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Research Article

Integrodifferential Inequality for Stability of Singularly Perturbed Impulsive Delay Integrodifferential Equations

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The exponential stability of singularly perturbed impulsive delay integrodifferential equations (SPIDIDEs) is concerned. By establishing an impulsive delay integrodifferential inequality (IDIDI), some sufficient conditions ensuring the exponentially stable of any solution of SPIDIDEs for sufficiently small $\varepsilon > 0$ are obtained. A numerical example shows the effectiveness of our theoretical results.

1. Introduction

Integrodifferential equations (IDEs) arise from many areas of science (from physics, biology, medicine, etc.), which have extensive scientific backgrounds and realistic mathematical models, and hence have been emerging as an important area of investigation in recent years, see [1–6]. Correspondingly, the stability of impulsive delay integrodifferential equations has been studied quite well, for example, [7–9]. However, besides delay and impulsive effects, singular perturbation likewise exists in a wide models for physiological processes or diseases [10]. And many good results on the stability of singularly perturbed delay differential equations have been reported, see, for example, [11–14]. Therefore, it is necessary to consider delay, impulse and singular perturbation on the stability of integrodifferential equations. However, to the best of our knowledge, there are no results on the problems of the exponential stability of solutions for SPIDIDEs due to some theoretical and technical difficulties. Based on this, this article is devoted to the discussion of this problem.

Applying differential inequalities, in [14–17], authors investigated the stability of impulsive differential equations. In [14], Zhu et al. established a delay differential inequality with impulsive initial conditions and derived some sufficient conditions ensuring the exponential stability of solutions for the singular perturbed impulsive delay differential equations (SPIDDEs). In this paper, we will improve the inequality established in [14] such that it is effective for SPIDIDEs. By establishing an IDIDI, some sufficient conditions ensuring the exponential stability of any solution of SPIDIDEs for sufficiently small $\varepsilon > 0$ are obtained. The results extend and improve the earlier publications, and which will be shown by the Remarks 3.2 and 3.5 provided later. An example is given to illustrate the theory.

2. Preliminaries

Throughout this letter, unless otherwise specified, let R^n be the space of n-dimensional real column vectors and $R^{m\times n}$ be the set of $m \times n$ real matrices. $\mathcal{N} \triangleq \{1,2,\dots,n\}$. For $A, B \in R^{m\times n}$ or $A, B \in R^n$, $A \geq B(A \leq B, A > B, A < B)$ means that each pair of corresponding elements of A and B satisfies the inequality "$\geq (\leq, >, <)$". Especially, A is called a nonnegative matrix if $A \geq 0$, and z is called a positive vector if $z > 0$.

$C[X, Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. In particular, let $C \triangleq C[(-\infty, 0], R^n]$ denote the family of all bounded continuous R^n-valued functions ϕ defined on $(-\infty, 0]$ with the norm $\|\phi\| = \sup_{-\infty<\theta\leq 0}|\phi(\theta)|$, where $|\cdot|$ is Euclidean norm of R^n.

$PC[I, R^n] \triangleq \{\varphi : I \to R^n \mid \varphi(t^+) = \varphi(t)$ for $t \in I, \varphi(t^-)$ exist for $t \in I, \varphi(t^-) = \varphi(t)$ for all but points $t_k \in I\}$, where $I \subset R$ is an interval, $\varphi(t^+)$ and $\varphi(t^-)$ denote the left limit and right limit of scalar function $\varphi(t)$, respectively. Especially, let $PC \triangleq PC[(-\infty, 0], R^n]$.

For $x \in R^n$ and $\varphi \in C$ or $\varphi \in PC$, we define

$$[x]^+ = (|x_1|, \dots, |x_n|)^T, \qquad [A]^+ = \left(|a_{ij}|\right)_{n\times n}, \qquad [\varphi(t)]_\tau = \left([\varphi_1(t)]_\tau, \dots, [\varphi_n(t)]_\tau\right)^T,$$
$$[\varphi(t)]_\tau^+ = \left[[\varphi(t)]^+\right]_\tau, \qquad [\varphi_i(t)]_\tau = \sup_{-\tau\leq s\leq 0}\{\varphi_i(t+s)\}, \quad i \in \mathcal{N}, \tag{2.1}$$
$$D^+\varphi(t) = \lim_{s\to 0^+}\sup\frac{\varphi(t+s)-\varphi(t)}{s}.$$

In this paper, we consider a class of SPIDIDEs described by

$$\varepsilon\dot{x}(t) = A(t)x(t) + f(t, x(t-\tau(t))) + \int_{-\infty}^{t} R(t-s)G(x(s))ds, \quad t \geq t_0,\ t \neq t_k,$$
$$x(t_k) = J_k\left(t_k, x\left(t_k^-\right)\right), \quad k = 1, 2, \dots, \tag{2.2}$$

with the initial conditions

$$x(t_0 + \theta) = \phi(\theta) \in PC, \quad \theta \in (-\infty, 0], \tag{2.3}$$

where $0 \le \tau(t) \le \tau$, $x(t) = (x_1(t), \ldots, x_n(t))^T \in PC[R, R^n]$, $A(t) = (a_{ij}(t))_{n\times n} \in PC[R, R^{n\times n}]$, $J_k \in C[R \times R^n, R^n]$, $R(t) = (r_{ij}(t))_{n\times n} \in PC[R^+, R^{n\times n}]$, $\varepsilon \in (0, \varepsilon_0]$ is a small parameter, and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = \infty$.

Definition 2.1. The solution of (2.2) is said to be exponentially stable for sufficiently small ε if there exist finite constant vectors $K > 0$ and $\sigma > 0$, which are independent of $\varepsilon \in (0, \varepsilon_0]$ for some ε_0, and a constant $\lambda > 0$ such that $[x(t) - y(t)]^+ \le Ke^{-\lambda(t-t_0)}$ for $t \ge t_0$ and for any initial perturbation satisfying $\sup_{s\in(-\infty,0]}[\phi(s) - \varphi(s)]^+ < \sigma$. Here $y(t)$ is the solution of (2.2) corresponding to the initial condition φ.

3. Main Results

In order to prove the main result in this paper, we first need the following technique lemma.

Lemma 3.1. *Assume that* $0 \le u(t) = (u_1(t), \ldots, u_n(t))^T \in R^n$, $t \ge t_0$ *satisfy*

$$D^+u(t) \le P(t)u(t) + Q(t)[u(t)]_\tau + \int_0^\infty W(s)u(t-s)ds, \quad t \ge t_0,$$
$$u(t_0 + \theta) = \varphi(\theta) \in PC, \quad \theta \in (-\infty, 0], \tag{3.1}$$

where $P(t) = (p_{ij}(t))_{n\times n} \ge 0$ *for* $t \ge t_0$ *and* $i \ne j$, $Q(t) = (q_{ij}(t))_{n\times n} \ge 0$ *for* $t \ge t_0$, $W(s) = (w_{ij}(s))_{n\times n} \ge 0$.

If there exist a positive constant ξ *and a positive vector* $z = (z_1, \ldots, z_n)^T \in R^n$ *and two positive diagonal matrices* $L = \operatorname{diag}\{L_1, \ldots, L_n\}$, $H = \operatorname{diag}\{h_1, \ldots, h_n\}$ *with* $0 < h_i < 1$ *such that*

$$\left(Q(t) + HP(t) + L + \int_0^\infty W(s)e^{\xi s}ds\right)z < 0, \quad t \ge t_0. \tag{3.2}$$

Then one has

$$u(t) \le ze^{-\lambda(t-t_0)}, \quad t \ge t_0, \tag{3.3}$$

where the positive constant λ *is defined as*

$$0 < \lambda < \lambda_0 = \min_{1\le i\le n}\left\{\inf_{t\ge t_0}\lambda_i(t) : \lambda_i(t)z_i + \sum_{j=1}^n\left(p_{ij}(t) + q_{ij}(t)e^{\lambda_i(t)\tau} + \int_0^\infty w_{ij}(s)e^{\lambda_i(t)s}ds\right)z_j = 0\right\}, \tag{3.4}$$

for the given z.

Proof. Note that the result is trivial if $\tau = 0$. In the following, we assume that $\tau > 0$. Denote

$$F(\lambda_i(t)) = \lambda_i(t)z_i + \sum_{j=1}^{n}\left(p_{ij}(t) + q_{ij}(t)e^{\lambda_i(t)\tau} + \int_0^{\infty} w_{ij}(s)e^{\lambda_i(t)s}ds\right)z_j, \quad t \geq t_0,\ i \in \mathcal{N}, \tag{3.5}$$

then for any given $t \geq t_0$, we have

$$\begin{aligned}
F(0) &= \sum_{j=1}^{n}\left(p_{ij}(t) + q_{ij}(t) + \int_0^{\infty} w_{ij}(s)ds\right)z_j \\
&\leq \sum_{j=1}^{n} p_{ij}(t)z_j - h_i\sum_{j=1}^{n} p_{ij}(t)z_j \\
&= (1-h_i)\sum_{j=1}^{n} p_{ij}(t)z_j \\
&\leq -(1-h_i)\frac{L_i}{h_i}z_i \\
&< 0,
\end{aligned} \tag{3.6}$$

the first inequality and the second inequality are from (3.2), the last inequality is because $0 < h_i < 1$, $L_i > 0$, $z_i > 0$, $i \in \mathcal{N}$.

We also have

$$\lim_{\lambda_i(t)\to\infty} F(\lambda_i(t)) = \infty, \qquad F'(\lambda_i(t)) = z_i + \sum_{j=1}^{n}\left(q_{ij}(t)\tau e^{\lambda_i(t)\tau} + \int_0^{\infty} w_{ij}(s)se^{\lambda_i(t)s}ds\right)z_j > 0. \tag{3.7}$$

So by (3.6) and (3.7), for any $t \geq t_0$, there is a unique positive $\lambda_i(t)$ such that

$$\lambda_i(t)z_i + \sum_{j=1}^{n}\left(p_{ij}(t) + q_{ij}(t)e^{\lambda_i(t)\tau} + \int_0^{\infty} w_{ij}(s)e^{\lambda_i(t)s}ds\right)z_j = 0, \quad i \in \mathcal{N}. \tag{3.8}$$

Therefore, from the definition of λ_0, one can know that $\lambda_0 \geq 0$.

Next, we will show that $\lambda_0 \neq 0$.

If this is not true, fix v_i satisfying $0 < h_i < v_i < 1$ and $1 - h_i/v_i - h_i > 0$, $i \in \mathcal{N}$, there exist a $t^* \geq t_0$ and some integer l such that $\overline{\lambda}_l(t^*) < \delta$, where $0 < \delta < \min\{(1 - h_l/v_l - h_l)(L_l/h_l),\ (l/\tau)\ln(1/v_l), \xi\}$, such that

$$\overline{\lambda}_l(t^*)z_l + \sum_{j=1}^{n}\left(p_{lj}(t) + q_{lj}(t)e^{\overline{\lambda}_l(t^*)\tau} + \int_0^{\infty} w_{ij}(s)e^{\overline{\lambda}_l(t^*)s}ds\right)z_j = 0. \tag{3.9}$$

Then, we have

$$
\begin{aligned}
0 &= \overline{\lambda}_l(t^*)z_l + \sum_{j=1}^{n}\left(p_{lj}(t) + q_{lj}(t)e^{\overline{\lambda}_l(t^*)\tau} + \int_0^{\infty} w_{lj}(s)e^{\overline{\lambda}_l(t^*)s}ds\right)z_j \\
&< \delta z_l + \sum_{j=1}^{n}\left(p_{lj}(t) + q_{lj}(t)e^{\delta\tau}\right)z_j + \sum_{j=1}^{n}\int_0^{\infty} w_{lj}(s)z_j e^{\delta s}ds \\
&< \delta z_l + \sum_{j=1}^{n}\left(p_{lj}(t) + \frac{1}{v_l}q_{lj}(t)\right)z_j + \sum_{j=1}^{n}\int_0^{\infty} w_{lj}(s)z_j e^{\xi s}ds \\
&\le \delta z_l + \sum_{j=1}^{n}p_{lj}(t)z_j - \frac{h_l}{v_l}\sum_{j=1}^{n}p_{lj}(t)z_j - h_l\sum_{j=1}^{n}p_{lj}(t)z_j \\
&= \delta z_l + \left(1 - \frac{h_l}{v_l} - h_l\right)\sum_{j=1}^{n}p_{lj}(t)z_j \\
&\le \delta z_l - \left(1 - \frac{h_l}{v_l} - h_l\right)\frac{L_l}{h_l}z_l \\
&< 0,
\end{aligned}
\tag{3.10}
$$

this contradiction shows that $\lambda_0 > 0$, so there at least exists a positive constant λ_0 such that $0 < \lambda < \lambda_0$, that is, the definition of λ for (3.3) is reasonable.

Since $\varphi(t) \in PC$ is bounded, we always can choose a sufficiently large $z > 0$ such that

$$u(t) \le ze^{-\lambda(t-t_0)}, \quad -\infty < t \le t_0. \tag{3.11}$$

In order to prove (3.3), we first prove for any given $k > 1$,

$$u_i(t) < kz_i e^{-\lambda(t-t_0)} \equiv v_i(t), \quad t \ge t_0, \ i \in \mathcal{N}. \tag{3.12}$$

If (3.12) is not true, then by continuity of $u(t)$, there must exist some integer m and $\hat{t} > t_0$ such that

$$u_m\left(\hat{t}\right) = v_m\left(\hat{t}\right), \qquad D^+u_m\left(\hat{t}\right) \ge v_m'\left(\hat{t}\right), \tag{3.13}$$

$$u_i(t) \le v_i(t), \quad -\infty < t \le \hat{t}, \ i \in \mathcal{N}. \tag{3.14}$$

So, by (3.1), the equality of (3.13), (3.14) and $p_{ij}(t) \geq 0$ and $i \neq j$, $q_{ij}(t) \geq 0$, for $t \geq t_0$, and the definition of λ, we derive that

$$\begin{aligned}
D^+u_m\left(\hat{t}\right) &\leq \sum_{j=1}^{n}\left(p_{mj}\left(\hat{t}\right)u_j\left(\hat{t}\right) + q_{mj}\left(\hat{t}\right)u_j\left(\hat{t}-\tau\right)\right) + \sum_{j=1}^{n}\int_0^\infty w_{mj}(s)u_j\left(\hat{t}-s\right)ds \\
&\leq \sum_{j=1}^{n}\left(p_{mj}\left(\hat{t}\right)kz_je^{-\lambda(\hat{t}-t_0)} + q_{mj}\left(\hat{t}\right)kz_je^{-\lambda(\hat{t}-\tau-t_0)}\right) + \sum_{j=1}^{n}\int_0^\infty w_{mj}(s)kz_je^{-\lambda(\hat{t}-s-t_0)}ds \\
&= \sum_{j=1}^{n}\left(p_{mj}\left(\hat{t}\right) + q_{mj}\left(\hat{t}\right)e^{\lambda\tau} + \int_0^\infty w_{mj}(s)e^{\lambda s}ds\right)kz_je^{-\lambda(\hat{t}-t_0)} \\
&< \sum_{j=1}^{n}\left(p_{mj}\left(\hat{t}\right) + q_{mj}\left(\hat{t}\right)e^{\lambda_m(\hat{t})\tau} + \int_0^\infty w_{mj}(s)e^{\lambda_m(\hat{t})\tau}ds\right)kz_je^{-\lambda(\hat{t}-t_0)} \\
&= -\lambda_m\left(\hat{t}\right)z_mke^{-\lambda(\hat{t}-t_0)} \\
&< -\lambda z_mke^{-\lambda(\hat{t}-t_0)} \\
&= v_m'\left(\hat{t}\right),
\end{aligned} \tag{3.15}$$

which contradicts the inequality in (3.13), and so (3.12) holds for all $t \geq t_0$. Letting $k \to 1$, then (3.3) holds, and the proof is completed. □

Remark 3.2. If $W(s) = (w_{ij}(s))_{n\times n} = 0$ in Lemma 3.1, then we get [14, Lemma 1].

Theorem 3.3. *Assume that $A(t) = (a_{ij}(t))_{n\times n} \geq 0$ for $t \geq t_0$ and $i \neq j$, further suppose the following*

(H_1) *For any $x, y \in R^n$, there exist nonnegative matrices $U(t) = (u_{ij}(t))_{n\times n}$ and $B = (b_{ij})_{n\times n}$, $t \geq t_0$, such that*

$$\begin{aligned}
[f(t,x) - f(t,y)]^+ &\leq U(t)[x-y]^+, \quad t \geq t_0, \\
[G(x) - G(y)]^+ &\leq B[x-y]^+, \quad t \geq t_0.
\end{aligned} \tag{3.16}$$

(H_2) *For any $x, y \in R^n$,there exist nonnegative constant matrices M_k such that*

$$[J_k(t,x) - J_k(t,y)]^+ \leq M_k[x-y]^+, \quad t \geq t_0. \tag{3.17}$$

(H_3) *There exist a positive constant $\bar{\xi}$ and a positive vector $z = (z_1, z_2, \ldots, z_n)^T \in R^n$ and two positive diagonal matrices $V = \operatorname{diag}\{v_1, \ldots, v_n\}$, $S = \operatorname{diag}\{s_1, s_2, \ldots, s_n\}$, with $0 < s_i < 1$, $i \in \mathcal{N}$ such that*

$$\left(U(t) + SA(t) + V + \int_0^\infty N(s)e^{\bar{\xi}s}ds\right)z < 0, \quad t \geq t_0, \tag{3.18}$$

where $N(s) = (n_{ij}(s))_{n\times n} \geq 0$, $n_{ij}(s) = |r_{ij}(s)|b_{ij}$.

(H_4) *There exists a positive constant* η *satisfying*

$$\frac{\ln \eta_k}{t_k - t_{k-1}} \leq \eta < \lambda(\varepsilon), \quad k = 1, 2, \ldots, \tag{3.19}$$

where η_k *satisfy*

$$\eta_k \geq 1, \quad \eta_k z \geq M_k z, \tag{3.20}$$

and $\lambda(\varepsilon)$ *is defined as*

$$0 < \lambda(\varepsilon) < \lambda_0(\varepsilon) = \min_{1\leq i\leq n}\left\{\inf_{t\geq t_0} \lambda_i(t,\varepsilon) : \lambda_i z_i + \sum_{j=1}^{n}\left(\frac{a_{ij}(t)}{\varepsilon} + \frac{u_{ij}(t)}{\varepsilon}e^{\lambda_i \tau} + \int_0^{\infty}\frac{n_{ij}(s)}{\varepsilon}e^{\lambda_i s}ds\right)z_j = 0\right\}. \tag{3.21}$$

for the given z.

Then there exists a small $\varepsilon_0 > 0$ *such that the solution of* (2.2) *is exponentially stable for sufficiently small* $\varepsilon \in (0, \varepsilon_0]$.

Proof. By a similar argument with (3.4), one can know that the $\lambda(\varepsilon)$ defined by (3.21) is reasonable. For any $\phi, \varphi \in PC$, let $x(t)$, $y(t)$ be two solutions of (2.2) through (t_0, ϕ), (t_0, φ), respectively. Since $\phi, \varphi \in PC$ are bounded, we can always choose a positive vector z such that

$$[x(t) - y(t)]^+ \leq z e^{-\lambda(t-t_0)}, \quad t \in (-\infty, t_0]. \tag{3.22}$$

Calculating the upper right derivative $D^+[x(t)-y(t)]^+$ along the solution of (2.2), by condition (H_1), we have

$$\begin{aligned}
D^+&[x(t) - y(t)]^+ \\
&= \operatorname{Sgn}(x(t) - y(t))(x(t) - y(t))' \\
&\leq \operatorname{Sgn}(x(t) - y(t))\frac{A(t)}{\varepsilon}(x(t) - y(t)) + \frac{1}{\varepsilon}[f(t, x(t - \tau(t))) - f(t, y(t - \tau(t)))]^+ \\
&\quad + \frac{1}{\varepsilon}\int_{-\infty}^{t} R(t-s)[G(x(s)) - G(y(s))]^+ ds \\
&\leq \frac{A(t)}{\varepsilon}[x(t) - y(t)]^+ + \frac{U(t)}{\varepsilon}[x(t) - y(t)]_\tau^+ + \frac{1}{\varepsilon}\int_0^{\infty}[R(s)]^+ B[x(t-s) - y(t-s)]^+ ds.
\end{aligned} \tag{3.23}$$

From condition (H_3), we have

$$\left(\frac{U(t)}{\varepsilon} + S\frac{A(t)}{\varepsilon} + \frac{V}{\varepsilon} + \int_0^{\infty}\frac{N(s)}{\varepsilon}e^{\bar{\xi}s}ds\right)z < 0, \quad t \geq t_0. \tag{3.24}$$

Therefore, (3.23) and (3.24) imply that all the assumptions of Lemma 3.1 are true. So we have

$$[x(t)-y(t)]^{+} \leq z e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_0, t_1), \tag{3.25}$$

where $\lambda(\varepsilon)$ is determined by (3.21) and the positive constant vector z is determined by (3.18).

Using the discrete part of (2.2), condition (H_2), (3.20) and (3.25), we can obtain that

$$\begin{aligned}
[x(t_1)-y(t_1)]^{+} &= \left[J_1\left(t_1, x\left(t_1^{-}\right)\right) - J_1\left(t_1, y\left(t_1^{-}\right)\right)\right]^{+} \\
&\leq M_1\left[x\left(t_1^{-}\right) - y\left(t_1^{-}\right)\right]^{+} \\
&\leq M_1 z e^{-\lambda(\varepsilon)(t_1-t_0)} \\
&\leq \eta_1 z e^{-\lambda(\varepsilon)(t_1-t_0)},
\end{aligned} \tag{3.26}$$

and so, we have

$$[x(t)-y(t)]^{+} \leq \eta_1 z e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in (-\infty, t_1]. \tag{3.27}$$

By a similar argument with (3.25), we can use (3.27) derive that

$$[x(t)-y(t)]^{+} \leq \eta_1 z e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_1, t_2). \tag{3.28}$$

Therefore, by simple induction, we have

$$[x(t)-y(t)]^{+} \leq \eta_1 \cdots \eta_{k-1} z e^{-\lambda(\varepsilon)(t-t_0)}, \quad t \in [t_{k-1}, t_k), \ k = 1, 2, \ldots. \tag{3.29}$$

From (3.19) and (3.29), we obtain

$$[x(t)-y(t)]^{+} \leq z e^{-(\lambda(\varepsilon)-\eta)(t-t_0)}, \quad t \in [t_{k-1}, t_k), \ k = 1, 2, \ldots, \ \forall \varepsilon > 0. \tag{3.30}$$

For any $t \geq t_0$, let $\lambda_i(t, \varepsilon)$ be defined as the unique positive zero of

$$\lambda_i z_i + \sum_{j=1}^{n} \left(\frac{a_{ij}(t)}{\varepsilon} + \frac{u_{ij}(t)}{\varepsilon} e^{\lambda_i \tau} + \int_0^{\infty} \frac{n_{ij}(s)}{\varepsilon} e^{\lambda_i s} ds \right) z_j = 0. \tag{3.31}$$

Differentiate both sides of (3.31) with respect to the variable ε, we have

$$\frac{d}{d\varepsilon}\lambda_i(t,\varepsilon) = \frac{-\lambda_i z_i}{\varepsilon z_i + \sum_{j=1}^{n} u_{ij}(t)\tau e^{\lambda_i \tau} z_j + \sum_{j=1}^{n}\int_0^{\infty} n_{ij}(s) s z_j e^{\lambda_i s} ds} < 0, \tag{3.32}$$

so $\lambda_i(t,\varepsilon)$ is monotonically decreasing with respect to the variable ε, which implies that $\lambda_0(\varepsilon)$ is also monotonically decreasing with respect to the variable ε. So we can choose the $\lambda(\varepsilon)$ in (3.21) satisfying the same monotonicity with $\lambda_0(\varepsilon)$, for example, $\lambda(\varepsilon) = \lambda_0(\varepsilon) - \delta$, where $0 < \delta < \lambda_0(\varepsilon) - \lambda(\varepsilon)$. Hence we can deduce that there exists a small $\varepsilon_0 > 0$ such that the solution of (2.2) is exponentially stable for sufficiently small $\varepsilon \in (0, \varepsilon_0]$. The proof is completed. □

Remark 3.4. Suppose that $N(s) = (n_{ij}(s))_{n\times n} = 0$ in Theorem 3.3, then we can easily get [14, Theorem 1]. In fact, "$\eta_k \triangleq \max\{\|M_k\|, 1\}$" of condition (H_4) in [14, Theorem 1] ensure that the above (3.20) holds.

Remark 3.5. If $J_k(t,x) = x$, $t \ge t_0$, that is there have no impulses in (2.2), then by Theorem 3.3, we can obtain the following result.

Corollary 3.6. *Assume that $A(t) = (a_{ij}(t))_{n\times n} \ge 0$ for $t \ge t_0$ and $i \ne j$, $N(s) = (n_{ij}(s))_{n\times n} \ge 0$, further suppose that (H_1) and (H_3) hold. Then there exists a small $\varepsilon_0 > 0$ such that the solution of* (2.2) *is exponentially stable for sufficiently small $\varepsilon \in (0, \varepsilon_0]$.*

Remark 3.7. From Lemma 3.1 and the proof of Theorem 3.3, it is obvious that the results obtained in this paper still hold for $\varepsilon = 1$. So this type of exponential stability can obviously be applied to general impulsive delay integrodifferential equations.

Remark 3.8. When $\varepsilon = 1$ and $r_{ij}(s) = 0$, the global exponential stability criteria for (2.2) have been established in [18] by utilizing the Lyapunov functional method. However, the additional assumption that f_j is bounded is required in [18].

4. An Illustrative Example

In this section, we will give an example to illustrate the exponential stability of (2.2).

Example 4.1. Consider the following SPIDIDEs:

$$\begin{aligned}
\varepsilon \dot{x}_1(t) &= (-10 - \sin t)x_1(t) + (2 + \sin t)\arctan x_1(t - \tau(t)) \\
&\quad + (1 + \cos t)\arctan x_2(t - \tau(t)) + \int_{-\infty}^{t} e^{-(t-s)} x_1(s) ds, \quad t \ne t_k, \\
\varepsilon \dot{x}_2(t) &= (-8 + 2\cos t)x_1(t) + \sin^2 t \arctan x_1(t - \tau(t)) \\
&\quad + (1 - \cos t)\arctan x_2(t - \tau(t)) + \int_{-\infty}^{t} e^{-2(t-s)} x_2(s) ds, \quad t \ne t_k, \\
x_1(t_k) &= \alpha_{1k} x_1(t_k^-) - \beta_{1k} x_2(t_k^-), \\
x_2(t_k) &= \beta_{2k} x_1(t_k^-) + \alpha_{2k} x_2(t_k^-),
\end{aligned} \tag{4.1}$$

where $\alpha_{ik}, \beta_{ik} \ge 0$ are constants, $\tau(t) = e^{-t} \le 1 \triangleq \tau$, $t \ge t_0$, $t_k = t_{k-1} + 3k$, $k = 1, 2, \ldots$.

We can easily find that conditions (H_1) and (H_2) are satisfied with

$$A(t) = \begin{pmatrix} -10-\sin t & 0 \\ 0 & -8+2\cos t \end{pmatrix}, \qquad U(t) = \begin{pmatrix} 2+\sin t & 1+\cos t \\ \sin^2 t & 1-\cos t \end{pmatrix},$$
$$N(s) = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix}, \qquad M_k = \begin{pmatrix} \alpha_{1k} & \beta_{1k} \\ \beta_{2k} & \alpha_{2k} \end{pmatrix}. \tag{4.2}$$

So there exist $\bar{\xi} = 0.5$, $z = (1,1)^T$, $V = \operatorname{diag}\{1, 1/3\}$ and $S = \operatorname{diag}\{0.8, 0.5\}$ such that

$$\left(U(t) + SA(t) + V + \int_0^\infty N(s)e^{\bar{\xi}s}ds\right)z = \left(-2+0.2\sin t+\cos t, -2+\sin^2 t\right) < 0, \quad t \ge t_0. \tag{4.3}$$

Let $\eta_k = \max\{\alpha_{1k}+\beta_{1k}, \alpha_{2k}+\beta_{2k}\}$, we can obtain η_k satisfy $\eta_k z \ge M_k z$.

Case 1. Let $\alpha_{1k} = 0.2e^{0.3k}$, $\alpha_{2k} = 0.7e^{0.3k}$, $\beta_{1k} = 0.5e^{0.3k}$, $\beta_{2k} = 0.3e^{0.3k}$, then we obtain that there exists an $\eta = 0.1 > 0$ such that

$$\eta_k = e^{0.3k} \ge 1, \qquad \frac{\ln \eta_k}{t_k - t_{k-1}} = \frac{0.3k}{3k} = 0.1 = \eta, \tag{4.4}$$

and for $\varepsilon > 0$, the positive constant $\lambda(\varepsilon)$ is determined by the following equations:

$$\lambda_1(t) + \frac{1}{\varepsilon}\left(-10-\sin t + (3+\sin t+\cos t)e^{\lambda_1(t)} + \int_0^\infty e^{-s}e^{\lambda_1(s)}ds\right) = 0,$$
$$\lambda_2(t) + \frac{1}{\varepsilon}\left(-8+2\cos t + \left(1+\sin^2 t-\cos t\right)e^{\lambda_2(t)} + \int_0^\infty e^{-2s}e^{\lambda_2(s)}ds\right) = 0. \tag{4.5}$$

So for a given ε, we can obtain the corresponding λ by (4.5). By the proof of Theorem 3.3, we know that λ is monotonically decreasing with respect to the variable ε, then there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, we have $\lambda > \eta$. Therefore, all the conditions of Theorem 3.3 are satisfied, we conclude that the solution of (4.1) is exponentially stable for sufficiently small $\varepsilon > 0$.

Case 2. Let $\alpha_{1k} = \alpha_{2k} = 1$ and $\beta_{1k} = \beta_{2k} = 0$, then (4.1) becomes the singularly perturbed delay integrodifferential equations without impulses. So by Corollary 3.6, the solution of (4.1) is exponentially stable for sufficiently small $\varepsilon > 0$.

Remark 4.2. Obviously, the delay differential inequality which established in [14] is ineffective for studing the stability of SPIDIDEs (4.1).

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