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Research Article

Stability of Homomorphisms and Generalized Derivations on Banach Algebras

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We prove the generalized Hyers-Ulam stability of homomorphisms and generalized derivations associated to the following functional equation $f(2x+y)+f(x+2y)=f(3x)+f(3y)$ on Banach algebras.

1. Introduction

The first stability problem concerning group homomorphisms was raised from a question of Ulam [1]. *Let* $(G_1,*)$ *be a group and let* $(G_2,\diamond,d)$ *be a metric group with the metric* $d(\cdot,\cdot)$*. Given* $\varepsilon>0$*, does there exist* $\delta(\epsilon)>0$ *such that if a mapping* $h:G_1\to G_2$ *satisfies the inequality*

$$d\big(h(x*y),h(x)\diamond h(y)\big)<\delta \tag{1.1}$$

for all $x,y\in G_1$*, then there is a homomorphism* $H:G_1\to G_2$ *with*

$$d(h(x),H(x))<\epsilon \tag{1.2}$$

for all $x\in G_1$*?*

Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Aoki [3] and Rassias [4] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded (see also [5]).

Theorem 1.1 (Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p) \tag{1.3}$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.4}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\varepsilon}{2-2^p}\|x\|^p \tag{1.5}$$

for all $x \in E$. If $p < 0$ then inequality (1.3) *holds for $x, y \ne 0$ and* (1.5) *for $x \ne 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.*

In 1994, a generalization of the Rassias' theorem was obtained by Găvruţa [6], who replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. For the stability problems of various functional equations and mappings and their Pexiderized versions, we refer the readers to [7–15]. We also refer readers to the books in [16–19].

Let A be a real or complex algebra. A mapping $D : A \to A$ is said to be a *(ring) derivation* if

$$D(a+b) = D(a) + D(b), \qquad D(ab) = D(a)b + aD(b) \tag{1.6}$$

for all $a, b \in A$. If, in addition, $D(\lambda a) = \lambda D(a)$ for all $a \in A$ and all $\lambda \in \mathbb{F}$, then D is called a *linear derivation*, where $\mathbb{F}$ denotes the scalar field of A. Singer and Wermer [20] proved that if A is a commutative Banach algebra and $D : A \to A$ is a continuous linear derivation, then $D(A) \subseteq \mathrm{rad}(A)$. They also conjectured that the same result holds even D is a discontinuous linear derivation. Thomas [21] proved the conjecture. As a direct consequence, we see that there are no nonzero linear derivations on a semisimple commutative Banach algebra, which had been proved by Johnson [22]. On the other hand, it is not the case for ring derivations. Hatori and Wada [23] determined a representation of ring derivations on a semi-simple commutative Banach algebra (see also [24]) and they proved that only the zero operator is a ring derivation on a semi-simple commutative Banach algebra with the maximal ideal space without isolated points. The stability of derivations between operator algebras was first obtained by Šemrl [25]. Badora [26] and Miura et al. [8] proved the Hyers-Ulam-Rassias stability of ring derivations on Banach algebras. An additive mapping $D : A \to A$ is called a *Jordan derivation* in case $D(a^2) = D(a)a + aD(a)$ is fulfilled for all $a \in A$. Every derivation is a Jordan derivation. The converse is in general not true (see [27, 28]). The concept of generalized derivation has been introduced by M. Brešar [29]. Hvala [30] and Lee [31] introduced a concept of (θ, ϕ)-derivation (see also [32]). Let θ, ϕ be automorphisms of A. An additive mapping $F : A \to A$ is called a (θ, ϕ)*-derivation* in case $F(ab) = F(a)\theta(b) + \phi(a)F(b)$ holds for all pairs $a, b \in A$. An additive mapping $F : A \to A$ is called a (θ, ϕ)*-Jordan derivation* in case $F(a^2) = F(a)\theta(a) + \phi(a)F(a)$ holds for all $a \in A$. An additive mapping $F : A \to A$

is called a *generalized* (θ,ϕ)*-derivation* in case $F(ab) = F(a)\theta(b) + \phi(a)D(b)$ holds for all pairs $a, b \in A$, where $D : A \to A$ is a (θ,ϕ)-derivation. An additive mapping $F : A \to A$ is called a *generalized* (θ,ϕ)*-Jordan derivation* in case $F(a^2) = F(a)\theta(a) + \phi(a)D(a)$ holds for all $a \in A$, where $D : A \to A$ is a (θ,ϕ)-Jordan derivation. It is clear that every generalized (θ,ϕ)-derivation is a generalized (θ,ϕ)-Jordan derivation.

The aim of the present paper is to establish the stability problem of homomorphisms and generalized (θ,ϕ)-derivations by using the fixed point method (see [7, 33–35]).

Let E be a set. A function $d : E \times E \to [0,\infty]$ is called a *generalized metric* on E if d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in E$;

(iii) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in E$.

We recall the following theorem by Margolis and Diaz.

Theorem 1.2 (See [36]). *Let (E, d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in E$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.7}$$

for all nonnegative integers n or there exists a nonnegative integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n \ge n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in E : d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \le (1/(1-L))d(y, Jy)$ for all $y \in Y$.*

2. Stability of Homomorphisms

Daróczy et al. [37] have studied the functional equation

$$f(px + (1-p)y) + f((1-p)x + py) = f(x) + f(y), \tag{2.1}$$

where $0 < p < 1$ is a fixed parameter and $f : I \to \mathbb{R}$ is unknown, I is a nonvoid open interval and (2.1) holds for all $x, y \in I$. They characterized the equivalence of (2.1) and Jensen's functional equation in terms of the algebraic properties of the parameter p. For $p = 1/2$ in (2.1), we get the Jensen's functional equation. In the present paper, we establish the general solution and some stability results concerning the functional equation (2.1) in normed spaces for $p = 1/3$. This applied to investigate and prove the generalized Hyers-Ulam stability of homomorphisms and generalized derivations in real Banach algebras. In this section, we assume that $\mathcal{X}$ is a normed algebra and $\mathcal{Y}$ is a Banach algebra. For convenience, we use the following abbreviation for a given mapping $f : \mathcal{X} \to \mathcal{Y}$,

$$Df(x, y) := f(2x + y) + f(x + 2y) - f(3x) - f(3y) \tag{2.2}$$

for all $x, y \in \mathcal{X}$.

Lemma 2.1. *Let X and Y be linear spaces. A mapping $f : X \to Y$ with $f(0) = 0$ satisfies*

$$f(2x+y) + f(x+2y) = f(3x) + f(3y) \tag{2.3}$$

for all $x, y \in X$, if and only if f is additive.

Proof. Let f satisfy (2.3). Letting $y = 0$ in (2.3), we get

$$f(x) + f(2x) = f(3x) \tag{2.4}$$

for all $x \in X$. Hence

$$[f(x) + f(-x)] + [f(2x) + f(-2x)] = f(3x) + f(-3x) \tag{2.5}$$

for all $x \in X$. Letting $y = -x$ in (2.3), we get $f(x) + f(-x) = f(3x) + f(-3x)$ for all $x \in X$. Therefore by (2.5) we have $f(2x) + f(-2x) = 0$ for all $x \in X$. This means that f is odd. Letting $y = -2x$ in (2.3) and using the oddness of f, we infer that $f(2x) = 2f(x)$ for all $x \in X$. Hence by (2.4) we have $f(3x) = 3f(x)$ for all $x \in X$. Therefore it follows from (2.3) that f satisfies

$$f(2x+y) + f(x+2y) = 3[f(x) + f(y)] \tag{2.6}$$

for all $x, y \in X$. Replacing x and y by $(2y - x)/3$ and $(2x - y)/3$ in (2.6), respectively, we get

$$f(x) + f(y) = f(2x-y) + f(2y-x) \tag{2.7}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.7) and using the oddness of f, we get

$$f(2x+y) - f(x+2y) = f(x) - f(y) \tag{2.8}$$

for all $x, y \in X$. Adding (2.6) to (2.8), we get $f(2x+y) = 2f(x) + f(y)$ for all $x, y \in X$. Using the identity $f(2x) = 2f(x)$ and replacing x by $x/2$ in the last identity, we infer that $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. Hence f is additive. The converse is obvious. □

Theorem 2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$ for which there exist functions $\varphi, \psi : \mathcal{X}^2 \to [0, \infty)$ such that*

$$\lim_{k\to\infty} \frac{1}{2^k}\psi\left(2^k x, y\right) = \lim_{k\to\infty} \frac{1}{2^k}\psi\left(x, 2^k y\right) = \lim_{k\to\infty} \frac{1}{4^k}\psi\left(2^k x, 2^k y\right) = 0, \tag{2.9}$$

$$\|Df(x,y)\| \le \varphi(x,y), \tag{2.10}$$

$$\|f(xy) - f(x)f(y)\| \le \psi(x,y) \tag{2.11}$$

for all $x, y \in \mathcal{X}$. If there exists a constant $0 < L < 1$ such that

$$\varphi(2x, 2y) \le 2L\varphi(x,y) \tag{2.12}$$

for all $x, y \in \mathcal{X}$*, then there exists a unique (ring) homomorphism* $H : \mathcal{X} \to \mathcal{Y}$ *satisfying*

$$\|f(x) - H(x)\| \le \frac{1}{2 - 2L}\phi(x), \tag{2.13}$$

$$H(x)[H(y) - f(y)] = [H(x) - f(x)]H(y) = 0 \tag{2.14}$$

for all $x, y \in \mathcal{X}$*, where*

$$\phi(x) := \varphi\left(\frac{x}{2}, 0\right) + \varphi\left(-\frac{x}{2}, 0\right) + \varphi\left(\frac{x}{2}, -\frac{x}{2}\right) + \varphi\left(-\frac{x}{3}, \frac{2x}{3}\right). \tag{2.15}$$

Proof. By the assumption, we have

$$\lim_{k\to\infty} \frac{1}{2^k}\varphi\left(2^k x, 2^k y\right) = 0 \tag{2.16}$$

for all $x, y \in \mathcal{X}$. Letting $y = 0$ in (2.10), we get

$$\|f(x) + f(2x) - f(3x)\| \le \varphi(x, 0) \tag{2.17}$$

for all $x \in \mathcal{X}$. Hence

$$\|[f(x) + f(-x)] + [f(2x) + f(-2x)] - [f(3x) + f(-3x)]\| \le \varphi(x, 0) + \varphi(-x, 0) \tag{2.18}$$

for all $x \in \mathcal{X}$. Letting $y = -x$ in (2.10), we get

$$\|[f(x) + f(-x)] - [f(3x) + f(-3x)]\| \le \varphi(x, -x) \tag{2.19}$$

for all $x \in \mathcal{X}$. Therefore by (2.18) we have

$$\|f(x) + f(-x)\| \le \varphi\left(\frac{x}{2}, 0\right) + \varphi\left(-\frac{x}{2}, 0\right) + \varphi\left(\frac{x}{2}, -\frac{x}{2}\right) \tag{2.20}$$

for all $x \in \mathcal{X}$. Letting $y = -2x$ in (2.10), we get

$$\|f(x) - f(-x) - f(2x)\| \le \varphi\left(-\frac{x}{3}, \frac{2x}{3}\right) \tag{2.21}$$

for all $x \in \mathcal{X}$. Now, it follows from (2.20) and (2.21) that

$$\|f(2x) - 2f(x)\| \le \varphi\left(\frac{x}{2}, 0\right) + \varphi\left(-\frac{x}{2}, 0\right) + \varphi\left(\frac{x}{2}, -\frac{x}{2}\right) + \varphi\left(-\frac{x}{3}, \frac{2x}{3}\right) \tag{2.22}$$

for all $x \in \mathcal{X}$. Let $E := \{g : \mathcal{X} \to \mathcal{Y},\ g(0) = 0\}$. We introduce a generalized metric on E as follows:

$$d_{\phi}(g,h) := \inf\{C \in [0,\infty] : \|g(x) - h(x)\| \le C\phi(x) \text{ for all } x \in \mathcal{X}\}. \tag{2.23}$$

It is easy to show that (E, d_{ϕ}) is a generalized complete metric space [34].

Now we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = \frac{1}{2}g(2x), \quad \forall g \in E,\ x \in \mathcal{X}. \tag{2.24}$$

Let $g, h \in E$ and let $C \in [0,\infty]$ be an arbitrary constant with $d_{\phi}(g,h) \le C$. From the definition of d_{ϕ}, we have

$$\|g(x) - h(x)\| \le C\phi(x) \tag{2.25}$$

for all $x \in \mathcal{X}$. By the assumption and the last inequality, we have

$$\|(\Lambda g)(x) - (\Lambda h)(x)\| = \frac{1}{2}\|g(2x) - h(2x)\| \le \frac{C}{2}\phi(2x) \le CL\phi(x) \tag{2.26}$$

for all $x \in \mathcal{X}$. So $d_{\phi}(\Lambda g, \Lambda h) \le L d_{\phi}(g,h)$ for any $g, h \in E$. It follows from (2.22) that $d_{\phi}(\Lambda f, f) \le 1/2$. Therefore according to Theorem 1.2, the sequence $\{\Lambda^k f\}$ converges to a fixed point H of Λ, that is,

$$H : \mathcal{X} \longrightarrow \mathcal{Y}, \qquad H(x) = \lim_{k\to\infty}\left(\Lambda^k f\right)(x) = \lim_{k\to\infty}\frac{1}{2^k}f\left(2^k x\right) \tag{2.27}$$

and $H(2x) = 2H(x)$ for all $x \in \mathcal{X}$. Also H is the unique fixed point of Λ in the set $E_{\phi} = \{g \in E : d_{\phi}(f,g) < \infty\}$ and

$$d_{\phi}(H,f) \le \frac{1}{1-L}d_{\phi}(\Lambda f, f) \le \frac{1}{2-2L}, \tag{2.28}$$

that is, inequality (2.13) holds true for all $x \in \mathcal{X}$. It follows from the definition of H, (2.10), and (2.16) that $DH(x,y) = 0$ for all $x, y \in \mathcal{X}$. Since $H(0) = 0$, by Lemma 2.1 the mapping H is additive. So it follows from the definition of H, (2.9), and (2.11) that

$$\begin{aligned}\|H(xy) - H(x)H(y)\| &= \lim_{k\to\infty}\frac{1}{4^k}\left\|f\left(4^k xy\right) - f\left(2^k x\right)f\left(2^k y\right)\right\| \\ &\le \lim_{k\to\infty}\frac{1}{4^k}\varphi\left(2^k x, 2^k y\right) = 0\end{aligned} \tag{2.29}$$

for all $x, y \in \mathcal{X}$. So H is homomorphism. Similarly, we have from (2.9) and (2.11) that

$$H(xy) = H(x)f(y), \qquad H(xy) = f(x)H(y) \tag{2.30}$$

for all $x, y \in \mathcal{X}$. Since H is homomorphism, we get (2.14) from (2.30).

Finally it remains to prove the uniqueness of H. Let $H_1 : \mathcal{X} \to \mathcal{Y}$ another homomorphism satisfying (2.13). Since $d_\phi(f, H_1) \leq 1/(2-2L)$ and H_1 is additive, we get $H_1 \in E_\phi$ and $(\Lambda H_1)(x) = (1/2)H_1(2x) = H_1(x)$ for all $x \in \mathcal{X}$, that is, H_1 is a fixed point of Λ. Since H is the unique fixed point of Λ in E_ϕ, we get $H_1 = H$. □

We need the following lemma in the proof of the next theorem.

Lemma 2.3 (See [38]). *Let X and Y be linear spaces and $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.4. *Let X and Y be linear spaces. A mapping $f : X \to Y$ satisfies*

$$f(2\mu x + \mu y) + f(\mu x + 2\mu y) = \mu[f(3x) + f(3y)] \tag{2.31}$$

for all $x, y \in X$ and all $\mu \in \mathbb{T}^1$, if and only if f is $\mathbb{C}$-linear.

Proof. Let f satisfy (2.31). Letting $x = y = 0$ in (2.31), we get $f(0) = 0$. By Lemma 2.1, the mapping f is additive. Letting $y = 0$ in (2.31) and using the additivity of f, we get that $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1$. So by Lemma 2.4, the mapping f is $\mathbb{C}$-linear. The converse is obvious. □

The following theorem is an alternative result of Theorem 2.2 with similar proof.

Theorem 2.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exist functions $\varphi, \psi : \mathcal{X}^2 \to [0, \infty)$ such that*

$$\begin{gathered}
\lim_{k\to\infty} 2^k \varphi\left(\frac{1}{2^k}x, y\right) = \lim_{k\to\infty} 2^k \psi\left(x, \frac{1}{2^k}y\right) = \lim_{k\to\infty} 4^k \psi\left(\frac{1}{2^k}x, \frac{1}{2^k}y\right) = 0, \\
\| f(2\mu x + \mu y) + f(\mu x + 2\mu y) - \mu[f(3x) + f(3y)] \| \leq \varphi(x, y), \\
\| f(xy) - f(x)f(y) \| \leq \psi(x, y)
\end{gathered} \tag{2.32}$$

for all $x, y \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$. If there exists a constant $0 < L < 1$ such that

$$2\varphi\left(\frac{1}{2}x, \frac{1}{2}y\right) \leq L\varphi(x, y) \tag{2.33}$$

for all $x, y \in \mathcal{X}$, then there exists a unique homomorphism $H : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x) - H(x)\| \leq \frac{L}{2-2L}\phi(x),$$
$$H(x)[H(y) - f(y)] = [H(x) - f(x)]H(y) = 0 \tag{2.34}$$

for all $x, y \in \mathcal{X}$, where $\phi(x)$ is defined as in Theorem 2.2.

Proof. It follows from the assumptions that $\varphi(0,0) = 0$, and so $f(0) = 0$. The rest of the proof is similar to the proof of Theorem 2.2 and we omit the details. □

Corollary 2.6. *Let $p, q, \delta, \varepsilon$ be non-negative real numbers with $0 < p,\ q < 1$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping such that*

$$\|f(2\mu x + \mu y) + f(\mu x + 2\mu y) - \mu[f(3x) + f(3y)]\| \leq \delta + \varepsilon(\|x\|^p + \|y\|^p),$$
$$\|f(xy) - f(x)f(y)\| \leq \delta + \varepsilon(\|x\|^q + \|y\|^q) \tag{2.35}$$

for all $x, y \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique homomorphism $H : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x) - H(x)\| \leq \frac{4\delta}{2-2^p} + \frac{2^p + 4 \times 3^p + 4^p}{6^p(2-2^p)}\varepsilon\|x\|^p,$$
$$H(x)[H(y) - f(y)] = [H(x) - f(x)]H(y) = 0 \tag{2.36}$$

for all $x, y \in \mathcal{X}$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x,y) := \delta + \varepsilon(\|x\|^p + \|y\|^p), \qquad \psi(x,y) := \delta + \varepsilon(\|x\|^q + \|y\|^q) \tag{2.37}$$

for all $x, y \in \mathcal{X}$. Then we can choose $L = 2^{p-1}$ and we get the desired results. □

Corollary 2.7. *Let p, q, ε be non-negative real numbers with $p > 1$ and $q > 2$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping such that*

$$\|f(2\mu x + \mu y) + f(\mu x + 2\mu y) - \mu[f(3x) + f(3y)]\| \leq \varepsilon(\|x\|^p + \|y\|^p),$$
$$\|f(xy) - f(x)f(y)\| \leq \varepsilon(\|x\|^q + \|y\|^q) \tag{2.38}$$

for all $x, y \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique homomorphism $H : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x) - H(x)\| \leq \frac{2^p + 4 \times 3^p + 4^p}{6^p(2^p-2)}\varepsilon\|x\|^p,$$
$$H(x)[H(y) - f(y)] = [H(x) - f(x)]H(y) = 0 \tag{2.39}$$

for all $x, y \in \mathcal{X}$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x,y) := \varepsilon(\|x\|^p + \|y\|^p), \qquad \psi(x,y) := \varepsilon(\|x\|^q + \|y\|^q) \tag{2.40}$$

for all $x, y \in \mathcal{X}$. Then we can choose $L = 2^{1-p}$ and we get the desired results. □

3. Stability of Generalized (θ, ϕ)-Derivations

In this section, we assume that $\mathcal{Y}$ is a Banach algebra, and θ, ϕ are automorphisms of $\mathcal{Y}$. For convenience, we use the following abbreviation for given mappings $f, g : \mathcal{Y} \to \mathcal{Y}$:

$$\begin{aligned} D_{f,g}^{\theta,\phi}(x,y) &:= f(xy) - f(x)\theta(y) - \phi(x)g(y), \\ J_{f,g}^{\theta,\phi}(x) &:= f\left(x^2\right) - f(x)\theta(x) - \phi(x)g(x) \end{aligned} \tag{3.1}$$

for all $x, y \in \mathcal{Y}$. Now we prove the generalized Hyers-Ulam stability of generalized (θ, ϕ)-derivations and generalized (θ, ϕ)-Jordan derivations in Banach algebras.

Theorem 3.1. *Let $f, g : \mathcal{Y} \to \mathcal{Y}$ be mappings with $f(0) = g(0) = 0$ for which there exists a function $\varphi : \mathcal{Y}^2 \to [0, \infty)$ such that*

$$\|Df(x,y)\| \le \varphi(x,y), \tag{3.2}$$

$$\left\|J_{f,g}^{\theta,\phi}(x)\right\| \le \varphi(x,x), \tag{3.3}$$

$$\|Dg(x,y)\| \le \varphi(x,y), \tag{3.4}$$

$$\left\|J_{g,g}^{\theta,\phi}(x)\right\| \le \varphi(x,x) \tag{3.5}$$

for all $x, y \in \mathcal{Y}$. If there exists a constants $0 < L < 1$ such

$$4\varphi(x,y) \le L\varphi(2x,2y) \tag{3.6}$$

for all $x, y \in \mathcal{Y}$, then there exist a unique (θ, ϕ)-Jordan derivation $G : \mathcal{Y} \to \mathcal{Y}$ and a unique generalized (θ, ϕ)-Jordan derivation $F : \mathcal{Y} \to \mathcal{Y}$ satisfying

$$\begin{aligned} \|f(x) - F(x)\| &\le \frac{L}{4-2L}\phi(x), \\ \|g(x) - G(x)\| &\le \frac{L}{4-2L}\phi(x) \end{aligned} \tag{3.7}$$

for all $x \in \mathcal{Y}$, where $\phi(x)$ is defined as in Theorem 2.2.

Proof. It follows from the assumptions that

$$\lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{3.8}$$

for all $x, y \in \mathcal{Y}$. By the proof of Theorem 2.5, there exist unique additive mappings $F, G : \mathcal{Y} \to \mathcal{Y}$ satisfying (3.7) and

$$F(x) = \lim_{k\to\infty} 2^k f\left(\frac{1}{2^k}x\right), \qquad G(x) = \lim_{k\to\infty} 2^k g\left(\frac{1}{2^k}x\right) \tag{3.9}$$

for all $x \in \mathcal{Y}$. It follows from the definitions of F, G (3.3), and (3.8) that

$$\begin{aligned}\left\|J_{F,G}^{\theta,\phi}(x)\right\| &= \lim_{n\to\infty} 4^n \left\|J_{f,g}^{\theta,\phi}\left(\frac{x}{2^n}\right)\right\| \le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{x}{2^n}\right) = 0,\\ \left\|J_{G,G}^{\theta,\phi}(x)\right\| &= \lim_{n\to\infty} 4^n \left\|J_{g,g}^{\theta,\phi}\left(\frac{x}{2^n}\right)\right\| \le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{x}{2^n}\right) = 0\end{aligned} \tag{3.10}$$

for all $x \in \mathcal{Y}$. Hence

$$F\left(x^2\right) = F(x)\theta(x) + \phi(x)G(x), \qquad G\left(x^2\right) = G(x)\theta(x) + \phi(x)G(x) \tag{3.11}$$

for all $x \in \mathcal{Y}$. Hence G is a (θ, ϕ)-Jordan derivation and F is a generalized (θ, ϕ)-Jordan derivation. □

Remark 3.2. Applying Theorem 3.1 for the case $\varphi(x, y) := \varepsilon(\|x\|^p + \|y\|^p)$ ($\varepsilon \ge 0$ and $p > 2$), there exist a unique (θ, ϕ)-Jordan derivation $G : \mathcal{Y} \to \mathcal{Y}$ and a unique generalized (θ, ϕ)-Jordan derivation $F : \mathcal{Y} \to \mathcal{Y}$ satisfying

$$\begin{aligned}\|f(x) - F(x)\| &\le \frac{2^p + 4 \times 3^p + 4^p}{6^p(2^p - 2)} \varepsilon \|x\|^p,\\ \|g(x) - G(x)\| &\le \frac{2^p + 4 \times 3^p + 4^p}{6^p(2^p - 2)} \varepsilon \|x\|^p\end{aligned} \tag{3.12}$$

for all $x \in \mathcal{Y}$.

The following theorem is an alternative result of Theorem 3.1 with similar proof.

Theorem 3.3. *Let $f, g : \mathcal{Y} \to \mathcal{Y}$ be mappings with $f(0) = g(0) = 0$ for which there exists a function $\varphi : \mathcal{Y}^2 \to [0, \infty)$ satisfying (3.2)–(3.5). If there exists a constant $0 < L < 1$ such*

$$\varphi(2x, 2y) \le 2L\varphi(x, y) \tag{3.13}$$

for all $x, y \in \mathcal{Y}$, then there exist a unique (θ,ϕ)-Jordan derivation $G : \mathcal{Y} \to \mathcal{Y}$ and a unique generalized (θ,ϕ)-Jordan derivation $F : \mathcal{Y} \to \mathcal{Y}$ satisfying

$$\|f(x) - F(x)\| \le \frac{1}{2-2L}\phi(x),$$
$$\|g(x) - G(x)\| \le \frac{1}{2-2L}\phi(x) \tag{3.14}$$

for all $x \in \mathcal{Y}$, where $\phi(x)$ is defined as in Theorem 2.2.

Remark 3.4. Applying Theorem 3.3 for the case $\varphi(x,y) := \delta + \varepsilon(\|x\|^p + \|y\|^p)$ ($\delta, \varepsilon \ge 0$ and $0 < p < 1$), there exist a unique (θ,ϕ)-Jordan derivation $G : \mathcal{Y} \to \mathcal{Y}$ and a unique generalized (θ,ϕ)-Jordan derivation $F : \mathcal{Y} \to \mathcal{Y}$ satisfying

$$\|f(x) - F(x)\| \le \frac{4\delta}{2-2^p} + \frac{2^p + 4 \times 3^p + 4^p}{6^p(2-2^p)}\varepsilon\|x\|^p,$$
$$\|g(x) - G(x)\| \le \frac{4\delta}{2-2^p} + \frac{2^p + 4 \times 3^p + 4^p}{6^p(2-2^p)}\varepsilon\|x\|^p \tag{3.15}$$

for all $x \in \mathcal{Y}$.

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