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Research Article

Quasivariational Inequalities for a Dynamic Competitive Economic Equilibrium Problem

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The aim of this paper is to consider a dynamic competitive economic equilibrium problem in terms of maximization of utility functions and of excess demand functions. This equilibrium problem is studied by means of a time-dependent quasivariational inequality which is set in the Lebesgue space $L^2([0,T],\mathbb{R})$. This approach allows us to obtain an existence result of time-dependent equilibrium solutions.

1. Introduction

The theory of variational inequality was born in the 1970s, driven by the solution given by G. Fichera to the Signorini problem on the elastic equilibrium of a body under unilateral constraints and by Stampacchia's work on defining the capacitory potential associated to a nonsymmetric bilinear form.

It is possible to attach to this theory a preliminary role in establishing a close relationship between theory and applications in a wide range of problems in mechanics, engineering, mathematical programming, control, and optimization [1–4]. In this paper, a dynamic competitive economic equilibrium problem by using a variational formulation is studied. It was Walras [5] who, in 1874, laid the foundations for the study of the general equilibrium theory, providing a succession of models, each taking into account more aspects of a real economy. The rigorous mathematical formulation of the general equilibrium problem, with possibly nonsmooth but convex data, was elaborated by Arrow and Debreu [6] in the 1954. In 1985, Border in [7] elaborated a variational inequality formulation of a Walrasian price equilibrium. By means of the variational formulation, Dafermos in [8] and Zhao in [9] proved some qualitative results for the solutions to the Walrasian problem in the static case. Moreover, Nagurney and Zhao [10] (see also Zhao [9], Dafermos and Zhao

[11]) considered the static Walrasian price equilibrium problem as a network equilibrium problem over an abstract network with very simple structure that consists of a single origin-destination pair of nodes and single links joining the two nodes. Furthermore, the characterization of Walrasian price equilibrium vectors as solutions of a variational inequality induces efficient algorithms for their computation (for further details see also Nagurney's book [12, Chapter 9], and its complete bibliography).

In [13] it was proven how, by introducing the Lagrange multipliers, a general economic equilibrium with utility function can be represented by a variational inequality problem. In recent years, some papers have been devoted to the study of the influence of time on the equilibrium problems in terms of variational inequality problems in suitable Lebesgue space [14–22]. We refer the interested reader to the book [23] where a variety of problems arising from economics, finance, or transportation science are formulated in Lebesgue spaces. In this paper, we have focused on the generalization of the dynamic case of the competitive economic equilibrium problem studied, in the static case, in [24–26].

The paper is organized as follows. In Section 2 we introduce the evolution in time of the competitive economic equilibrium problem in which the data depend on time $t \in [0,T]$ and we show how the governing equilibrium conditions can be formulated in terms of an evolutionary quasivariational inequality. By means of this characterization, in Section 3, we are able to give an existence result for the equilibrium solutions by using a two-step procedure. Firstly, we give the existence and uniqueness to the equilibrium consumption and for this equilibrium we achieve a regularity result. Then we are able to prove the existence of the competitive prices.

2. Walrasian Pure Exchange Model

During a period of time $[0,T]$, $T > 0$, we consider a marketplace consisting of l different goods indexed by $j = 1,\ldots,l$, $l > 1$, and n agents indexed by $a = 1,\ldots,n$.

Each agent $a = 1,2,\ldots,n$ is endowed at least with a positive quantity of commodity:

$$\forall a = 1,\ldots,n \quad \exists j : e_a^j(t) > 0 \quad \text{a.e.} \quad [0,T], \tag{2.1}$$

and we denote by

$$e_a(t) = \left(e_a^1(t), e_a^2(t), \ldots, e_a^l(t)\right) \tag{2.2}$$

the endowment vector relative to the agent a at the time t. The consumption relative to the agent a at the time t is

$$x_a(t) = \left(x_a^1(t), x_a^2(t), \ldots, x_a^l(t)\right), \tag{2.3}$$

where $x_a^j(t)$ is the nonnegative consumption relative to the commodity j. Furthermore,

$$x(t) \equiv (x_1(t), x_2(t), \ldots, x_n(t))^T \tag{2.4}$$

represents the consumption of the market at the time $t \in [0,T]$. We associate to each commodity j, $j = 1,2,\ldots,l$, at the time $t \in [0,T]$, a nonnegative price $p^j(t)$ and we denote by

$$p(t) = \left(p^1(t), p^2(t), \ldots, p^l(t)\right) \tag{2.5}$$

the price vector at the time t. We assume that the *free disposal* of commodities is assumed, that is, the a priori exclusion of negative prices. We choose the vectors e_a, x_a, and p in the Hilbert space $L^2([0,T],\mathbb{R}^l) = L$ and x in $L^2([0,T],\mathbb{R}^{n\times l})$.

In this economy, only pure exchanges are assumed: the only activity of each agent is to trade (that is buy and sell) his own commodities with each other agent. At the time t, agent's preferences for consuming different goods are given by his utility function $u_a(t, x_a(t))$ defined on $[0,T] \times \mathbb{R}^l$. In this market, the aim of each agent is to maximize their utility, in the period of time $[0,T]$, by performing pure exchanges of the given goods. There are natural constraints that the consumers must satisfy: the wealth of a consumer, in the period $[0,T]$, is his endowment, and the total amount of commodities that a consumer can buy in the period $[0,T]$ is at most equal to the total amount of commodities that the consumer sells off during the whole period $[0,T]$. This means that, for all $a = 1,\ldots,n$ and for all $p \in P$, one has the following maximization problem:

$$\mathcal{U}_a(\overline{x}_a) = \max_{x_a \in M_a(p)} \int_0^T u_a(t, x_a(t))\,dt, \tag{2.6}$$

where

$$\begin{aligned} M_a(p) &= \left\{ x_a \in L : x_a^j(t) \geq 0,\ \forall j = 1,\ldots,l,\ \text{a.e.}\ \ [0,T],\ \langle p, x_a - e_a\rangle_L \leq 0 \right\}, \\ p \in P &= \left\{ p \in L : p^j(t) \geq 0,\ \forall j = 1,\ldots,l,\ \sum_{j=1}^{l} p^j(t) = 1 \text{ a.e. } [0,T] \right\}. \end{aligned} \tag{2.7}$$

For each $a = 1,\ldots,n$ and $p \in P$, $M_a(p)$ is a closed and convex set of L.

We assume that the utility function, for each agent $a = 1,\ldots,n$, satisfies the following assumptions:

(U_1) $u_a(t,\cdot)$ is concave a.e. $[0,T]$,

(U_2) $u_a(t,\cdot) \in C^1(\mathbb{R}^l_+)$ a.e. $[0,T]$,

(U_3) for all $p \in P$: for all $x_a \in M_a(p), \nabla u_a(t, x_a(t)) \neq 0$ a.e. $[0,T]$; moreover for all $x_a \in M_a(p)$ such that $x_a^s(t) = 0$ in E, for all $E \subseteq [0,T]$, $m(E) > 0$, it results $\partial u_a(t, x_a(t))/\partial x_a^s > 0$ in E.

If there is $\overline{x}_a$, the solution to maximization problem (2.6), we pose $\overline{x}_a = \overline{x}_a(p)$ and $z(p) = \sum_{a=1}^{n}(\overline{x}_a(p) - e_a)$.

Then the definition of the dynamic competitive equilibrium problem for a pure exchange economy takes the following form.

Definition 2.1. Let $\overline{p} \in P$ and $\overline{x}(\overline{p}) \in M(\overline{p}) = \prod_{a=1}^{n} M_a(\overline{p})$. The pair $(\overline{p}, \overline{x}(\overline{p})) \in P \times M(\overline{p})$ is a dynamic competitive equilibrium if and only if for all $a = 1, \dots, n$,

$$\mathcal{U}_a(\overline{x}_a) = \max_{x_a \in M_a(\overline{p})} \int_0^T u_a(t, x_a(t))dt, \tag{2.8}$$

and for all $j = 1, 2, \dots, l$ and a.e. $[0, T]$:

$$z(\overline{p})(t) = \sum_{a=1}^{n} \left(\left(\overline{x}_a^j(\overline{p}) \right)(t) - e_a^j(t) \right) \leq 0. \tag{2.9}$$

Our purpose is to give the following characterization.

Theorem 2.2. *The pair $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ is a dynamic competitive equilibrium of a pure exchange economic market with utility function if and only if it is a solution to the evolutionary quasivariational inequality*

Find $(\overline{p}, \overline{x}(\overline{p})) \in P \times M(\overline{p})$ such that

$$\sum_{a=1}^{n} \langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle_L + \left\langle \sum_{a=1}^{n} (\overline{x}_a(\overline{p}) - e_a), p - \overline{p} \right\rangle_L \leq 0 \quad \forall (p, x) \in P \times M(\overline{p}). \tag{2.10}$$

Proof. Firstly, we observe that the pair $(\overline{p}, \overline{x}(\overline{p})) \in P \times M(\overline{p})$ is a solution to evolutionary quasivariational inequality (2.10) if and only if $\overline{x}(\overline{p})$ is a solution to evolutionary variational inequality

$$\langle \nabla u_a(\overline{x}_a(\overline{p})), x_a - \overline{x}_a(\overline{p}) \rangle_L \leq 0, \quad \forall x_a \in M_a(\overline{p}), \tag{2.11}$$

and $\overline{p}$ is a solution to evolutionary variational inequality

$$\left\langle \sum_{a=1}^{n} (\overline{x}_a(\overline{p}) - e_a), p - \overline{p} \right\rangle_L \leq 0 \quad \forall p \in P. \tag{2.12}$$

Now, we will prove the theorem by means of the following steps.

(1) For all $p \in P$, $\overline{x}_a \in M_a(p)$ is a solution to the problem (2.6) if and only if $\overline{x}_a$ is a solution to the variational problem

$$\langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle_L \leq 0, \quad \forall x_a \in M_a(p). \tag{2.13}$$

In fact, let us assume that $\overline{x}_a$ is a solution to problem (2.6); for all $x_a \in M_a(p)$ we can define the functional

$$F(\lambda) = \int_0^T u_a(t, \lambda \overline{x}_a(t) + (1 - \lambda) x_a(t))dt, \quad \forall \lambda \in [0, 1]. \tag{2.14}$$

For all $\lambda \in [0,1]$, it results in the following:

$$F(\lambda) \leq \max_{x_a \in M_a(p)} \int_0^T u_a(t, x_a(t))dt = \int_0^T u_a(t, \overline{x}_a(t))dt = F(1), \tag{2.15}$$

then $F(\cdot)$ admits the maximum solution when $\lambda = 1$ and $F'(1) \geq 0$. Hence we can consider the derivative of $F(\cdot)$ with respect to λ:

$$\begin{aligned} F'(\lambda) &= \frac{\partial}{\partial\lambda} \int_0^T u_a(t, \lambda\overline{x}_a(t) + (1-\lambda)x_a(t))dt \\ &= \int_0^T \sum_{j=1}^{l} \frac{\partial u_a(t, \lambda\overline{x}_a(t) + (1-\lambda)x_a(t))}{\partial x_a^j} \left(\overline{x}_a^j(t) - x_a^j(t)\right)dt, \end{aligned} \tag{2.16}$$

and we obtain

$$\begin{aligned} F'(1) &= \int_0^T \sum_{j=1}^{l} \frac{\partial u_a(t, \overline{x}_a(t))}{\partial x_a^j} \left(\overline{x}_a^j(t) - x_a^j(t)\right)dt \\ &= \langle \nabla u_a(\overline{x}_a), \overline{x}_a - x_a \rangle_L \geq 0 \quad \forall x_a \in M_a(p), \end{aligned} \tag{2.17}$$

namely, the variational inequality (2.13).

Conversely, let us assume that $\overline{x}_a \in M_a(p)$ is a solution to variational problem (2.13). Since $u_a(t, x_a)$ is concave a.e. $[0,T]$, the functional

$$\mathcal{U}_a(x_a) = \int_0^T u_a(t, x_a(t))dt \tag{2.18}$$

is concave, then for all $x_a \in M_a(p)$, the following estimate holds:

$$\mathcal{U}_a(\lambda x_a + (1-\lambda)\overline{x}_a) \geq \lambda\mathcal{U}_a(x_a) + (1-\lambda)\mathcal{U}_a(\overline{x}_a), \quad \forall\lambda \in [0,1], \tag{2.19}$$

namely, for all $\lambda \in (0,1]$:

$$\frac{\mathcal{U}_a(\overline{x}_a + \lambda(x_a - \overline{x}_a)) - \mathcal{U}_a(\overline{x}_a)}{\lambda} \geq \mathcal{U}_a(x_a) - \mathcal{U}_a(\overline{x}_a). \tag{2.20}$$

When $\lambda \to 0^+$, the left-hand side of (2.20) converges to

$$\left[\frac{\partial}{\partial\lambda}\mathcal{U}_a(\overline{x}_a + \lambda(x_a - \overline{x}_a))\right]_{\lambda=0} = \sum_{j=1}^{l} \frac{\partial\mathcal{U}_a(\overline{x}_a)}{\partial x_a^j}\left(x_a^j - \overline{x}_a^j\right) = \langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle_L; \tag{2.21}$$

so, from (2.20) and since $\overline{x}_a$ is a solution to variational inequality (2.13), it follows that

$$0 \geq \langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle_L \geq \mathcal{U}_a(x_a) - \mathcal{U}_a(\overline{x}_a), \quad \forall x_a \in M_a(p). \tag{2.22}$$

Hence $\overline{x}_a$ is a solution to the problem (2.6).

(2) The solution to variational inequality (2.13) belongs to the set

$$\Gamma_a(p) = \left\{ x_a \in L : x_a^j(t) \geq 0 \text{ a.e. } [0,T],\ j = 1,\dots,l,\ \langle p, x_a - e_a \rangle_L = 0 \right\}. \tag{2.23}$$

In fact, first of all, let us show that there exists

$$\begin{gathered} x'_a \in C = \{x_a \in L : x_a(t) \geq 0 \text{ a.e. } [0,T]\} \text{ such that} \\ \int_0^T u_a(t, x'_a(t))dt > \int_0^T u_a(t, \overline{x}_a(t))dt. \end{gathered} \tag{2.24}$$

Ab absurdum, let us assume that for all $x_a \in C$ it results in

$$\int_0^T u_a(t, x_a(t))dt \leq \int_0^T u_a(t, \overline{x}_a(t))dt. \tag{2.25}$$

Then $\overline{x}_a$ is the maximal point of the problem (2.6) on C and from step (1):

$$\langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle_L \leq 0, \quad \forall x_a \in C. \tag{2.26}$$

By (2.26) it follows that $\overline{x}_a^j(t) > 0$ a.e. $[0,T]$ for all $j = 1,\dots,l$. In fact, let us suppose that there exist j^* and $E \subseteq [0,T]$ with $m(E) > 0$ such that $\overline{x}_a^{j^*}(t) = 0$ in E. Let us assume in (2.26) $x_a \in C$ such that

$$x_a^j(t) = \begin{cases} \overline{x}_a^j(t) & \text{in } [0,T] \setminus E,\ j = 1,\dots,l, \\ \overline{x}_a^j(t) & \text{in } E,\ \text{for } j = 1,\dots,l,\ j \neq j^*, \\ x_a^{j^*}(t) \geq 0 & \text{in } E \text{ for } j = j^*,\ x_a^{j^*} \in L^2(E), \end{cases} \tag{2.27}$$

we get

$$\int_E \frac{\partial u_a(t, \overline{x}_a(t))}{\partial x_a^{j^*}} x_a^{j^*}(t)dt \leq 0, \quad \forall x_a^{j^*} \in L^2(E),\ x_a^{j^*}(t) \geq 0 \text{ in } E. \tag{2.28}$$

From (2.28) it derives that $\partial u_a(t, \overline{x}_a(t))/\partial x_a^{j^*} \leq 0$ in E, that is, assumption (U_3) is contradicted. Hence from (2.26) it results in that $\overline{x}_a^j(t) > 0$ a.e. $[0, T]$ for all $j = 1, \ldots, l$.

Let us fix $i = 1, \ldots, l$, since $\overline{x}_a^i(t) > 0$ a.e. $[0, T]$, we can choose

$$x_a^j(t) = \begin{cases} \overline{x}_a^i(t) \pm \varepsilon(t) & \text{in } [0, T], \text{ for } j = i, \\ \overline{x}_a^j(t) & \text{in } [0, T], \text{ for } j = 1, \ldots, l, \ j \neq i, \end{cases} \tag{2.29}$$

where $\varepsilon(t) \in L^2([0, T])$ and $0 < \varepsilon(t) \leq \overline{x}_a^i(t)$ a.e. $[0, T]$. From (2.26) we get

$$\int_0^T \sum_{j=1}^{l} \frac{\partial u_a(t, \overline{x}_a(t))}{\partial x_a^j} \left(x_a^j(t) - \overline{x}_a^j(t) \right) dt = \int_0^T \frac{\partial u_a(t, \overline{x}_a(t))}{\partial x_a^i} (\pm \varepsilon(t)) dt \leq 0, \tag{2.30}$$

namely,

$$\int_0^T \frac{\partial u_a(t, \overline{x}_a(t))}{\partial x_a^i} \varepsilon(t) dt = 0, \quad \forall \varepsilon(t) : 0 < \varepsilon(t) \leq \overline{x}_a^i(t). \tag{2.31}$$

Hence,

$$\frac{\partial u_a(t, \overline{x}_a(t))}{\partial x_a^j} = 0 \quad \text{a.e.} \quad [0, T]. \tag{2.32}$$

Condition (2.32) contradicts the assumption (U_3) and the estimate (2.24) is proved.

Now, let us show that

$$\langle p, \overline{x}_a - e_a \rangle_L = 0. \tag{2.33}$$

Ab absurdum, let us assume that $\langle p, \overline{x}_a - e_a \rangle_L < 0$ and choose $\delta > 0$ and $\lambda > 0$ such that

$$\delta + \langle p, \overline{x}_a - e_a \rangle_L < 0, \qquad \lambda < \frac{\delta}{\|x'_a - \overline{x}_a\|_L \|p\|_L}. \tag{2.34}$$

We have

$$\begin{aligned} \langle p, \lambda x'_a + (1 - \lambda)\overline{x}_a - e_a \rangle_L &= \lambda \langle p, x'_a - \overline{x}_a \rangle_L + \langle p, \overline{x}_a - e_a \rangle_L \\ &\leq \lambda \|p\|_L \|x'_a - \overline{x}_a\|_L + \langle p, \overline{x}_a - e_a \rangle_L < \delta + \langle p, \overline{x}_a - e_a \rangle_L < 0, \end{aligned} \tag{2.35}$$

namely, $\lambda x'_a + (1-\lambda)\overline{x}_a \in M_a(p)$. Furthermore, being $u_a(t, x_a(t))$ concave a.e. $[0,T]$ and by (2.24), we have

$$\int_0^T u_a\left(t, \lambda x'_a(t) + (1-\lambda)\overline{x}_a(t)\right)dt > \int_0^T u_a(t, \overline{x}_a(t))dt \quad \forall \lambda \in]0,1], \tag{2.36}$$

that is, $\lambda x'_a + (1-\lambda)\overline{x}_a$ is a solution to maximization problem (2.6) against the assumption on $\overline{x}_a$.

Then for all $a = 1, \ldots, l$, each solution to evolutionary variational inequality (2.13), satisfies the following condition:

$$\left\langle p, \overline{x}_a - e_a \right\rangle_L = 0, \quad \forall p \in P, \tag{2.37}$$

that is, the well-known Walras law.

(3) It holds that $\overline{p} \in P$ satisfies condition (2.9) if and only if it is a solution to variational inequality

$$\left\langle \sum_{a=1}^{n} (\overline{x}_a(\overline{p}) - e_a), p - \overline{p} \right\rangle_L \le 0 \quad \forall p \in P. \tag{2.38}$$

In fact, for the readers' convenience we report the proof of Theorem 1 of [18]. We observe that from Walras' law, the variational inequality (2.38) is equivalent to

$$\left\langle z(\overline{p}), p \right\rangle_L \le 0 \quad \forall p \in P, \tag{2.39}$$

where $z(p) = \sum_{a=1}^{n}(\overline{x}_a(p) - e_a)$. Let $\overline{p} \in P$ be an equilibrium price vector, that is, it satisfies (2.9). We have $z^j(\overline{p}(t)) \le 0$ a.e. $[0,T]$ for each $j = 1, 2, \ldots, l$ and because $p \in P$, it results in $p^j(t) \ge 0$ a.e. $[0,T]$ for each $j = 1, 2, \ldots, l$. Therefore, $z^j(\overline{p}(t)) \cdot p^j(t) \le 0$ a.e. $[0,T]$ for each $j = 1, 2, \ldots, l$, namely, $\overline{p}$ is a solution to variational inequalities (2.39) and (2.38). Viceversa, let $\overline{p} \in P$ be a solution to variational inequality (2.39) (or (2.38)). Suppose that there exist an index i and a subset $E \subseteq [0,T]$ with $m(E) > 0$ such that

$$z^i\left(\overline{p}(t)\right) > 0 \quad \forall t \in E. \tag{2.40}$$

Let us assume in (2.39), $p \in P$ such that

$$p^j(t) = \begin{cases} \overline{p}^j(t) & \text{a.e. } [0,T] \setminus E, \\ \varepsilon & \text{in } E \text{ for } j \neq i, \ j = 1, \ldots, l, \\ 1 - (l-1)\varepsilon & \text{in } E \text{ for } j = i, \end{cases} \tag{2.41}$$

where

$$\varepsilon \begin{cases} \in \left]0, \dfrac{1}{l-1}\right[& \text{if } \displaystyle\int_E \sum_{\substack{j=1 \\ j\neq i}}^{l} \left(z^j(\overline{p}(t)) - z^i(\overline{p}(t))\right)dt \geq 0, \\ < \min\left\{ \dfrac{1}{l-1}, \dfrac{\int_E z^i(\overline{p}(t))dt}{\int_E \sum_{j=1, j\neq i}^{l}\left(z^i(\overline{p}(t)) - z^j(\overline{p}(t))\right)dt} \right\} & \\ & \text{if } \displaystyle\int_E \sum_{\substack{j=1 \\ j\neq i}}^{l} \left(z^j(\overline{p}(t)) - z^i(\overline{p}(t))\right)dt < 0. \end{cases} \tag{2.42}$$

We have

$$\begin{aligned} &\int_{[0,T]\setminus E} \sum_{j=1}^{l} z^j(\overline{p}(t)) \cdot \left(\overline{p}^j(t)\right)dt + \int_E \sum_{\substack{j=1 \\ j\neq i}}^{l} z^j(\overline{p}(t)) \cdot \varepsilon\, dt + \int_E z^i(\overline{p}(t)) \cdot (1-(l-1)\varepsilon)dt \\ &\quad = \varepsilon \int_E \sum_{\substack{j=1 \\ j\neq i}}^{l} \left(z^j(\overline{p}(t)) - z^i(\overline{p}(t))\right)dt + \int_E z^i(\overline{p}(t))dt \leq 0. \end{aligned} \tag{2.43}$$

If $\int_E \sum_{j=1, j\neq i}^{l}(z^j(\overline{p}(t)) - z^i(\overline{p}(t))) \geq 0$, the above estimate does not hold.

If $\int_E \sum_{j=1, j\neq i}^{l}(z^j(\overline{p}(t)) - z^i(\overline{p}(t)))dt < 0$, by the choice of ε, it results in that the estimate is false. Then (2.40) cannot occur and we get $z^j(\overline{p}(t))dt \leq 0$ a.e. $[0,T]$, for all $j = 1, \ldots, l$. □

3. Existence Results

In this section we are concerned with the problem of the existence of the dynamic competitive equilibrium, by using the variational theory.

3.1. Existence and Regularity of the Equilibrium Consumption

Firstly, for all price $p \in P$ and for all $a = 1, \ldots, n$, let us consider evolutionary variational inequality (2.13) that is equivalent to

$$\langle -\nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle_L \geq 0, \quad \forall x_a \in M_a(p). \tag{3.1}$$

We suppose that the operator $-\nabla u_a(x_a)$ is an affine operator:

$$-\nabla u_a(t, x_a(t)) = A_a(t)x_a(t) + B_a(t), \tag{3.2}$$

for each $t \in [0,T]$, where $A_a : [0,T] \to \mathbb{R}^{l\times l}$ and $B : [0,T] \to \mathbb{R}^l_+$, with A_a a bounded and positive defined matrix:

$$\exists M > 0 : \|A_a(t)\| \leq M \quad \text{a.e. } [0,T],$$
$$\exists \nu > 0 : \langle A_a(t)x_a(t), x_a(t)\rangle \geq \nu\|x_a(t)\|^2 \quad \forall x_a(t) \in M_a(t,p(t)) \quad \text{a.e. } [0,T]. \tag{3.3}$$

Since $A_a(t)$ is a positive defined matrix for all $p \in P$, there exists a unique solution $\overline{x}_a(p)$ to evolutionary variational inequality (3.1). Then the excess demand function arises

$$z : P \longrightarrow L,$$
$$p \longrightarrow z(p) = \sum_{a=1}^{n}(\overline{x}_a(p) - e_a). \tag{3.4}$$

Now, our goal is to give a regularity result for the evolutionary variational inequality (3.1), in particular, we prove that $\overline{x}_a(\cdot)$ is continuous on P. In order to achieve the continuity result, we need to recall the concept of set convergence in the sense of Mosco (see, e.g., [27]).

Definition 3.1 (see [27]). Let $(V, \|\cdot\|)$ be an Hilbert space $\mathbf{K} \subset V$ a closed, nonempty, convex set. A sequence of nonempty, closed, convex sets $\mathbf{K}_n$ converges to $\mathbf{K}$ as $n \to +\infty$, that is, $\mathbf{K}_n \to \mathbf{K}$, if and only if

(M1) for any $H \in \mathbf{K}$, there exists a sequence $\{H_n\}_{n\in\mathbb{N}}$ strongly converging to H in V such that H_n lies in $\mathbf{K}_n$ for all n,

(M2) for any $\{H_{k_n}\}_{n\in\mathbb{N}}$ weakly converging to H in V, such that H_{k_n} lies in $\mathbf{K}_{k_n}$ for all n, then the weak limit H belongs to $\mathbf{K}$.

Definition 3.2 (see, e.g., [28]). A sequence of operators $A_n : \mathbf{K}_n \to V'$ converges to an operator $A : \mathbf{K} \to V'$ if

$$\|A_nH_n - A_nF_n\|_* \leq M\|H_n - F_n\|, \quad \forall H_n, F_n \in \mathbf{K}_n,$$
$$\langle A_nH_n - A_nF_n, H_n - F_n\rangle \geq \nu\|H_n - F_n\|^2, \quad \forall H_n, F_n \in \mathbf{K}_n \tag{3.5}$$

hold with fixed constants $M, \nu > 0$ and

(M3) the sequence $\{A_nH_n\}_{n\in\mathbb{N}}$ strongly converges to AH in V', for any sequence $\{H_n\}_{n\in\mathbb{N}} \subset \mathbf{K}_n$ strongly converging to $H \in \mathbf{K}$.

Now, we remember an abstract result due to Mosco on stability of solutions to a variational inequality. More precisely, let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$, find $H \in \mathbf{K}$ such that

$$\langle AH + B, F - H\rangle \geq 0, \quad \forall F \in \mathbf{K}. \tag{3.6}$$

Theorem 3.3 (see, e.g., [28]). *Let $\mathbf{K}_n \to \mathbf{K}$ in sense of Mosco (M1)-(M2), $A_n \to A$ in the sense of (M3), and $B_n \to B$ in V'. Then the unique solutions H_n of*

$$H_n \in \mathbf{K}_n : \langle A_nH_n + B_n, F_n - H_n\rangle \geq 0, \quad \forall F_n \in \mathbf{K}_n, \tag{3.7}$$

converge strongly to the solution H *of the limit problem* (3.6)*, that is,*

$$H_n \longrightarrow H \quad in\ V. \tag{3.8}$$

Theorem 3.4. *For all* $\{p_n\}_{n\in\mathbb{N}} \subseteq P$ *strongly converging to* p*, then* $M_a(p_n) \rightarrow M_a(p)$ *in Mosco's sense.*

Let $p \in P$ fixed and let $\{p_n\}_{n\in\mathbb{N}} \subseteq P$ be a sequence such that $p_n \rightarrow p \in P$. We prove that $M_a(p_n) \rightarrow M_a(p)$ in Mosco's sense, that is, it is enough to show that (M1) and (M2) hold.

Let $x_a(p) \in M_a(p)$. We pose

$$x_a(p_n) = x_a(p) - \eta_n \quad \forall n \in \mathbb{N}, \tag{3.9}$$

namely,

$$x_a^j(p_n)(t) = x_a^j(p)(t) - \eta_n^j(t) \quad \text{a.e. } [0,T],\ \forall j = 1,\dots,l,\ \forall n \in \mathbb{N}, \tag{3.10}$$

such that $\{\eta_n\}_{n\in\mathbb{N}} \subseteq L^2([0,T],\mathbb{R}^l)$, $\eta_n \rightarrow 0$, and

$$x_a^j(p)(t) - e_a^j(t) \leq \eta_n^j(t) \leq x_a^j(p)(t), \quad \text{a.e. } [0,T],\ \forall j = 1,\dots,l,\ \forall n \in \mathbb{N}. \tag{3.11}$$

Let us verify that $x_a(p_n) \in M_a(p_n)$ for all $n \in \mathbb{N}$.

From the right-hand side of (3.11) it results in what follows:

$$x_a^j(p_n)(t) = x_a^j(p)(t) - \eta_n^j(t) \geq 0 \quad \text{a.e. } [0,T],\ \forall j = 1,\dots,l,\ \forall n \in \mathbb{N}. \tag{3.12}$$

Moreover, from the left-hand side of (3.11) for all $j = 1,\dots,l$ and for all $n \in N$, it results in

$$x_a^j(p_n)(t) - e_a^j(t) = x_a^j(p)(t) - \eta_n^j(t) - e_a^j(t) \leq 0 \quad \text{a.e. } [0,T], \tag{3.13}$$

then, because $p_n \in P$, $p_n^j(t) \geq 0$ a.e. $[0,T]$, from (3.13), we have

$$p_n^j(t)\left(x_a^j(p_n)(t) - e_a^j(t)\right) \leq 0, \quad \text{a.e. } [0,T],\ \forall j = 1,\dots,l; \tag{3.14}$$

hence,

$$\langle p_n, x_a(p_n) - e_a\rangle_L = \int_0^T \sum_{j=1}^{l} p_n^j(t)\left(x_a^j(p_n)(t) - e_a^j(t)\right)dt \leq 0. \tag{3.15}$$

For all $n \in \mathbb{N}$, we have $x_a(p_n) \in M_a(p_n)$. Furthermore, by

$$\left\|x_a(p_n) - x_a(p)\right\|_L = \left\|x_a(p) - \eta_n - x_a(p)\right\|_L = \left\|\eta_n\right\|_L, \tag{3.16}$$

because $\eta_n \to 0$, it follows that

$$x_a(p_n) \longrightarrow x_a(p). \tag{3.17}$$

Hence, (M1) holds.

We prove (M2). Let $\{x_a(p_{k_n})\}$ a sequence such that $x_a(p_{k_n}) \in M_a(p_{k_n})$ is weakly convergent to $x_a(p)$. We prove that $x_a(p) \in M_a(p)$:

$$\begin{aligned} x_a(p_{k_n}) \rightharpoonup x_a(p) &\Longleftrightarrow \forall g \in L\langle x_a(p_{k_n}), g\rangle_L \longrightarrow \langle x_a(p), g\rangle_L \\ &\Longleftrightarrow \forall g \in L\int_0^T \sum_{j=1}^{l} x_a^j(p_{k_n})(t)\cdot g^j(t)dt \longrightarrow \int_0^T \sum_{j=1}^{l} x_a^j(p)(t)\cdot g^j(t)dt. \end{aligned} \tag{3.18}$$

By choosing $g^j(t) \geq 0$ for all $j = 1,\ldots,l$ a.e. $[0,T]$, one has

$$\int_0^T \sum_{j=1}^{l} x_a^j(p_{k_n})(t)\cdot g^j(t)dt \geq 0 \Longrightarrow \int_0^T \sum_{j=1}^{l} x_a^j(p)(t)\cdot g^j(t)dt \geq 0. \tag{3.19}$$

From (3.19), it follows that $(x_a^j(p))(t) \geq 0$ for all $j = 1,\ldots,l$ a.e. $[0,T]$; in fact if there exist j^* and $E \subseteq [0,T]$, $m(E) > 0$ such that $(x_a^{j^*}(p))(t) < 0$ for all $t \in E$, by choosing $g \in L$ such that

$$g^j(t) = \begin{cases} g^j(t) > 0 & \text{in } E \text{ if } j = j^*, \\ 0 & \text{in } [0,T] \setminus E\ \forall j, \\ 0 & \text{in } E \text{ if } j \neq j^*, \end{cases} \tag{3.20}$$

condition (3.19) is contradicted. Then $x_a^j(p)(t) \geq 0$ for all $j = 1,\ldots,l$ a.e. $[0,T]$.

Furthermore, from

$$x_a(p_{k_n}) \rightharpoonup x_a(p), \qquad p_n \longrightarrow p, \tag{3.21}$$

it results in what follows:

$$\langle p_{k_n}, x_a(p_{k_n}) - e_a\rangle_L \longrightarrow \langle p, x_a(p) - e_a\rangle_L. \tag{3.22}$$

Since $x_a(p_{k_n}) \in M_a(p_{k_n})$, one has

$$\langle p_{k_n}, x_a(p_{k_n}) - e_a\rangle_L \leq 0 \quad \forall n \in \mathbb{N} \Longrightarrow \langle p, x_a(p) - e_a\rangle_L \leq 0, \tag{3.23}$$

hence $x_a(p) \in M_a(p)$. So condition (M2) holds.

Then we have proved that for all $\{p_n\} \subseteq P$ such that $p_n \to p$, it results in that $M_a(p_n)$ converging to $M_a(p)$ in Mosco's sense.

Theorem 3.5. *Let $(-\nabla u_a(x_a))$ be an affine operator of form* (3.40). *Then $\overline{x}_a(p)$ is continuous on P.*

Proof. We have that $A_n \to A$ in the sense of (M3). In fact,

(a) for each $n \in \mathbb{N}$ and $x_a(p_n), y_a(p_n) \in M_a(p_n)$, since A_a is bounded in $[0,T]$, there exists $K > 0$ such that

$$\begin{aligned}\left\| A_a(x_a(p_n) - y_a(p_n))\right\|_L &\leq \|A_a\|_L \cdot \left\|x_a(p_n) - y_a(p_n)\right\|_L \\ &= K \cdot \left\|x_a(p_n) - y_a(p_n)\right\|_L;\end{aligned} \tag{3.24}$$

(b) by positivity of the matrix A_a, for each $n \in \mathbb{N}$ and $x_a(p_n), y_a(p_n) \in M_a(p_n)$, there exists $\nu > 0$ such that

$$\langle A_a(x_a(p_n)) - A_a(y_a(p_n)), x_a(p_n) - y_a(p_n)\rangle_L \geq \nu\left\|x_a(p_n) - y_a(p_n)\right\|_L^2; \tag{3.25}$$

(c) for each sequence $\{x_a(p_n)\}_{n\in\mathbb{N}}$, with $x_a(p_n) \in M_a(p_n)$, strongly converging to $x_a(p) \in M_a(p)$, the sequence $\{A_a(x_a(p_n))\}$ strongly converges to $A_a(x_a(p))$, in fact,

$$\begin{aligned}\left\|A_a(x_a(p_n)) - A_a(x_a(p))\right\|_L &= \left\|A_a(x_a(p_n) - x_a(p))\right\|_L \\ &\leq \|A_a\|_L \cdot \left\|x_a(p_n) - x_a(p)\right\|_L;\end{aligned} \tag{3.26}$$

because $\|x_a(p_n) - x_a(p)\|_L \to 0$, then $\|A_a(x_a(p_n)) - A_a(x_a(p))\|_L \to 0$.

By Theorem 3.3, the sequence $\{\overline{x}_a(p_n)\}$, where, for all $n \in \mathbb{N}$, $\overline{x}_a(p_n)$ is the unique solution of

$$\langle A_a(\overline{x}_a(p_n)) + B_a, x_a(p_n) - \overline{x}_a(p_n)\rangle_L \geq 0, \quad \forall x_a(p_n) \in M_a(p_n), \tag{3.27}$$

converges strongly to the solution $\overline{x}_a(p)$ of the limit problem (3.1), that is,

$$\overline{x}_a(p_n) \longrightarrow \overline{x}_a(p) \quad \text{in } M_a(p). \tag{3.28}$$

Hence, we have proved that for all p_n strongly converging to p, $\overline{x}_a(p_n)$ strongly converges to $\overline{x}_a(p)$, then $\overline{x}_a(p)$ is continuous on P. □

3.2. Existence of Competitive Prices and Existence of Equilibrium

Let us assume the following regularity condition:

$$\lim_{|h|\to 0} \int_0^T \left\|p(t+h) - p(t)\right\|^2 dt = 0 \quad \text{uniformly in } p, \tag{3.29}$$

namely, for all $\varepsilon > 0 \, \exists \delta > 0$ such that $\|p(t+h) - p(t)\|_L < \varepsilon$ for all $h \in \mathbb{R}$, $|h| < \delta$ and for all $p \in P$. This is condition interpreted as the uniform integral continuity of price and for example it is satisfied by all the functions:

$$p(t) \in \Pi_{C,\alpha} = \left\{ p(t) \in L^2\left([0,T], \mathbb{R}^l\right) : \int_0^T \|p(t+h) - p(t)\|^2 dt \le C\|h\|^{2\alpha} \right\}, \tag{3.30}$$

where C is a positive constant and $\alpha \in]0,1]$ (see, e.g., [29]). Let us consider the evolutionary variational inequality:

$$\left\langle \sum_{a=1}^{n} (e_a - \overline{x}_a(\overline{p})), p - \overline{p} \right\rangle_L \ge 0 \quad \forall p \in \widetilde{P}, \tag{3.31}$$

where

$$\begin{aligned} \widetilde{P} = \Bigg\{ & p \in L^2\left(\mathbb{R}, \mathbb{R}^l\right) : p^j(t) \ge 0 \;\forall j,\; \sum_{j=1}^{l} p^j(t) = 1 \text{ a.e. } [0,T], \\ & \lim_{h \to 0} \int_0^T \|p(t+h) - p(t)\|^2 dt = 0 \text{ uniformly in } p,\; p(t) = 0 \text{ if } t \notin [0,T] \Bigg\}. \end{aligned} \tag{3.32}$$

In order to prove an existence result of solutions to (3.31), we recall the following.

Theorem 5.1 of [15]. *Let E be a real topological vector space and let $K \subseteq E$ be a convex and nonempty. Let $C : K \to E^*$ be such that for all $y \in K$,*

$$\xi \longrightarrow \langle C(\xi), y - \xi \rangle_E \quad \text{is usc on } K \text{ (hemicontinuity on } K), \tag{3.33}$$

and there exist $A \subseteq K$ nonempty, compact and $B \subseteq K$ compact such that for every $H \in K \setminus A$, there exists $F \in B$ with $\langle C(H), F - H \rangle_E < 0$, there exists $\overline{x} \in A$ such that

$$\langle C(\overline{x}), y - \overline{x} \rangle_E \ge 0 \quad \forall y \in K. \tag{3.34}$$

Theorem 3.6. *Let $\mathcal{F}$ be a bounded set of $L^2(\mathbb{R})$. Let us suppose that*

$$\lim_{h \to 0} \|p(t+h) - p(t)\|_{L^2}^2 = 0 \quad \text{uniformly in } p \in \mathcal{F}, \tag{3.35}$$

with $p(t) = 0$ if $t \notin [0,T]$. Then $\mathcal{F}|_{[0,T]}$ has compact closure in $L^2([0,T])$.

Theorem 3.6 is the L^2-version of Ascoli's theorem, due to Riesz, Fréchet, and Kolmogorov (see, e.g., [30]). Now, we can prove the following.

Theorem 3.7. *Let us consider evolutionary variational inequality* (3.31). *There exists at least one $\overline{p}$ solution to* (3.31).

Proof. Let us observe that since $\widetilde{P}$ is a closed and bounded set, by Theorem 3.6, it follows that $\widetilde{P}$ is a compact set. Then, in Theorem 5.1 we can choose $A = \widetilde{P}$ and $B = \emptyset$ and it results in that the excess demand function $\sum_{a=1}^{n}(e_a - \overline{x}_a(\overline{p}))$ is strongly hemicontinuous, that is, for all $q \in \widetilde{P}$, the function

$$p \longrightarrow \left\langle \sum_{a=1}^{n}(e_a - \overline{x}_a(p)), q - p \right\rangle_L \tag{3.36}$$

is strongly continuous. In fact, for all $\{p_n\}_{n\in\mathbb{N}}$ such that $p_n \to p$, by Theorem 3.5, $\overline{x}_a(p_n) \to \overline{x}_a(p)$. So, for all $q \in \widetilde{P}$, we have

$$\sum_{a=1}^{n}(e_a - \overline{x}_a(p_n)) \longrightarrow \sum_{a=1}^{n}(e_a - \overline{x}_a(p)), \qquad q - p_n \longrightarrow q - p, \tag{3.37}$$

hence

$$\left\langle \sum_{a=1}^{n}(e_a - \overline{x}_a(p_n)), q - p_n \right\rangle_L \longrightarrow \left\langle \sum_{a=1}^{n}(e_a - \overline{x}_a(p)), q - p \right\rangle_L; \tag{3.38}$$

namely, for all $q \in \widetilde{P}$, the function

$$p \longrightarrow \left\langle \sum_{a=1}^{n}(e_a - \overline{x}_a(p)), q - p \right\rangle_L \tag{3.39}$$

is continuous. By [15, Theorem 5.1] the evolutionary variational inequality (3.31) admits a solution. □

Finally, we have following existence result of dynamic competitive equilibrium for a pure exchange economy.

Theorem 3.8. *Let the operator $-\nabla u_a(x_a)$ an affine operator:*

$$-\nabla u_a(t, x_a(t)) = A_a(t)x_a(t) + B_a(t), \tag{3.40}$$

for each $t \in [0,T]$, where $A_a : [0,T] \to \mathbb{R}_+^{l\times l}$ and $B : [0,T] \to \mathbb{R}_+^l$, with A_a a bounded and positive defined matrix. Then there exists $(\overline{x}(\overline{p}), \overline{p}) \in M(\overline{p}) \times \widetilde{P}$ solution to evolutionary quasivariational inequality

$$\begin{aligned} &\sum_{a=1}^{n}\langle \nabla u_a(t, \overline{x}_a), x_a - \overline{x}_a\rangle_L + \left\langle \sum_{a=1}^{n}(\overline{x}_a(\overline{p}) - e_a), p - \overline{p} \right\rangle_L \\ &\quad \le 0 \quad \forall (x,p) \in \widetilde{P} \times M(\overline{p}), \end{aligned} \tag{3.41}$$

namely, there exists at least a dynamic competitive equilibrium.

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