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Research Article

Global Exponential Stability of Periodic Oscillation for Nonautonomous BAM Neural Networks with Distributed Delay

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We derive a new criterion for checking the global stability of periodic oscillation of bidirectional associative memory (BAM) neural networks with periodic coefficients and distributed delay, and find that the criterion relies on the Lipschitz constants of the signal transmission functions, weights of the neural network, and delay kernels. The proposed model transforms the original interacting network into matrix analysis problem which is easy to check, thereby significantly reducing the computational complexity and making analysis of periodic oscillation for even large-scale networks.

1. Introduction

The bidirectional associative memory (BAM) neural network which was first introduced by Kosko in 1987 [1, 2] is formed by neurons arranged in two layers. The neurons in one layer are fully interconnected to the neurons in the other layer, while there are no interconnections among neurons in the same layer. Through iterations of forward and backward information flows between the two layers, it performs a two-way associative search for stored bipolar vector pairs and generalizes the single-layer autoassociative hebbian correlation to a two-layer pattern matched heteroassociative.

As it is well known, research on neural dynamical systems not only involves a discussion of stability properties, but also involves many dynamic behavior such as periodic oscillatory behavior, bifurcation, and chaos [3–19]. In the application of neural networks to some practical problems, the properties of equilibrium points play important roles. An equilibrium point can be looked as a special periodic solution of neural networks with arbitrary period. In this sense, the analysis of periodic solutions of neural networks could be more general than that of equilibrium points. There are some results on the existence and stability of periodic solution of BAM neural networks. Liu et al. [20, 21] obtained several sufficient conditions which ensure existence and stability of periodic solution for BAM neural networks with periodic coefficients and time-varying delays. Subsequently, Guo et al. [22] obtained some sufficient conditions ensuring the existence, uniqueness, and stability of the periodic solution for BAM neural networks with periodic variable coefficients and variable delays, and they also estimated the exponentially convergent rate. Song et al. obtained several sufficient conditions which ensure existence and stability of periodic solution for BAM neural networks with periodic coefficients and periodic time-varying delays [23]. Moreover, neural networks usually has a spatial extent due to the presence of an amount of parallel pathways with a variety of axon sizes and lengths. Thus, the delays in neural networks are usually continuously distributed. Recently, there are some authors studied the BAM neural networks with distributed delays and constants coefficients [24–26].

Until recently, few studies have considered periodic solution for the BAM neural networks with periodic coefficients and distributed delays. Zhou et al. considered the periodic solution for the BAM neural networks with period coefficients and continuously distributed delays [27]. However, the result in Zhou et al. contains two limitations, one made for periodic T and the other is $\min_{1\leqslant i,j\leqslant n}\{(\overline{a}_i - \sum_{j=1}^{m}\overline{d}_{ji}N_{ji}),\ (\overline{c}_j - \sum_{i=1}^{n}\overline{b}_{ij}M_{ij})\} > 0$, which also in the Wang et al. [28]. This limitation is being removed by this paper. Based on the continuation theorem of Mawhin's coincidence degree theory, the nonsingular M-matrix and Lyapunov functionals, we derive a new global exponential stability criterion in matrix form for periodic oscillation of BAM neural networks with period coefficients and distributed delay. Moreover, our criterion is easy to check out.

The paper is organized as follows. Our model and some preliminaries are given in Section 2. The existence of periodic solution is proved in Section 3. The exponential stability of periodic oscillator is considered in Section 4. An example is shown in Section 5. Several summary remarks are finally given in Section 6.

2. Preliminaries

In this paper, we study the BAM neural networks with periodic coefficients and continuously distributed delays modeled by the following system:

$$\begin{aligned}\dot{u}_i(t) &= -a_i(t)u_i(t) + \sum_{j=1}^{m} b_{ij}(t)h_{ij}\left(\int_0^{t_{ij}} f_{ij}(s)v_j(t-s)ds\right) + I_i(t),\\ \dot{v}_j(t) &= -c_j(t)v_j(t) + \sum_{i=1}^{n} d_{ji}(t)e_{ji}\left(\int_0^{\tau_{ji}} g_{ji}(s)u_i(t-s)ds\right) + L_j(t),\end{aligned} \tag{2.1}$$

where $i = 1,2,\ldots,n$; $j = 1,2,\ldots,m$; $a_i(t) > 0$ and $c_j(t) > 0$ denote the rate with which the cells i and j reset their potential to the resting state when isolated from the other cells and inputs; $b_{ij}(t)$ and $d_{ji}(t)$ are connection weights of the neural network; $I_i(t)$, $L_j(t)$ denote the ith and the jth component of an external input source introduced from outside the network to the ith cell and jth cell at time t, respectively. Moreover, the jth cell has an impact on the ith cell in the time of t_{ij} and the jth cell has an impact on the ith cell in the time of τ_{ji}.

If $u_i(t)$ and $v_j(t)$ satisfy system (2.1) and $u_i(t+T) = u_i(t)$, $v_j(t+T) = v_j(t)$, then they are T-periodic solutions of system (2.1). The initial conditions associated with system (2.1) are given as follows:

$$u_i(s) = \phi_i(s), \quad v_j(s) = \varphi_j(s), \quad s \in (-\infty, 0], \tag{2.2}$$

where $\phi_i(s)$, $\varphi_j(s)$ are continuous function ($i = 1, 2, \dots, n; j = 1, 2, \dots, m$).

Throughout this paper, we make the following assumptions.

Assumption 2.1. $a_i(t)$, $b_{ij}(t)$, $c_j(t)$, $d_{ji}(t)$, $I_i(t)$ and $J_j(t)$ are continuous T-periodic functions on $\mathbf{R}$. In addition, $a_i^+ = \sup_{t\in\mathbf{R}}|a_i(t)| < +\infty$, $a_i^- = \inf_{t\in\mathbf{R}}|a_i(t)| > 0$, $b_{ij}^+ = \sup_{t\in\mathbf{R}}|b_{ij}(t)| < +\infty$, $c_i^+ = \sup_{t\in\mathbf{R}}|c_j(t)| < +\infty$, $c_i^- = \inf_{t\in\mathbf{R}}|c_j(t)| > 0$, $d_{ji}^+ = \sup_{t\in\mathbf{R}}|d_{ji}(t)| < +\infty$, $I_i^+ = \sup_{t\in\mathbf{R}}|I_i(t)| < +\infty$, and $L_j^+ = \sup_{t\in\mathbf{R}}|L_j(t)| < +\infty$.

Assumption 2.2. Signal transmission functions $h_{ij}(u)$, $e_{ji}(v)$ are bounded on $\mathbf{R}$, and there exist number $M_{ij} > 0$ and $N_{ji} > 0$ such that

$$\left|h_{ij}(u) - h_{ij}(v)\right| \leqslant M_{ij}|u - v|, \qquad \left|e_{ji}(u) - e_{ji}(v)\right| \leqslant N_{ji}|u - v| \tag{2.3}$$

for each $u, v \in \mathbf{R}$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Assumption 2.3. The delay kernels $f_{ij}(s), g_{ji}(s) : [0, +\infty) \to [0, +\infty)$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m$) are continuous and integrable and satisfy

$$\int_0^{+\infty} f_{ij}(s)ds = 1, \qquad \int_0^{+\infty} g_{ji}(s)ds = 1. \tag{2.4}$$

Assumption 2.4. The delay kernels $f_{ij}(s)$, $g_{ji}(s) : [0, +\infty) \to [0, +\infty)$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m$) satisfy

$$\int_0^{+\infty} e^{\alpha s} f_{ij}(s)ds \leqslant 1, \qquad \int_0^{+\infty} e^{\alpha s} g_{ji}(s)ds \leqslant 1, \tag{2.5}$$

where α is a bounded positive real number.

Now, we give some useful notations, definitions, and lemmas as follows: $\|u\| = (\int_0^T |u(s)|^2 ds)^{1/2}$ and $\overline{v} = (1/T)\int_0^T v(s)ds$, where $u(s) \in C(\mathbf{R}, \mathbf{R})$, $v(s)$ is T-periodic function.

Assume that $\mathbb{T}^{n\times n} = \{\mathbf{A} = (a_{ij})_{n\times n} : a_{ij} \leqslant 0, i \neq j\}$, then we have the following.

Lemma 2.5 (see [17, 29, 30]). *Let $\mathbf{A} \in \mathbb{T}^{n\times n}$. Then, each of the following conditions is equivalent to the statement '$\mathbf{A}$ is a nonsingular M-matrix':*

(a) *all of the principal minors of* $\mathbf{A}$ *are positive;*

(b) *the real parts of all the eigenvalue of* $\mathbf{A}$ *are positive;*

(c) $\mathbf{A}$ *is inverse-positive; that is,* $\mathbf{A}^{-1}$ *exists and* $\mathbf{A}^{-1} \geqslant 0$;

(d) *there is a vector* x *(or* y*), whose elements are all positive, such that the elements of* $\mathbf{A}x$ *(or* $\mathbf{A}^T y$*) are all positive;*

(e) $\mathbf{A}$ *has all positive diagonal elements and there exists a positive diagonal matrix* $\mathbf{D}$ *such that* $\mathbf{AD}$ *is strictly diagonally dominant; that is,*

$$a_{ii}d_i > \sum_{i\neq j}\left|a_{ij}\right|d_j, \quad i = 1,2,\ldots,n. \tag{2.6}$$

In the following, we introduce some concepts and results from the book by Gaines and Mawhin [31].

Let X and Z be two Banach spaces, $L : \operatorname{Dom} L \subset X \to Z$ a linear mapping, and $N : X \to Z$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} ImL < +\infty$ and ImL is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $ImP = \operatorname{Ker} L$, $\operatorname{Ker} Q = ImL = Im(I-Q)$, it follows that mapping $L|_{\operatorname{Dom} L \bigcap \operatorname{Ker} P} : (I-P)X \to ImL$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Since ImQ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : ImQ \to \operatorname{Ker} L$.

Lemma 2.6 (Mawhin's continuation theorem). *Let X and Z be two Banach spaces and L be a Fredholm mapping of index zero. Assume that $\Omega \subset X$ is an open bounded set and $N : X \to Z$ is a continuous operator which is L-compact on $\overline{\Omega}$. Then $Lx = Nx$ has at least one solution in* $\operatorname{Dom} L \bigcap \overline{\Omega}$*, if the following conditions are satisfied:*

(a) *for each* $\lambda \in (0,1)$, $x \in \partial\Omega \bigcap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;

(b) *for each* $x \in \partial\Omega \bigcap \operatorname{Ker} L$, $QNx \neq 0$;

(c) $\deg\{JQNx, \Omega \bigcap \operatorname{Ker} L, 0\} \neq 0$,

where $J : ImQ \to \operatorname{Ker} L$ *is an isomorphism.*

3. Existence of Periodic Solutions

Now we give the following sufficient conditions on the existence of periodic solutions.

Theorem 3.1. *Assume that Assumptions 2.1–2.3 hold. Then, system* (2.1) *has at least one T-periodic solution, if*

$$\mathbb{P} = \begin{pmatrix} I_n & \mathbb{P}_{12} \\ \mathbb{P}_{21} & I_n \end{pmatrix} \tag{3.1}$$

is a nonsingular M-matrix, where I_n *is unite matrix and* $\mathbb{P}_{12} = (p_{ij})_{n\times m}$, $p_{ij} = -b_{ij}^{+}M_{ij}/a_i^{-}$; $\mathbb{P}_{21} = (q_{ji})_{m\times n}$, $q_{ji} = -d_{ji}^{+}N_{ji}/c_j^{-}$.

Proof. Let $X = Z = \{x(t) = (u_1(t), \ldots, u_n(t), v_1(t), \ldots, v_m(t))^\top \in C(\mathbf{R}, \mathbf{R}^{n+m}) \mid u_i(t) = u_i(t + T),\ v_j(t) = v_j(t+T),\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m\}$, then X is a Banach space with the norm $\|x\|_1 = \sum_{i=1}^n \max_{t\in[0,T]}|u_i(t)| + \sum_{j=1}^m \max_{t\in[0,T]}|v_j(t)|$.

Let $L : \operatorname{Dom} L \subset X \to Z$, $P : X \bigcap \operatorname{Dom} L \to \operatorname{Ker} L$, $Q : X \to X/ImL$, and $N : X \to Z$ be given by the following:

$$Lx = (\dot{u}_1(t), \dot{u}_2(t), \ldots, \dot{u}_n(t), \dot{v}_1(t), \dot{v}_2(t), \ldots, \dot{v}_m(t))^\top,$$

$$Px = Qx = (\overline{u}_1(t), \overline{u}_2(t), \ldots, \overline{u}_n(t), \overline{v}_1(t), \overline{v}_2(t), \ldots, \overline{v}_m(t))^\top,$$

$$Nx = \begin{pmatrix} -a_1(t)u_1(t) + \sum_{j=1}^m b_{1j}(t)h_{1j}\left(\int_0^{t_{1j}} f_{1j}(s)v_j(t-s)ds\right) + I_1(t) \\ \cdots \\ -a_n(t)u_n(t) + \sum_{j=1}^m b_{nj}(t)h_{nj}\left(\int_0^{t_{nj}} f_{nj}(s)v_j(t-s)ds\right) + I_n(t) \\ -c_1(t)v_1(t) + \sum_{i=1}^n d_{1i}(t)e_{1i}\left(\int_0^{\tau_{1i}} g_{1i}(s)u_i(t-s)ds\right) + L_1(t) \\ \cdots \\ -c_m(t)v_m(t) + \sum_{i=1}^n d_{mi}(t)e_{mi}\left(\int_0^{\tau_{mi}} g_{mi}(s)u_i(t-s)ds\right) + L_m(t) \end{pmatrix}. \tag{3.2}$$

It is easy to see that L is a linear operator with $\operatorname{Ker} L = \{x(t) \mid x(t) = \kappa \in \mathbf{R}^{n+m}\}$. $ImL = \{x(t) \mid x(t) \in Z, \int_0^T x(t)dt = 0\}$ is closed in Z, and $\dim \operatorname{Ker} L = \operatorname{codim} ImL = n + m$. Therefore, L is a Fredholm mapping of index zero. It is easy to prove that P and Q are two projectors, and $ImP = \operatorname{Ker} L$, $ImL = \operatorname{Ker} Q = Im(I - Q)$. By using the Arzelá-Ascoli theorem, it is easy to prove that for every bounded subset $\Omega \in X$, $K_p(I - Q)N$ are relatively compact on $\overline{\Omega}$ in X, that is, N is L-compact on $\overline{\Omega}$.

Consider the operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0, 1), \tag{3.3}$$

that is

$$\begin{aligned} \dot{u}_i(t) &= -\lambda a_i(t)u_i(t) + \lambda\sum_{j=1}^m b_{ij}(t)h_{ij}\left(\int_0^{t_{ij}} f_{ij}(s)v_j(t-s)ds\right) + \lambda I_i(t), \\ \dot{v}_j(t) &= -\lambda c_j(t)v_j(t) + \lambda\sum_{i=1}^n d_{ji}(t)e_{ji}\left(\int_0^{\tau_{ji}} g_{ji}(s)u_i(t-s)ds\right) + \lambda L_j(t), \end{aligned} \tag{3.4}$$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Denote $\mu_i = \lambda(\sum_{j=1}^m b_{ij}^+ \sup_{s\in\mathbf{R}}|h_{ij}(s)| + I_i^+)$. By using (3.4), we obtain

$$-\mu_i \leqslant \dot{u}_i(t) + \lambda a_i(t)u_i(t) \leqslant \mu_i. \tag{3.5}$$

Multiplying both sides of (3.5) by $e^{\lambda\int_0^t a_i(s)ds}$, we have

$$-\mu_i e^{\lambda\int_0^t a_i(s)ds} \leqslant \left(u_i(t)e^{\lambda\int_0^t a_i(s)ds}\right)' \leqslant \mu_i e^{\lambda\int_0^t a_i(s)ds}. \tag{3.6}$$

Integrating the inequality above from 0 to ν $(\nu \geqslant 0)$ [27], we obtain

$$-\mu_i \int_0^{\nu} e^{\lambda\int_0^t a_i(s)ds} dt \leqslant u_i(\nu)e^{\lambda\int_0^{\nu} a_i(s)ds} - u_i(0) \leqslant \mu_i \int_0^{\nu} e^{\lambda\int_0^t a_i(s)ds} dt. \tag{3.7}$$

Hence,

$$-\mu_i \int_0^{\nu} e^{-\lambda\int_t^{\nu} a_i(s)ds} dt + u_i(0)e^{-\lambda\int_0^{\nu} a_i(s)ds} \leqslant u_i(\nu) \leqslant \mu_i \int_0^{\nu} e^{-\lambda\int_t^{\nu} a_i(s)ds} dt + u_i(0)e^{-\lambda\int_0^{\nu} a_i(s)ds} \tag{3.8}$$

for $\nu \geqslant t \geqslant 0$. So we have

$$-\left(|u_i(0)| - \frac{\mu_i}{\lambda a_i^-}\right)e^{-\lambda a_i^- \nu} - \frac{\mu_i}{\lambda a_i^-} \leqslant u_i(\nu) \leqslant \left(|u_i(0)| - \frac{\mu_i}{\lambda a_i^-}\right)e^{-\lambda a_i^- \nu} + \frac{\mu_i}{\lambda a_i^-}, \tag{3.9}$$

that is,

$$|u_i(\nu)| \leqslant \left|\left(|u_i(0)| - \frac{\mu_i}{\lambda a_i^-}\right)\right| + \frac{\mu_i}{\lambda a_i^-}, \tag{3.10}$$

which implies that u_i is bounded and similarly v_j. By the Assumption 2.3, we know that

$$\int_0^{t_{ij}} f_{ij}(s)v_j(t-s)ds \tag{3.11}$$

is uniformly convergent. Therefore, the following iterated integral:

$$\int_0^T u_i(t)\int_0^{t_{ij}} f_{ij}(s)v_j(t-s)ds\, dt \tag{3.12}$$

can be changed integrating order.

Suppose that $(u_1(t),\ldots,u_n(t),v_1(t),\ldots,v_m(t))^{\top} \in X$ is any periodic solution of system (2.1) for a certain $\lambda \in (0,1)$. Multiplying both sides of

$$\dot{u}_i(t) = -\lambda a_i(t)u_i(t) + \lambda\sum_{j=1}^{m} b_{ij}(t)h_{ij}\left(\int_0^{t_{ij}} f_{ij}(s)v_j(t-s)ds\right) + \lambda I_i(t) \tag{3.13}$$

by $u_i(t)$ and integrating from 0 to T, we obtain

$$\int_0^T \lambda a_i(t)u_i^2(t)dt = \int_0^T \lambda \sum_{j=1}^{m} b_{ij}(t)u_i(t)h_{ij}\left(\int_0^{t_{ij}} f_{ij}(s)v_j(t-s)ds\right)dt + \int_0^T \lambda I_i(t)u_i(t)dt. \tag{3.14}$$

From Assumptions 2.1, 2.2, and 2.3 and noting that $(\int_0^T |v_j(t-s)|^2 dt)^{1/2} = (\int_0^T |v_j(t)|^2 dt)^{1/2} = \|v_j\|$, we have

$$\begin{aligned}
a_i^- \int_0^T u_i^2(t)dt &\leqslant \int_0^T a_i(t)u_i^2(t)dt \\
&= \int_0^T \sum_{j=1}^{m} b_{ij}(t)u_i(t)\left[h_{ij}\left(\int_0^{t_{ij}} f_{ij}(s)v_j(t-s)ds\right) - h_{ij}(0)\right]dt \\
&\quad + \int_0^T \sum_{j=1}^{m} b_{ij}(t)h_{ij}(0)u_i(t)dt + \int_0^T I_i(t)u_i(t)dt \\
&\leqslant \sum_{j=1}^{m} b_{ij}^+ M_{ij} \int_0^{+\infty} f_{ij}(s) \int_0^T \left|v_j(t-s)\right| |u_i(t)| dt\, ds \\
&\quad + \sqrt{T}\left(\sum_{j=1}^{m} b_{ij}^+ \left|h_{ij}(0)\right| + I_i^+\right)\|u_i\| \\
&\leqslant \sum_{j=1}^{m} b_{ij}^+ M_{ij} \left\|v_j\right\| \|u_i\| + \sqrt{T}\left(\sum_{j=1}^{m} b_{ij}^+ \left|h_{ij}(0)\right| + I_i^+\right)\|u_i\|,
\end{aligned} \tag{3.15}$$

that is,

$$\|u_i\| \leqslant \frac{1}{a_i^-}\sum_{j=1}^{m} b_{ij}^+ M_{ij}\left\|v_j\right\| + \frac{\sqrt{T}}{a_i^-}\left(\sum_{j=1}^{m} b_{ij}^+ \left|h_{ij}(0)\right| + I_i^+\right). \tag{3.16}$$

By similar argument, we have

$$\left\|v_j\right\| \leqslant \frac{1}{c_j^-}\sum_{i=1}^{n} d_{ji}^+ N_{ji}\|u_i\| + \frac{\sqrt{T}}{c_j^-}\left(\sum_{i=1}^{n} d_{ji}^+ \left|e_{ji}(0)\right| + L_j^+\right). \tag{3.17}$$

It follows (3.16) and (3.17) that

$$\mathbb{P}y \leqslant s, \tag{3.18}$$

where $\mathbb{P} = \begin{pmatrix} I_n & \mathbb{P}_{12} \\ \mathbb{P}_{21} & I_n \end{pmatrix}$, I_n is a unite matrix and $\mathbb{P}_{12} = (p_{ij})_{n\times m}$, $p_{ij} = -b_{ij}^+ M_{ij}/a_i^-$; $\mathbb{P}_{21} = (q_{ji})_{m\times n}$, $q_{ji} = -d_{ji}^+ N_{ji}/c_j^-$, $y = (\|u_1\|, \|u_2\|, \dots, \|u_n\|, \|v_1\|, \|v_2\|, \dots, \|v_m\|)^\top$, $s = (s_1, s_2, \dots, s_{n+m})^\top$, $s_i = \sqrt{T}(\sum_{j=1}^m b_{ij}^+ |h_{ij}(0)| + I_i^+)/a_i^-$, $s_{n+j} = \sqrt{T}(\sum_{i=1}^n d_{ji}^+ |e_{ji}(0)| + L_j^+)/c_j^-$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$.

Application of Lemma 2.5 yields

$$y \leqslant \mathbb{P}^{-1} s \triangleq (r_1, r_2, \dots, r_{n+m})^\top, \tag{3.19}$$

which implies that

$$\|u_i\| \leqslant r_i, \quad \|v_j\| \leqslant r_{n+j}. \tag{3.20}$$

It is not difficult to check that there exist $t_i^*, t_{n+j}^* \in [0, T]$ such that

$$u_i(t_i^*) \leqslant \frac{r_i}{\sqrt{T}}, \qquad v_j\left(t_{n+j}^*\right) \leqslant \frac{r_{n+j}}{\sqrt{T}} \tag{3.21}$$

for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Multiplying both sides of (3.13) by $\dot{u}_i(t)$ $(i = 1, 2, \dots, n)$ and integrating from 0 to T, we obtain

$$\begin{aligned}
\|\dot{u}_i\|^2 &= \lambda \sum_{j=1}^m \int_0^T b_{ij}(t)\dot{u}_i(t) h_{ij}\left(\int_0^{t_{ij}} f_{ij}(s) v_j(t-s)\,ds\right) dt \\
&\quad - \lambda \int_0^T a_i(t)\dot{u}_i(t) u_i(t)\,dt + \lambda \int_0^T I_i(t)\dot{u}_i(t)\,dt \\
&= \lambda \sum_{j=1}^m \int_0^T b_{ij}(t)\dot{u}_i(t)\left[h_{ij}\left(\int_0^{t_{ij}} f_{ij}(s) v_j(t-s)\,ds\right) dt - h_{ij}(0)\right] \\
&\quad + \lambda \sum_{j=1}^m \int_0^T b_{ij}(t) h_{ij}(0)\dot{u}_i(t)\,dt - \lambda \int_0^T a_i(t) u_i(t)\dot{u}_i(t)\,dt + \lambda \int_0^T I_i(t)\dot{u}_i(t)\,dt \\
&\leqslant \sum_{j=1}^m b_{ij}^+ M_{ij} \int_0^{+\infty} f_{ij}(s)\left(\int_0^T |\dot{u}_i(t)|^2 dt\right)^{1/2} \left(\int_0^T |v_j(t-s)|^2 dt\right)^{1/2} ds \\
&\quad + \sqrt{T} \sum_{j=1}^m b_{ij}^+ \left|h_{ij}(0)\right| \|\dot{u}_i\| + a_i^+ \left(\int_0^T |\dot{u}_i(t)|^2 dt\right)^{1/2} \left(\int_0^T |u_i(t)|^2 dt\right)^{1/2} + \sqrt{T} I_i^+ \|\dot{u}_i\| \\
&\leqslant \sum_{j=1}^m b_{ij}^+ M_{ij} \|\dot{u}_i\| \left\|v_j\right\| + a_i^+ \|\dot{u}_i\| \|u_i\| + \sqrt{T} \sum_{j=1}^m b_{ij}^+ \left|h_{ij}(0)\right| \|\dot{u}_i\| + \sqrt{T} I_i^+ \|\dot{u}_i\|.
\end{aligned} \tag{3.22}$$

Therefore, $\|\dot{u}_i\| \leqslant \sum_{j=1}^m b_{ij}^+ M_{ij} \|v_j\| + a_i^+ \|u_i\| + \sqrt{T}(\sum_{j=1}^m b_{ij}^+ |h_{ij}(0)| + I_i^+)$.

It is easy check that

$$\begin{aligned}
u_i(t) &= u_i(t^*) + \int_{t^*}^{T} \dot{u}_i(t)dt \leqslant \frac{r_i}{\sqrt{T}} + \sqrt{T}\|\dot{u}_i\| \\
&\leqslant \frac{r_i}{\sqrt{T}} + \sqrt{T}\left[\sum_{j=1}^{m} b_{ij}^{+} M_{ij} \|v_j\| + a_i^{+}\|u_i\| + \sqrt{T}\left(\sum_{j=1}^{m} b_{ij}^{+}\left|h_{ij}(0)\right| + I_i^{+}\right)\right] \\
&\leqslant \frac{r_i}{\sqrt{T}} + \sqrt{T}\left[\sum_{j=1}^{m} b_{ij}^{+} M_{ij} r_{n+j} + a_i^{+} r_i + \sqrt{T}\left(\sum_{j=1}^{m} b_{ij}^{+}\left|h_{ij}(0)\right| + I_i^{+}\right)\right] \triangleq \frac{r_i}{\sqrt{T}} + \zeta_i, \\
v_j(t) &\leqslant \frac{r_{n+j}}{\sqrt{T}} + \sqrt{T}\left[\sum_{i=1}^{n} d_{ji}^{+} N_{ji} r_i + c_j^{+} r_{n+j} + \sqrt{T}\left(\sum_{i=1}^{n} d_{ji}^{+}\left|e_{ji}(0)\right| + L_j^{+}\right)\right] \triangleq \frac{r_{n+j}}{\sqrt{T}} + \zeta_{n+j}.
\end{aligned} \tag{3.23}$$

Let $\xi_i = r_i/\sqrt{T} + \zeta_i + \epsilon$ and $\xi_{n+j} = r_{n+j}/\sqrt{T} + \zeta_{n+j} + \epsilon \quad (0 < \epsilon \ll 1)$. We take

$$\Omega = \left\{x = (u_1(t), \ldots, u_n(t), v_1(t), \ldots, v_m(t))^{\top} \in X \mid |u_i(t)| < \xi_i,\ \left|v_j(t)\right| < \xi_{n+j}\right\}. \tag{3.24}$$

Obviously, condition (a) of Lemma 2.6 is satisfied. When $x \in \partial\Omega \bigcap \operatorname{Ker} L = \partial\Omega \bigcap \mathbf{R}^{n+m}$, x is a constant vector in R^{n+m} with $|u_i(t)| = \xi_i, |v_j(t)| = \xi_{n+j}$. Then, we have

$$\begin{aligned}
u_i(QNx)_i &= -\overline{a}_i u_i^2 + u_i \sum_{j=1}^{m} \overline{b}_{ij} h_{ij}(v_j) + u_i \overline{I}_i, \\
v_j(QNx)_{n+j} &= -\overline{c}_j v_j^2 + v_j \sum_{i=1}^{n} \overline{d}_{ji} e_{ji}(u_i) + v_j \overline{L}_j.
\end{aligned} \tag{3.25}$$

We claim that there exists some i or j $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$ such that

$$u_i(QNx)_i < 0 \quad \text{or} \quad v_j(QNx)_{n+j} < 0. \tag{3.26}$$

If $u_i(QNx)_i \geqslant 0$ and $v_j(QNx)_{n+j} \geqslant 0$, then, we obtain

$$\begin{aligned}
\overline{a}_i u_i^2 &\leqslant u_i \sum_{j=1}^{m} \overline{b}_{ij} h_{ij}(v_j) + u_i I_i^{+} \\
&= \sum_{j=1}^{m} u_i \overline{b}_{ij}\left[h_{ij}(v_j) - h_{ij}(0)\right] + u_i \sum_{j=1}^{m} \overline{b}_{ij} h_{ij}(0) + u_i I_i^{+} \\
&\leqslant |u_i| \sum_{j=1}^{m} b_{ij}^{+} M_{ij}\left|v_j\right| + |u_i| \sum_{j=1}^{m} b_{ij}^{+}\left|h_{ij}(0)\right| + |u_i| I_i^{+}.
\end{aligned} \tag{3.27}$$

Which implies

$$\xi_i \leqslant \frac{1}{\overline{a}_i}\sum_{j=1}^{m} b_{ij}^{+} M_{ij}\xi_{n+j} + \frac{1}{\overline{a}_i}\left(\sum_{j=1}^{m} b_{ij}^{+}\left|h_{ij}(0)\right| + I_i^{+}\right) \leqslant \frac{1}{a_i^{-}}\sum_{j=1}^{m} b_{ij}^{+} M_{ij}\xi_{n+j} + \frac{s_i}{\sqrt{T}}. \tag{3.28}$$

By a similar argument, we obtain

$$\xi_{n+j} \leqslant \frac{1}{\overline{c}_j}\sum_{i=1}^{n} d_{ji}^{+} N_{ji}\xi_i + \frac{1}{\overline{c}_j}\left(\sum_{i=1}^{n} d_{ji}^{+}\left|e_{ij}(0)\right| + L_j^{+}\right) \leqslant \frac{1}{c_j^{-}}\sum_{i=1}^{n} d_{ji}^{+} N_{ji}\xi_i + \frac{s_{n+j}}{\sqrt{T}}. \tag{3.29}$$

On the other hand,

$$\xi > \frac{r}{\sqrt{T}} + \zeta > \frac{r}{\sqrt{T}} = \frac{\mathbb{P}^{-1}s}{\sqrt{T}}, \tag{3.30}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_{n+m})$, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{n+m})$, and $r = (r_1, r_2, \ldots, r_{n+m})$. It follows that there exists some i or j such that

$$\xi_i - \frac{1}{a_i^{-}}\sum_{j=1}^{m} b_{ij}^{+} M_{ij}\left(\xi_{n+j}\right) > \frac{s_i}{\sqrt{T}} \tag{3.31}$$

or

$$\xi_{n+j} - \frac{1}{c_j^{-}}\sum_{i=1}^{n} d_{ji}^{+} N_{ji}(\xi_i) > \frac{s_{n+j}}{\sqrt{T}}, \tag{3.32}$$

which is contradiction with (3.28) and (3.29). Then, there exists some i or j ($1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$) such that $u_i(QNx)_i < 0$ or $v_j(QNx)_{n+j} < 0$. Therefore,

$$\|QNx\|_1 = \sum_{i=1}^{n}\left|(QNx)_i\right| + \sum_{j=1}^{m}\left|(QNx)_{n+j}\right| > 0. \tag{3.33}$$

This indicates that condition (b) of Lemma 2.6 is satisfied.

Define $\mathbb{H}(x,\theta) = -\theta x + (1-\theta)QNx$, $\theta \in [0,1]$, where $x = (u_1(t), \ldots, u_n(t), v_1(t), \ldots, v_m(t))^{\top} \in \mathbf{R}^{n+m}$. When $x \in \operatorname{Ker} L \bigcap \partial\Omega$, we have

$$\begin{aligned} \|\mathbb{H}(x,\theta)\|_1 &= \sum_{i=1}^{n}\left|\mathbb{H}(u_i,\theta)\right| + \sum_{j=1}^{m}\left|\mathbb{H}(v_j,\theta)\right| \\ &= \sum_{i=1}^{n}\left|-\theta u_i + (1-\theta)(QNx)_i\right| + \sum_{j=1}^{m}\left|-\theta v_j + (1-\theta)(QNx)_j\right| > 0. \end{aligned} \tag{3.34}$$

According to the invariant of homology, we have

$$\deg\left\{JQNx, \Omega \bigcap \operatorname{Ker} L, 0\right\} \neq 0, \tag{3.35}$$

where $J : ImQ \rightarrow \operatorname{Ker} L$ is an isomorphism. Therefore, according to the continuation theorem of Gaines and Mawhin, system (2.1) has at least one T-periodic solution. The proof is completed. □

Remark 3.2. In [27], the period T was assumed to be $T < \min\{1/a_i^+, 1/c_j^+\}$. In [28], the inequations $\min_{1\leqslant i,j\leqslant n}\{(\overline{a}_i - \sum_{j=1}^m \overline{d}_{ji}N_{ji}),\ (\overline{c}_j - \sum_{i=1}^n \overline{b}_{ij}M_{ij})\} > 0$ must hold and also in [27]. Here, the limitation is being removed.

4. Global Exponential Stability of Periodic Solution

In this section, we discuss the global exponential stability of the periodic solution of system (2.1). Under the assumptions of Theorem 3.1, system (2.1) has at least one T-periodic solution

$$x^*(t) = \left(u_1^*(t), \ldots, u_n^*(t), v_1^*(t), \ldots, v_m^*(t)\right)^{\top}. \tag{4.1}$$

Now, we give the following definition about global exponential of periodic solution:

Definition 4.1. The periodic solution $x^*(t)$ of model (2.1) is said to be globally exponentially stable, if there exist positive constants β, M such that

$$\left(\sum_{i=1}^{n}|u(t) - u^*(t)|^2 + \sum_{j=1}^{m}|v(t) - v^*(t)|^2\right)^{1/2} \leq M\|x_0 - x_0^*\|_2 e^{-\beta t} \tag{4.2}$$

for all $t > 0$, where x_0, x_0^* represent the history of x and x^* on $(-\infty, 0)$, respectively, and $\|x_0 - x_0^*\|_2 = \{\sum_{i=1}^n (\sup_{-\infty<t\leqslant 0}|\phi_i(t) - u_i^*(t)|)^2 + \sum_{j=1}^m (\sup_{-\infty<t\leqslant 0}|\varphi_j(t) - v_j^*(t)|)^2\}^{1/2}$.

Definition 4.2. A matrix M is said to be diagonally dominant if $|m_{ii}| \geqslant \sum_{j\neq i}|m_{ij}|$ for all i, $|m_{ii}| > \sum_{j\neq i}|m_{ij}|$ for at least one i, where m_{ij} denotes the entry in the ith row and jth column.

Theorem 4.3. *Assume that Assumptions 2.1, 2.2, 2.4 hold and $\mathbb{P}$ is a nonsingular M-matrix, where $\mathbb{P}$ is the same as in Theorem 3.1. System* (2.1) *has a unique T-periodic solution, which is globally exponentially stable, if $(\mathbb{Q} - \alpha I)^T \in \mathbb{T}$ and is a weakly diagonally dominant matrix as*

$$\mathbb{Q} = \begin{pmatrix} \mathbb{Q}_{11} & \mathbb{Q}_{12} \\ \mathbb{Q}_{21} & \mathbb{Q}_{22} \end{pmatrix}, \tag{4.3}$$

where $\mathbb{Q}_{11} = \operatorname{diag}(2a_1^- - \sum_{j=1}^m b_{1j}^+ M_{1j}, 2a_2^- - \sum_{j=1}^m b_{2j}^+ M_{2j}, \ldots, 2a_n^- - \sum_{j=1}^m b_{nj}^+ M_{nj})$*;* $\mathbb{Q}_{12} = (m_{ij})_{n\times m}$*,* $m_{ij} = -b_{ij}^+ M_{ij}$*;* $\mathbb{Q}_{21} = (n_{ji})_{m\times n}$*,* $n_{ji} = -d_{ji}^+ N_{ji}$*;* $\mathbb{Q}_{22} = \operatorname{diag}(2c_1^- - \sum_{i=1}^n d_{1i}^+ N_{1i}, 2c_2^- - \sum_{i=1}^n d_{2i}^+ N_{2i}, \ldots, 2c_m^- - \sum_{i=1}^n d_{mi}^+ N_{mi})$.

Proof. Let $z(t) = (u_1(t) - u_1^*(t), \dots, u_n(t) - u_n^*(t), v_1(t) - v_1^*(t), \dots, v_m(t) - v_m^*(t))^\top$. Then,

$$
\begin{aligned}
z_i(t)\dot{z}_i(t) &= -a_i(t)|z_i(t)|^2 + \sum_{j=1}^{m} b_{ij}(t) z_i(t) h_{ij}\left\{\int_0^{t_{ij}} f_{ij}(s)\left[z_{n+j}(t-s) + v_j^*(t-s)\right]ds\right\} \\
&\quad - \sum_{j=1}^{m} b_{ij}(t) z_i(t) h_{ij}\left(\int_0^{t_{ij}} f_{ij}(s) v_j^*(t-s)ds\right) \\
&\leqslant -a_i^-|z_i(t)|^2 + \sum_{j=1}^{m} b_{ij}^+ M_{ij}\int_0^{t_{ij}} f_{ij}(s) z_i(t) z_{n+j}(t-s)ds \\
&\leqslant -a_i^-|z_i(t)|^2 + \sum_{j=1}^{m} b_{ij}^+ M_{ij}\int_0^{+\infty} f_{ij}(s)\frac{|z_i(t)|^2 + \left|z_{n+j}(t-s)\right|^2}{2}ds \\
&\leqslant -a_i^-|z_i(t)|^2 + \frac{1}{2}\sum_{j=1}^{m} b_{ij}^+ M_{ij}\left(|z_i(t)|^2 + \int_0^{+\infty} f_{ij}(s)\left|z_{n+j}(t-s)\right|^2 ds\right), \\
z_{n+j}(t)\dot{z}_{n+j}(t) &\leqslant -c_j^-\left|z_{n+j}(t)\right|^2 + \frac{1}{2}\sum_{i=1}^{n} d_{ji}^+ N_{ji}\left(\left|z_{n+j}(t)\right|^2 + \int_0^{+\infty} g_{ji}(s)|z_i(t-s)|^2 ds\right)
\end{aligned}
\tag{4.4}
$$

for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

We define the following Lyapunov functionals:

$$
\begin{aligned}
\widehat{V}_i(t) &= \frac{e^{\alpha t}}{\alpha}|z_i(t)|^2 + \frac{1}{\alpha}\sum_{j=1}^{m} b_{ij}^+ M_{ij}\int_0^{+\infty}\int_{t-s}^{t} f_{ij}(s)\left|z_{n+j}(\gamma)\right|^2 e^{\alpha(\gamma+s)}d\gamma ds, \\
\widehat{V}_{n+j}(t) &= \frac{e^{\alpha t}}{\alpha}\left|z_{n+j}(t)\right|^2 + \frac{1}{\alpha}\sum_{i=1}^{n} d_{ji}^+ N_{ji}\int_0^{+\infty}\int_{t-s}^{t} g_{ji}(s)\left|z_i(\gamma)\right|^2 e^{\alpha(\gamma+s)}d\gamma ds.
\end{aligned}
\tag{4.5}
$$

Calculating the Dini upper right derivative of $\widehat{V}_j(t)$ and $\widehat{V}_{n+j}(t)$ along the solution of (2.1), and estimating it via the assumptions [32], we have

$$
\begin{aligned}
D^+\widehat{V}_i(t) &\leqslant e^{\alpha t}|z_i(t)|^2 + \frac{e^{\alpha t}}{\alpha}\left\{-2a_i^-|z_i(t)|^2 + \sum_{j=1}^{m} b_{ij}^+ M_{ij}\left[|z_i(t)|^2 + \int_0^{+\infty} f_{ij}(s)\left|z_{n+j}(t-s)\right|^2 ds\right]\right\} \\
&\quad + \frac{e^{\alpha t}}{\alpha}\left\{\sum_{j=1}^{m} b_{ij}^+ M_{ij}\left[\int_0^{+\infty} f_{ij}(s)\left|z_{n+j}(t)\right|^2 e^{\alpha s}ds - \int_0^{+\infty} f_{ij}(s)\left|z_{n+j}(t-s)\right|^2 ds\right]\right\} \\
&= \frac{e^{\alpha t}}{\alpha}\left(\alpha + \sum_{j=1}^{m} b_{ij}^+ M_{ij} - 2a_i^-\right)|z_i(t)|^2 + \frac{e^{\alpha t}}{\alpha}\sum_{j=1}^{m} b_{ij}^+ M_{ij}\left|z_{n+j}(t)\right|^2\int_0^{+\infty} f_{ij}(s)e^{\alpha s}ds \\
&\leqslant \frac{e^{\alpha t}}{\alpha}\left\{\left(\alpha + \sum_{j=1}^{m} b_{ij}^+ M_{ij} - 2a_i^-\right)|z_i(t)|^2 + \sum_{j=1}^{m} b_{ij}^+ M_{ij}\left|z_{n+j}(t)\right|^2\right\},
\end{aligned}
$$

$$D^+\widehat{V}_{n+j}(t) \leqslant \frac{e^{\alpha t}}{\alpha}\left\{\left(\alpha+\sum_{i=1}^{n}d_{ji}^{+}N_{ji}-2c_j^{-}\right)\left|z_{n+j}(t)\right|^2+\sum_{i=1}^{n}d_{ji}^{+}N_{ji}|z_i(t)|^2\right\}. \quad (4.6)$$

Let

$$\mathbb{Q}=\begin{pmatrix}\mathbb{Q}_{11} & \mathbb{Q}_{12}\\ \mathbb{Q}_{21} & \mathbb{Q}_{22}\end{pmatrix}, \quad (4.7)$$

where $\mathbb{Q}_{11} = \operatorname{diag}(2a_1^- - \sum_{j=1}^m b_{1j}^+ M_{1j}, 2a_2^- - \sum_{j=1}^m b_{2j}^+ M_{2j}, \ldots, 2a_n^- - \sum_{j=1}^m b_{nj}^+ M_{nj})$; $\mathbb{Q}_{12} = (m_{ij})_{n\times m}$, $m_{ij} = -b_{ij}^+ M_{ij}$; $\mathbb{Q}_{21} = (n_{ji})_{m\times n}$, $n_{ji} = -d_{ji}^+ N_{ji}$; $\mathbb{Q}_{22} = \operatorname{diag}(2c_1^- - \sum_{i=1}^n d_{1i}^+ N_{1i}, 2c_2^- - \sum_{i=1}^n d_{2i}^+ N_{2i}, \ldots, 2c_m^- - \sum_{i=1}^n d_{mi}^+ N_{mi})$.

Consider the Lyapunov functional

$$\mathbb{V}(t)=\sum_{k=1}^{n+m}\widehat{V}_k(t). \quad (4.8)$$

When $(\mathbb{Q}-\alpha I)^T \in \mathbb{T}$ and is a weakly diagonally dominant matrix, calculating the Dini upper right derivative of $\mathbb{V}$ along the solution of (2.1), we have

$$\begin{aligned}
D^+\mathbb{V}(t) &= D^+\sum_{k=1}^{n+m}\widehat{V}_k(t)\\
&\leqslant \frac{e^{\alpha t}}{\alpha}\left\{\left(-2a_i^- +\sum_{j=1}^{m}b_{ij}^{+}M_{ij}+\alpha\right)|z_i(t)|^2+\sum_{j=1}^{m}b_{ij}^{+}M_{ij}\left|z_{n+j}(t)\right|^2\right\}\\
&\quad+\frac{e^{\alpha t}}{\alpha}\left\{\left(-2c_j^- +\sum_{i=1}^{n}d_{ji}^{+}N_{ji}+\alpha\right)\left|z_{n+j}(t)\right|^2+\sum_{i=1}^{n}d_{ji}^{+}N_{ji}|z_i(t)|^2\right\}\\
&=\frac{e^{\alpha t}}{\alpha}\left\{-2a_i^- +\sum_{j=1}^{m}b_{ij}^{+}M_{ij}+\sum_{i=1}^{n}d_{ji}^{+}N_{ji}+\alpha\right\}|z_i(t)|^2\\
&\quad+\frac{e^{\alpha t}}{\alpha}\left\{-2c_j^- +\sum_{i=1}^{n}d_{ji}^{+}N_{ji}+\sum_{j=1}^{m}b_{ij}^{+}M_{ij}+\alpha\right\}|z_{n+i}(t)|^2<0.
\end{aligned} \quad (4.9)$$

Therefore, for $\mathbb{V}(t) \leqslant \mathbb{V}(0)$,

$$\begin{aligned}
\mathbb{V}(0)=\frac{1}{\alpha}\Bigg\{&\sum_{k=1}^{n+m}z_i^2(0)+\sum_{j=1}^{m}b_{ij}^{+}M_{ij}\int_0^{+\infty}\int_{-s}^{0}f_{ij}(s)\left|z_{n+j}(\gamma)\right|^2e^{\alpha(\gamma+s)}d\gamma ds\\
&+\sum_{i=1}^{n}d_{ji}^{+}N_{ji}\int_0^{+\infty}\int_{-s}^{0}g_{ji}(s)\left|z_i(\gamma)\right|^2e^{\alpha(\gamma+s)}d\gamma ds\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{\alpha}\left\{n+m+\sum_{j=1}^{m} b_{ij}^{+} M_{ij} \int_{0}^{+\infty} \int_{-s}^{0} f_{ij}(s) e^{\alpha(\gamma+s)} d\gamma ds\right. \\
&\qquad \left.+\sum_{i=1}^{n} d_{ji}^{+} N_{ji} \int_{0}^{+\infty} \int_{-s}^{0} g_{ji}(s) e^{\alpha(\gamma+s)} d\gamma ds\right\} \max_{k\in\{1,\ldots,m+n\}} \sup_{\gamma\leqslant 0} z_{k}^{2}(\gamma) \\
&\leqslant \frac{1}{\alpha}(n+m) \max_{k\in\{1,\ldots,m+n\}} \sup_{\gamma\leqslant 0} z_{k}^{2}(\gamma) \\
&\leqslant \frac{1}{\alpha}(n+m)\left\{\sum_{i=1}^{n}\left(\sup_{-\infty<t\leqslant 0}\left|\phi_{i}(t)-u_{i}^{*}(t)\right|\right)^{2}+\sum_{j=1}^{m}\left(\sup_{-\infty<t\leqslant 0}\left|\varphi_{j}(t)-v_{j}^{*}(t)\right|\right)^{2}\right\}.
\end{aligned}
\tag{4.10}
$$

Thus,

$$
\begin{aligned}
e^{\alpha t}&\left(\sum_{i=1}^{n}|u(t)-u^{*}(t)|^{2}+\sum_{j=1}^{m}|v(t)-v^{*}(t)|^{2}\right)<\alpha\mathbb{V}(t)<\alpha\mathbb{V}(0) \\
&\leqslant (n+m)\left\{\sum_{i=1}^{n}\left(\sup_{-\infty<t\leqslant 0}\left|\phi_{i}(t)-u_{i}^{*}(t)\right|\right)^{2}+\sum_{j=1}^{m}\left(\sup_{-\infty<t\leqslant 0}\left|\varphi_{j}(t)-v_{j}^{*}(t)\right|\right)^{2}\right\},
\end{aligned}
\tag{4.11}
$$

that is,

$$
\left(\sum_{i=1}^{n}|u(t)-u^{*}(t)|^{2}+\sum_{j=1}^{m}|v(t)-v^{*}(t)|^{2}\right)^{1/2}<\sqrt{n+m}\left\|x_{0}-x_{0}^{*}\right\|_{2} e^{-\alpha t/2}. \tag{4.12}
$$

This means that periodic solution of system (2.1) is globally exponentially stable. The proof is completed. □

Remark 4.4. For system (2.1), when delay kernels $f_{ij}(s)$, $g_{ji}(s)$ are δ-functions, that is, we take $f_{ij}(s)=\delta(s-\varrho_i)$ and $g_{ji}(s)=\delta(s-\sigma_j)$, then system (2.1) can be reduced to a fixed time delay system and all above theorems are hold.

5. Example

In this section, we present an example to show the effectiveness and correctness of our theoretical results.

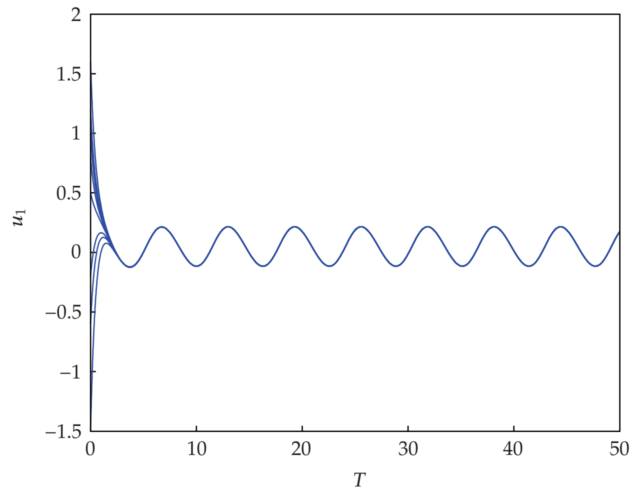

Figure 1: Time evolution of u_1 of system (5.1).

Consider the following BAM neural networks:

$$\dot{u}_i(t) = -a_i(t)u_i(t) + \sum_{j=1}^{3} b_{ij}(t)h_{ij}\left(\int_0^{t_{ij}} f_{ij}(s)v_j(t-s)ds\right) + I_i(t),$$
$$\dot{v}_j(t) = -c_j(t)v_j(t) + \sum_{i=1}^{3} d_{ji}(t)e_{ji}\left(\int_0^{\tau_{ji}} g_{ji}(s)u_i(t-s)ds\right) + L_j(t). \tag{5.1}$$

We select a set of parameters as $\alpha = 4/5$, $a_i(t) = 2 + 0.1\sin t$, $c_j(t) = 2 + 0.1\cos t$, $f_{ij}(s) = g_{ji}(s) = e^{-s}$, $I_i(t) = -0.2 + 0.4\cos t$ and $L_j(t) = 0.3 + 0.3\sin t$, $h_{ij}(s) = \tanh(s)$, $e_{ji}(s) = 1/(1+s)$, $t_{ij} = \tau_{ji} = 20$,

$$(b_{ij}(t))_{3\times 3} = \begin{pmatrix} 0.1 + 0.1\sin t & 0.2 + 0.1\sin t & 0.3 + 0.1\sin t \\ 0.5 + 0.1\sin t & 0.4 + 0.1\sin t & 0.3 + 0.1\sin t \\ 0.3 + 0.1\sin t & 0.2 + 0.1\sin t & -0.5 + 0.1\sin t \end{pmatrix},$$
$$(d_{ji}(t))_{3\times 3} = \begin{pmatrix} 0.2 + 0.1\cos t & 0.4 + 0.1\cos t & 0.1 + 0.1\cos t \\ -0.2 + 0.1\cos t & 0.3 + 0.1\cos t & 0.5 + 0.1\cos t \\ 0.5 + 0.1\cos t & 0.2 + 0.1\cos t & 0.3 + 0.1\cos t \end{pmatrix}. \tag{5.2}$$

Remark 5.1. A frequently used model for distributed time delays in biological, neural networks applications is to choose for the Gamma kernel due to mathematical difficulties. In this paper, we set a weak delay kernel, that is, an exponential kernel.

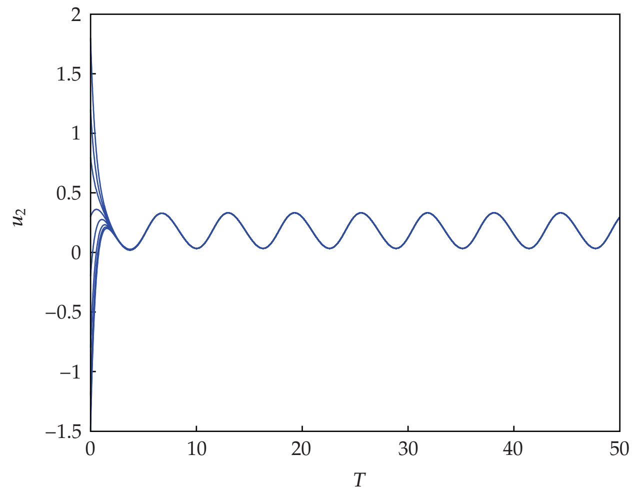

Figure 2: Time evolution of u_2 of system (5.1).

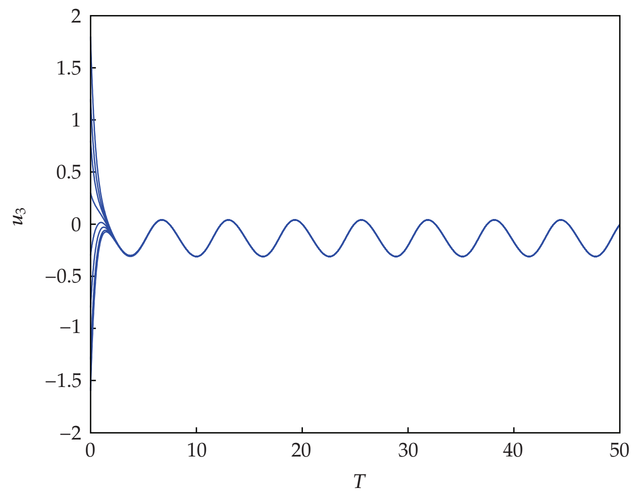

Figure 3: Time evolution of u_3 of system (5.1).

It is easy to check that $\max h'_{ij} \leqslant 1$ and $\max e'_{ji} \leqslant 1$. Let $M_{ij} = 1$ and $N_{ji} = 1$. By some computations, we obtain

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 & -0.1053 & -0.1579 & -0.2105 \\ 0 & 1 & 0 & -0.3158 & -0.2632 & -0.2105 \\ 0 & 0 & 1 & -0.3158 & -0.1579 & -0.2105 \\ -0.1579 & -0.2632 & -0.1053 & 1 & 0 & 0 \\ -0.1579 & -0.2105 & -0.3158 & 0 & 1 & 0 \\ -0.3158 & -0.1579 & -0.2105 & 0 & 0 & 1 \end{pmatrix},$$

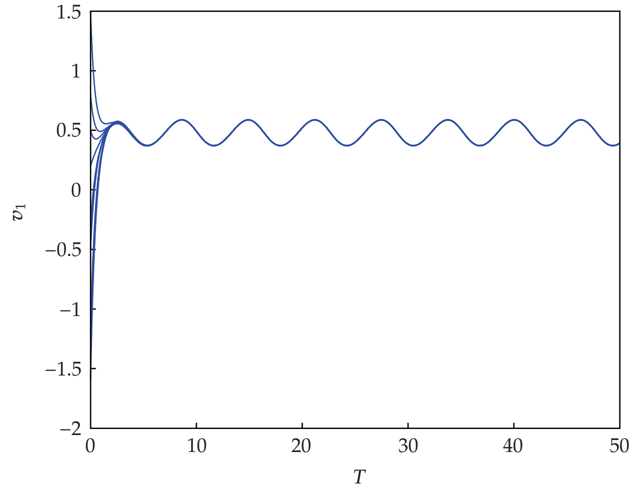

Figure 4: Time evolution of v_1 of system (5.1).

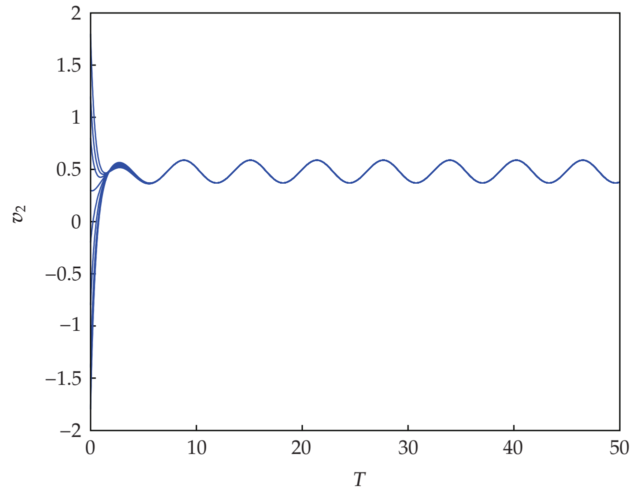

Figure 5: Time evolution of v_2 of system (5.1).

$$\mathbb{Q} = \begin{pmatrix} 2.9 & 0 & 0 & -0.2 & -0.3 & -0.4 \\ 0 & 2.3 & 0 & -0.6 & -0.5 & -0.4 \\ 0 & 0 & 2.5 & -0.6 & -0.3 & -0.4 \\ -0.3 & -0.5 & -0.2 & 2.8 & 0 & 0 \\ -0.3 & -0.4 & -0.6 & 0 & 2.5 & 0 \\ -0.6 & -0.3 & -0.4 & 0 & 0 & 2.5 \end{pmatrix}. \tag{5.3}$$

It is easy to check that $\mathbb{P}$ is a nonsingular M-matrix and $(\mathbb{Q} - 0.8I)^T$ is a weakly diagonally dominant matrix. From Theorem 4.3 system (5.1) has T-periodic oscillation, which

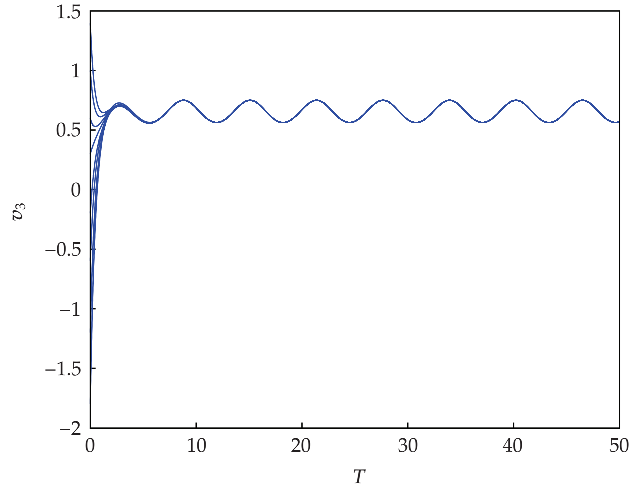

Figure 6: Time evolution of v_3 of system (5.1).

is globally exponentially stable. In Figures 1, 2, 3, 4, 5, and 6, we plot the trajectories of u_i and v_j, respectively.

6. Conclusion

In this paper, we derive a new criterion for checking the global stability of periodic oscillation of BAM neural networks with distributed delay and periodic external input sources and find that the criterion rely on the Lipschitz constants of the signal transmission functions, weights of the neural network and delay kernels by using the continuation theorem of Mawhin's coincidence degree theory, the nonsingular M-matrix and Lyapunov function. The proposed model transforms the original interacting network into matrix analysis, thereby significantly reducing the computational complexity and making analysis of periodic oscillation for even large-scale networks. Most importantly, our result is very practical in the design of BAM neural networks.

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