f) \geq 1$ such that $K f_n \gamma_n \geq f_m \gamma_m$ holds for all $n \geq m \geq 1$, [6].

We may associate A two lower triangular matrices $\overline{A}$ and $\widehat{A}$ as follows:

$$\begin{aligned} \overline{a}_{nv} &= \sum_{r=v}^{n} a_{nr}, \quad n, v = 0, 1, \ldots, \\ \widehat{a}_{nv} &= \overline{a}_{nv} - \overline{a}_{n-1,v}, \quad n = 1, 2, \ldots, \end{aligned} \tag{1.5}$$

where

$$\widehat{a}_{00} = \overline{a}_{00} = a_{00}. \tag{1.6}$$

Given any sequence $\{x_n\}$, the notation $x_n \asymp O(1)$ means $x_n = O(1)$ and $1/x_n = O(1)$. For any matrix entry a_{nv}, $\Delta_v a_{nv} := a_{nv} - a_{n,v+1}$.

Quite recently, Savaş [1] obtained sufficient conditions for $\sum a_n \lambda_n$ to be summable $|A,\delta|_k$, $k \geq 1, 0 \leq \delta < 1/k$ as follows.

Theorem 1.1. *Let A be a lower triangular matrix with nonnegative entries satisfying*

$$a_{n-1,v} \geq a_{nv} \quad \text{for } n \geq v+1, \tag{1.7}$$

$$\overline{a}_{n0} = 1, \quad n = 0,1,\ldots, \tag{1.8}$$

$$na_{nn} \asymp O(1), \quad n \longrightarrow \infty, \tag{1.9}$$

$$\sum_{v=1}^{n-1} a_{vv}\widehat{a}_{n,v+1} = O(a_{nn}), \tag{1.10}$$

$$\sum_{n=v+1}^{m+1} n^{\delta k}|\Delta_v \widehat{a}_{nv}| = O\left(v^{\delta k}a_{vv}\right), \tag{1.11}$$

$$\sum_{n=v+1}^{m+1} n^{\delta k}\widehat{a}_{n,v+1} = O\left(v^{\delta k}\right), \tag{1.12}$$

and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences such that

$$|\Delta\lambda_n| \leq \beta_n, \tag{1.13}$$

$$\beta_n \longrightarrow 0, \quad n \longrightarrow \infty. \tag{1.14}$$

If $\{X_n\}$ is a quasi-β-power increasing sequence for some $0 < \beta < 1$ such that

$$|\lambda_n|X_n = O(1), \quad n \longrightarrow \infty, \tag{1.15}$$

$$\sum_{n=1}^{\infty} nX_n\left|\Delta\beta_n\right| < \infty, \tag{1.16}$$

$$\sum_{n=1}^{m} n^{\delta k-1}|s_n|^k = O(X_m), \quad m \longrightarrow \infty, \tag{1.17}$$

then the series $\sum a_n\lambda_n$ is summable $|A,\delta|_k$, $k \geq 1, 0 \leq \delta < 1/k$.

Theorem 1.1 enhanced a theorem of Savas [7] by replacing an almost increasing sequence with a quasi-β-power increasing sequence for some $0 < \beta < 1$. It should be noted that if $\{X_n\}$ is an almost increasing sequence, then (1.15) implies that the sequence $\{\lambda_n\}$ is bounded. However, when $\{X_n\}$ is a quasi-β-power increasing sequence or a quasi-f-increasing sequence, (1.15) does not imply $|\lambda_m| = O(1), m \to \infty$. For example, since $X_m = m^{-\beta}$ is a quasi-β-power increasing sequence for $0 < \beta < 1$ and if we take $\lambda_m = m^{\delta}, 0 < \delta < \beta < 1$, then $|\lambda_m|X_m = m^{\delta-\beta} = O(1), m \to \infty$ holds but $|\lambda_m| = m^{\delta} \neq O(1)$ (see [8]). Therefore, we remark that condition $\{\lambda_n\} \in bv_0$ should be added to the statement of Theorem 1.1.

The goal of this paper is to prove the following theorem by using quasi-f-increasing sequences. Our main result includes the moderated version of Theorem 1.1. We will show that the crucial condition of our proof, $\{\lambda_n\} \in bv_0$, can be deduced from another condition of the theorem. Also, we shall eliminate condition (1.15) in our theorem; however we shall deduce this condition from the conditions of our theorem.

2. The Main Results

We now shall prove the following theorems.

Theorem 2.1. *Let A satisfy conditions* (1.7)–(1.12)*, and let* $\{\beta_n\}$ *and* $\{\lambda_n\}$ *be sequences satisfying conditions* (1.13) *and* (1.14) *of Theorem 1.1 and*

$$\sum_{n=1}^{m}\lambda_n = o(m), \quad m \longrightarrow \infty. \tag{2.1}$$

If $\{X_n\}$ *is a quasi-f-increasing sequence and conditions* (1.17) *and*

$$\sum_{n=1}^{\infty} nX_n(\beta,\mu)|\Delta\beta_n| < \infty \tag{2.2}$$

are satisfied, then the series $\sum a_n\lambda_n$ *is summable* $|A,\delta|_k, k \geq 1, 0 \leq \delta < 1/k$*, where* $\{f_n\} := \{n^{\beta}(\log n)^{\mu}\}, \mu \geq 0, 0 \leq \beta < 1$*, and* $X_n(\beta,\mu) := (n^{\beta}(\log n)^{\mu}X_n)$.

Theorem 2.1 includes the following theorem with the special case $\mu = 0$. Theorem 2.2 moderates the hypotheses of Theorem 1.1.

Theorem 2.2. *Let A satisfy conditions* (1.7)–(1.12)*, and let* $\{\beta_n\}$ *and* $\{\lambda_n\}$ *be sequences satisfying conditions* (1.13), (1.14)*, and* (2.1)*. If* $\{X_n\}$ *is a quasi-β-power increasing sequence for some* $0 \leq \beta < 1$ *and conditions* (1.17) *and*

$$\sum_{n=1}^{\infty} nX_n(\beta)|\Delta\beta_n| < \infty \tag{2.3}$$

are satisfied, where $X_n(\beta) := (n^{\beta}X_n)$*, then the series* $\sum a_n\lambda_n$ *is summable* $|A,\delta|_k, k \geq 1, 0 \leq \delta < 1/k$.

Remark 2.3. The crucial condition, $\{\lambda_n\} \in bv_0$, and condition (1.15) do not appear among the conditions of Theorems 2.1 and 2.2. By Lemma 3.3, under the conditions on $\{X_n\}, \{\beta_n\}$, and $\{\lambda_n\}$ as taken in the statement of Theorem 2.1, also in the statement of Theorem 2.2 with the special case $\mu = 0$, conditions $\{\lambda_n\} \in bv_0$ and (1.15) hold.

3. Lemmas

We shall need the following lemmas for the proof of our main Theorem 2.1.

Lemma 3.1 (see [9]). *Let* $\{\varphi_n\}$ *be a sequence of real numbers and denote*

$$\Phi_n := \sum_{k=1}^{n}\varphi_k, \qquad \Psi_n := \sum_{k=n}^{\infty}|\Delta\varphi_k|. \tag{3.1}$$

If $\Phi_n = o(n)$*, then there exists a natural number* $\mathbb{N}$ *such that*

$$|\varphi_n| \leq 2\Psi_n \tag{3.2}$$

for all $n \geq \mathbb{N}$*.*

Lemma 3.2 (see [8]). *If* $\{X_n\}$ *is a quasi-*f*-increasing sequence, where* $\{f_n\} = \{n^\beta(\log n)^\mu\}, \mu \geq 0, 0 \leq \beta < 1$*, then conditions* (2.1) *of Theorem 2.1,*

$$\sum_{n=1}^{m}|\Delta\lambda_n| = o(m), \quad m \longrightarrow \infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty} nX_n(\beta,\mu)|\Delta|\Delta\lambda_n|| < \infty, \tag{3.4}$$

where $X_n(\beta,\mu) = (n^\beta(\log n)^\mu X_n)$*, imply conditions* (1.15) *and*

$$\lambda_n \longrightarrow 0, \qquad n \longrightarrow \infty. \tag{3.5}$$

Lemma 3.3. *If* $\{X_n\}$ *is a quasi-*f*-increasing sequence, where* $\{f_n\} = \{n^\beta(\log n)^\mu\}, \mu \geq 0, 0 \leq \beta < 1$*, then, under conditions* (1.13), (1.14), (2.1)*, and* (2.2)*, conditions* (1.15) *and* (3.5) *are satisfied.*

Proof. It is clear that (1.13) and (1.14)⇒(3.3). Also, (1.13) and (2.2)⇒(3.4). By Lemma 3.2, under conditions (1.13)-(1.14) and (2.1)–(2.2), we have (1.15) and (3.5). □

Lemma 3.4. *Let* $\{X_n\}$ *be a quasi-*f*-increasing sequence, where* $\{f_n\} = \{n^\beta(\log n)^\mu\}, \mu \geq 0, 0 \leq \beta < 1$*. If conditions* (1.13), (1.14)*, and* (2.2) *are satisfied, then*

$$n\beta_n X_n = O(1), \tag{3.6}$$

$$\sum_{n=1}^{\infty}\beta_n X_n < \infty. \tag{3.7}$$

Proof. It is clear that if $\{X_n\}$ is quasi-f-increasing, then $\{n^\beta(\log n)^\mu X_n\}$ is quasi-increasing. Since $\beta_n \to 0, n \to \infty$, from the fact that $\{n^{1-\beta}(\log n)^{-\mu}\}$ is increasing and (2.2), we have

$$\begin{aligned} n\beta_n X_n &= nX_n\sum_{k=n}^{\infty}|\Delta\beta_k| \\ &= O(1)n^{1-\beta}(\log n)^{-\mu}\sum_{k=n}^{\infty}k^\beta(\log k)^\mu X_k|\Delta\beta_k| \\ &= O(1)\sum_{k=n}^{\infty}kX_k|\Delta\beta_k| = O(1). \end{aligned} \tag{3.8}$$

Again using (2.2),

$$\begin{aligned}\sum_{n=1}^{\infty}\beta_n X_n &= O(1)\sum_{n=1}^{\infty}X_n\sum_{k=n}^{\infty}|\Delta\beta_k| \\ &= O(1)\sum_{k=1}^{\infty}|\Delta\beta_k|\sum_{n=1}^{k}n^{\beta}(\log n)^{\mu}X_n n^{-\beta}(\log n)^{-\mu} \\ &= O(1)\sum_{k=1}^{\infty}k^{\beta}(\log k)^{\mu}X_k|\Delta\beta_k|\sum_{n=1}^{k}n^{-\beta}(\log n)^{-\mu} \\ &= O(1)\sum_{k=1}^{\infty}kX_k(\beta,\mu)|\Delta\beta_k| = O(1).\end{aligned} \tag{3.9}$$

□

4. Proof of Theorem 2.1

Let y_n denote the nth term of the A-transform of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$y_n = \sum_{i=0}^{n}a_{ni}s_i = \sum_{v=0}^{n}\overline{a}_{nv}\lambda_v a_v. \tag{4.1}$$

Then, for $n \geq 1$, we have

$$Y_n := y_n - y_{n-1} = \sum_{v=0}^{n}\widehat{a}_{nv}\lambda_v a_v. \tag{4.2}$$

Applying Abel's transformation, we may write

$$Y_n = \sum_{v=1}^{n-1}\Delta_v(\widehat{a}_{nv}\lambda_v)\sum_{r=1}^{v}a_r + \widehat{a}_{nn}\lambda_n\sum_{v=1}^{n}a_v. \tag{4.3}$$

Since

$$\Delta_v(\widehat{a}_{nv}\lambda_v) = \lambda_v\Delta_v\widehat{a}_{nv} + \Delta\lambda_v\widehat{a}_{n,v+1}, \tag{4.4}$$

we have

$$\begin{aligned}Y_n &= a_{nn}\lambda_n s_n + \sum_{v=1}^{n-1}\Delta_v\widehat{a}_{nv}\lambda_v s_v + \sum_{v=1}^{n-1}\widehat{a}_{n,v+1}\Delta\lambda_v s_v \\ &= Y_{n,1} + Y_{n,2} + Y_{n,3}, \text{ say.}\end{aligned} \tag{4.5}$$

Since

$$|Y_{n,1} + Y_{n,2} + Y_{n,3}|^k \le 3^k\left(|Y_{n,1}|^k + |Y_{n,2}|^k + |Y_{n,3}|^k\right), \tag{4.6}$$

to complete the proof, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|Y_{n,r}|^k < \infty, \quad \text{for } r = 1,2,3. \tag{4.7}$$

Since $\{\lambda_n\}$ is bounded by Lemma 3.3, using (1.9), we have

$$\begin{aligned} I_1 &= \sum_{n=1}^{m} n^{\delta k+k-1}|Y_{n,1}|^k = \sum_{n=1}^{m} n^{\delta k+k-1}|a_{nn}\lambda_n s_n|^k \\ &\le \sum_{n=1}^{m} n^{\delta k}(na_{nn})^{k-1} a_{nn}|\lambda_n|^{k-1}|\lambda_n||s_n|^k \\ &= O(1)\sum_{n=1}^{m} n^{\delta k} a_{nn}|\lambda_n||s_n|^k. \end{aligned} \tag{4.8}$$

Using properties (1.15), in view of Lemma 3.3, and (3.7), from (1.9), (1.13), and (1.17),

$$\begin{aligned} I_1 &= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|\sum_{v=1}^{n} v^{\delta k} a_{vv}|s_v|^k + O(1)|\lambda_m|\sum_{v=1}^{m} v^{\delta k} a_{vv}|s_v|^k \\ &= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|\sum_{v=1}^{n} v^{\delta k-1}|s_v|^k + O(1)|\lambda_m|\sum_{v=1}^{m} v^{\delta k-1}|s_v|^k \\ &= O(1)\sum_{n=1}^{m-1}\beta_n X_n + O(1)|\lambda_m|X_m = O(1) \quad \text{as } m \longrightarrow \infty. \end{aligned} \tag{4.9}$$

Applying Hölder's inequality,

$$\begin{aligned} I_2 &= \sum_{n=2}^{m+1} n^{\delta k+k-1}|Y_{n,2}|^k = O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v \hat{a}_{nv}||\lambda_v||s_v|\right)^k \\ &= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta_v \hat{a}_{nv}||\lambda_v|^k|s_v|^k\left(\sum_{v=1}^{n-1}|\Delta_v \hat{a}_{nv}|\right)^{k-1}. \end{aligned} \tag{4.10}$$

Using (1.9) and (1.11) and boundedness of $\{\lambda_n\}$,

$$\begin{aligned}
I_2 &= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{v=1}^{n-1}|\Delta_v \widehat{a}_{nv}||s_v|^k|\lambda_v|^{k-1}|\lambda_v| \\
&= O(1)\sum_{v=1}^{m}|\lambda_v||s_v|^k\sum_{n=v+1}^{m+1} n^{\delta k}|\Delta_v \widehat{a}_{nv}| \\
&= O(1)\sum_{v=1}^{m} v^{\delta k}a_{vv}|\lambda_v||s_v|^k = O(1), \quad \text{as} \quad m \longrightarrow \infty,
\end{aligned} \tag{4.11}$$

as in the proof of I_1.

Finally, again using Hölder's inequality, from (1.9), (1.10), and (1.12),

$$\begin{aligned}
I_3 &= \sum_{n=2}^{m+1} n^{\delta k+k-1}|Y_{n,3}|^k = O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}\widehat{a}_{n,v+1}|\Delta\lambda_v||s_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\sum_{v=1}^{n-1}\widehat{a}_{n,v+1}|\Delta\lambda_v|^k|s_v|^k a_{vv}^{1-k}\left(\sum_{v=1}^{n-1}a_{vv}\widehat{a}_{n,v+1}\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k}\sum_{v=1}^{n-1}\widehat{a}_{n,v+1}|\Delta\lambda_v|^k|s_v|^k a_{vv}^{1-k} \\
&= O(1)\sum_{v=1}^{m}|\Delta\lambda_v|^k|s_v|^k a_{vv}^{1-k}\sum_{n=v+1}^{m+1} n^{\delta k}\widehat{a}_{n,v+1} \\
&= O(1)\sum_{v=1}^{m}(v|\Delta\lambda_v|)^k v^{\delta k}a_{vv}|s_v|^k.
\end{aligned} \tag{4.12}$$

By Lemma 3.1, condition (3.3), in view of Lemma 3.3, implies that

$$n|\Delta\lambda_n| \leq 2n\sum_{k=n}^{\infty}|\Delta|\Delta\lambda_k|| \leq 2\sum_{k=n}^{\infty}k|\Delta|\Delta\lambda_k|| \tag{4.13}$$

holds. Thus, by Lemma 3.3, (3.4) implies that $\{n|\Delta\lambda_n|\}$ is bounded. Therefore, from (1.9) and (1.13),

$$\begin{aligned}
I_3 &= O(1)\sum_{v=1}^{m}(v|\Delta\lambda_v|)^{k-1}v|\Delta\lambda_v|v^{\delta k}a_{vv}|s_v|^k \\
&= O(1)\sum_{v=1}^{m}v\beta_v v^{\delta k-1}|s_v|^k.
\end{aligned} \tag{4.14}$$

Using Abel transformation and (1.17),

$$I_3 = O(1)\sum_{v=1}^{m-1}\left|\Delta(v\beta_v)\right|\left(\sum_{r=1}^{v} r^{\delta k-1}|s_r|^k\right) + O(1)m\beta_m\sum_{v=1}^{m} v^{\delta k-1}|s_v|^k$$
$$= O(1)\sum_{v=1}^{m-1}\left|\Delta(v\beta_v)\right|X_v + O(1)m\beta_m X_m. \tag{4.15}$$

Since

$$\Delta(v\beta_v) = v\beta_v - (v+1)\beta_{v+1} = v\Delta\beta_v - \beta_{v+1}, \tag{4.16}$$

we have

$$I_3 = O(1)\sum_{v=1}^{m-1} vX_v\left|\Delta\beta_v\right| + O(1)\sum_{v=1}^{m-1} X_{v+1}\beta_{v+1} + O(1)mX_m\beta_m$$
$$= O(1), \quad \text{as } m \longrightarrow \infty, \tag{4.17}$$

by virtue of (2.2) and properties (3.6) and (3.7) of Lemma 3.4.

So we obtain (4.7). This completes the proof.

5. Corollaries and Applications to Weighted Means

Setting $\delta = 0$ in Theorems 2.1 and 2.2 yields the following two corollaries, respectively.

Corollary 5.1. *Let A satisfy conditions* (1.7)–(1.10)*, and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences satisfying conditions* (1.13)*,* (1.14)*, and* (2.1)*. If $\{X_n\}$ is a quasi-f-increasing sequence, where $\{f_n\} := \{n^\beta(\log n)^\mu\}$, $\mu \geq 0, 0 \leq \beta < 1$, and conditions* (2.2) *and*

$$\sum_{n=1}^{m}\frac{1}{n}|s_n|^k = O(X_m), \quad m \longrightarrow \infty, \tag{5.1}$$

are satisfied, then the series $\sum a_n\lambda_n$ is summable $|A|_k$, $k \geq 1$.

Proof. If we take $\delta = 0$ in Theorem 2.1, then condition (1.17) reduces condition (5.1). In this case conditions (1.11) and (1.12) are obtained by conditions (1.7)–(1.10). □

Corollary 5.2. *Let A satisfy conditions* (1.7)–(1.10)*, and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences satisfying conditions* (1.13)*,* (1.14)*, and* (2.1)*. If $\{X_n\}$ is a quasi-β-power increasing sequence for some $0 \leq \beta < 1$ and conditions* (2.3) *and* (5.1) *are satisfied, then the series $\sum a_n\lambda_n$ is summable $|A|_k$, $k \geq 1$.*

A weighted mean matrix, denoted by $(\overline{N}, p_n)$, is a lower triangular matrix with entries $a_{nv} = p_v/P_n$, where $\{p_n\}$ is nonnegative sequence with $p_0 > 0$ and $P_n := \sum_{v=0}^{n} p_v \to \infty$, as $n \to \infty$.

Corollary 5.3. *Let* $\{p_n\}$ *be a positive sequence satisfying*

$$np_n \asymp O(P_n), \quad \text{as } n \longrightarrow \infty, \tag{5.2}$$

$$\sum_{n=v+1}^{m+1} n^{\delta k} \frac{p_n}{P_n P_{n-1}} = O\left(\frac{v^{\delta k}}{P_v}\right), \tag{5.3}$$

and let $\{\beta_n\}$ *and* $\{\lambda_n\}$ *be sequences satisfying conditions* (1.13), (1.14), *and* (2.1). *If* $\{X_n\}$ *is a quasi-*f*-increasing sequence, where* $\{f_n\} := \{n^{\beta}(\log n)^{\mu}\}$, $\mu \geq 0$, $0 \leq \beta < 1$, *and conditions* (1.17) *and* (2.2) *are satisfied, then the series,* $\sum a_n \lambda_n$ *is summable* $|\overline{N}, p_n, \delta|_k$ *for* $k \geq 1$ *and* $0 \leq \delta < 1/k$.

Proof. In Theorem 2.1 set $A = (\overline{N}, p_n)$. It is clear that conditions (1.7), (1.8), and (1.10) are automatically satisfied. Condition (1.9) becomes condition (5.2), and conditions (1.11) and (1.12) become condition (5.3) for weighted mean method. □

Corollary 5.3 includes the following result with the special case $\mu = 0$.

Corollary 5.4. *Let* $\{p_n\}$ *be a positive sequence satisfying* (5.2) *and* (5.3), *and let* $\{X_n\}$ *be a quasi-*β*-power increasing sequence for some* $0 \leq \beta < 1$. *Then under conditions* (1.13), (1.14), (1.17), (2.1), *and* (2.3), $\sum a_n \lambda_n$ *is summable* $|\overline{N}, p_n, \delta|_k$, $k \geq 1$, $0 \leq \delta < 1/k$.

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