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Research Article

A New General Integral Operator Defined by Al-Oboudi Differential Operator

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We define a new general integral operator using Al-Oboudi differential operator. Also we introduce new subclasses of analytic functions. Our results generalize the results of Breaz, Güney, and Sălăgean.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S} := \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}$.

For $f \in \mathcal{A}$, Al-Oboudi [1] introduced the following operator:

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \quad \lambda \geq 0, \tag{1.3}$$

$$D^k f(z) = D_\lambda\left(D^{k-1} f(z)\right), \quad (k \in \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{1.4}$$

If f is given by (1.1), then from (1.3) and (1.4) we see that

$$D^k f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^k a_n z^n, \quad (k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{1.5}$$

with $D^k f(0) = 0$.

Remark 1.1. When $\lambda = 1$, we get Sălăgean's differential operator [2].

Now we introduce new classes $\mathcal{S}_k(\delta, b, \lambda)$ and $\mathcal{K}_k(\delta, b, \lambda)$ as follows.

A function $f \in \mathcal{A}$ is in the classes $\mathcal{S}_k(\delta, b, \lambda)$, where $\delta \in [0,1)$, $b \in \mathbb{C} - \{0\}$, $\lambda \geq 0$, $k \in \mathbb{N}_0$, if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{D^{k+1} f(z)}{D^k f(z)} - 1\right)\right\} > \delta \tag{1.6}$$

or equivalently

$$\operatorname{Re}\left\{1 + \frac{\lambda}{b}\left(\frac{z(D^k f(z))'}{D^k f(z)} - 1\right)\right\} > \delta \tag{1.7}$$

for all $z \in \mathbb{U}$.

A function $f \in \mathcal{A}$ is in the classs $\mathcal{K}_k(\delta, b, \lambda)$, where $\delta \in [0,1)$, $b \in \mathbb{C} - \{0\}$, $\lambda \geq 0$, $k \in \mathbb{N}_0$, if and only if

$$\operatorname{Re}\left\{1 + \frac{\lambda}{b}\frac{z\left(D^k f(z)\right)''}{\left(D^k f(z)\right)'}\right\} > \delta \tag{1.8}$$

for all $z \in \mathbb{U}$.

We note that $f \in \mathcal{K}_k(\delta, b, \lambda)$ if and only if $zf' \in \mathcal{S}_k(\delta, b, \lambda)$.

Remark 1.2. (i) For $k = 0$ and $\lambda = 1$, we have the classes

$$\mathcal{S}_0(\delta, b, 1) \equiv \mathcal{S}_\delta^*(b), \quad \mathcal{K}_0(\delta, b, 1) \equiv \mathcal{C}_\delta(b) \tag{1.9}$$

introduced by Frasin [3].

(ii) For $b = 1$ and $\lambda = 1$, we have the class

$$\mathcal{S}_k(\delta, 1, 1) \equiv \mathcal{S}_k(\delta) \tag{1.10}$$

of k-starlike functions of order δ defined by Sălăgean [2].

(iii) In particular, the classes

$$\mathcal{S}_0(\delta, 1, 1) \equiv \mathcal{S}^*(\delta), \quad \mathcal{K}_0(\delta, 1, 1) \equiv \mathcal{K}(\delta) \tag{1.11}$$

are the classes of starlike functions of order δ and convex functions of order δ in $\mathbb{U}$, respectively.

(iv) Furthermore, the classes

$$\mathcal{S}_0(0,1,1) \equiv \mathcal{S}^*, \quad \mathcal{K}_0(0,1,1) \equiv \mathcal{K} \tag{1.12}$$

are familiar classes of starlike and convex functions in $\mathbb{U}$, respectively.

(v) For $\lambda = 1$, we get

$$\mathcal{K}_k(\delta, b, 1) \equiv \mathcal{S}_{k+1}(\delta, b, 1). \tag{1.13}$$

Let us introduce the new subclasses $\mathcal{US}_k(\alpha,\delta,b,\lambda)$, $\mathcal{UK}_k(\alpha,\delta,b,\lambda)$ and $\mathcal{SH}_k(\alpha,b,\lambda)$, $\mathcal{KH}_k(\alpha,b,\lambda)$ as follows.

A function $f \in \mathcal{A}$ is in the class $\mathcal{US}_k(\alpha,\delta,b,\lambda)$ if and only if f satisfies

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{D^{k+1}f(z)}{D^k f(z)} - 1\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{D^{k+1}f(z)}{D^k f(z)} - 1\right)\right| + \delta \quad (z \in \mathbb{U}) \tag{1.14}$$

or equivalently

$$\operatorname{Re}\left\{1 + \frac{\lambda}{b}\left(\frac{z(D^k f(z))'}{D^k f(z)} - 1\right)\right\} > \alpha\left|\frac{\lambda}{b}\left(\frac{z(D^k f(z))'}{D^k f(z)} - 1\right)\right| + \delta, \tag{1.15}$$

where $\alpha \geq 0$, $\delta \in [-1,1)$, $\alpha + \delta \geq 0$, $b \in \mathbb{C} - \{0\}$, $\lambda \geq 0$, $k \in \mathbb{N}_0$.

A function $f \in \mathcal{A}$ is in the class $\mathcal{UK}_k(\alpha,\delta,b,\lambda)$ if and only if f satisfies

$$\operatorname{Re}\left\{1 + \frac{\lambda}{b}\frac{z(D^k f(z))''}{(D^k f(z))'}\right\} > \alpha\left|\frac{\lambda}{b}\frac{z(D^k f(z))''}{(D^k f(z))'}\right| + \delta, \tag{1.16}$$

where $\alpha \geq 0$, $\delta \in [-1,1)$, $\alpha + \delta \geq 0$, $b \in \mathbb{C} - \{0\}$, $\lambda \geq 0$, $k \in \mathbb{N}_0$.

We note that $f \in \mathcal{UK}_k(\alpha,\delta,b,\lambda)$ if and only if $zf' \in \mathcal{US}_k(\alpha,\delta,b,\lambda)$.

Remark 1.3. (i) For $\alpha = 0$, we have

$$\mathcal{US}_k(0,\delta,b,\lambda) \equiv \mathcal{S}_k(\delta,b,\lambda), \quad \mathcal{UK}_k(0,\delta,b,\lambda) \equiv \mathcal{K}_k(\delta,b,\lambda). \tag{1.17}$$

(ii) For $b = 1$ and $\lambda = 1$, we have the class

$$\mathcal{US}_k(\alpha,\delta,1,1) \equiv \mathcal{US}_k(\alpha,\delta). \tag{1.18}$$

of k-uniform starlike functions of order δ and type α, [4].

(iii) For $\lambda = 1$, we have

$$\mathcal{UK}_k(\alpha,\delta,b,1) \equiv \mathcal{US}_{k+1}(\alpha,\delta,b,1). \tag{1.19}$$

(iv) For $b = 1$ and $\lambda = 1$, we have

$$\mathcal{UK}_k(\alpha, \delta, 1, 1) \equiv \mathcal{US}_{k+1}(\alpha, \delta). \tag{1.20}$$

Geometric Interpretation

$f \in \mathcal{US}_k(\alpha, \delta, b, \lambda)$ and $f \in \mathcal{UK}_k(\alpha, \delta, b, \lambda)$ if and only if $1 + (\lambda/b)((z(D^k f(z))'/D^k f(z)) - 1)$ and $1 + (\lambda/b)(z(D^k f(z))''/(D^k f(z))')$, respectively, take all the values in the conic domain $R_{\alpha,\delta}$ which is included in the right-half plane such that

$$R_{\alpha,\delta} = \left\{u + iv : u > \alpha\sqrt{(u-1)^2 + v^2} + \delta\right\}. \tag{1.21}$$

From elementary computations we see that $\partial R_{\alpha,\delta}$ represents the conic sections symmetric about the real axis. Thus $R_{\alpha,\delta}$ is an elliptic domain for $\alpha > 1$, a parabolic domain for $\alpha = 1$, a hyperbolic domain for $0 < \alpha < 1$ and a right-half plane $u > \delta$ for $\alpha = 0$.

A function $f \in \mathcal{A}$ is in the class $\mathcal{SH}_k(\alpha, b, \lambda)$ if and only if f satisfies

$$\left|1 + \frac{1}{b}\left(\frac{D^{k+1} f(z)}{D^k f(z)} - 1\right) - 2\alpha(\sqrt{2} - 1)\right| < \operatorname{Re}\left\{\sqrt{2}\left(1 + \frac{1}{b}\left(\frac{D^{k+1} f(z)}{D^k f(z)} - 1\right)\right)\right\} + 2\alpha(\sqrt{2} - 1) \quad (z \in \mathbb{U}), \tag{1.22}$$

where $\alpha > 0$, $b \in \mathbb{C} - \{0\}$, $\lambda \geq 0$, $k \in \mathbb{N}_0$.

A function $f \in \mathcal{A}$ is in the class $\mathcal{KH}_k(\alpha, b, \lambda)$ if and only if f satisfies

$$\left|1 + \frac{\lambda}{b}\frac{z(D^k f(z))''}{(D^k f(z))'} - 2\alpha(\sqrt{2} - 1)\right| < \operatorname{Re}\left\{\sqrt{2}\left(1 + \frac{\lambda}{b}\frac{z(D^k f(z))''}{(D^k f(z))'}\right)\right\} + 2\alpha(\sqrt{2} - 1) \quad (z \in \mathbb{U}), \tag{1.23}$$

where $\alpha > 0$, $b \in \mathbb{C} - \{0\}$, $\lambda \geq 0$, $k \in \mathbb{N}_0$.

We note that $f \in \mathcal{KH}_k(\alpha, b, \lambda)$ if and only if $zf' \in \mathcal{SH}_k(\alpha, b, \lambda)$.

Remark 1.4. (i) For $b = 1$ and $\lambda = 1$, we have the classes

$$\begin{gathered} \mathcal{SH}_k(\alpha, 1, 1) \equiv \mathcal{SH}_k(\alpha), \\ \mathcal{KH}_k(\alpha, 1, 1) \equiv \mathcal{SH}_{k+1}(\alpha, 1, 1) \equiv \mathcal{SH}_{k+1}(\alpha) \end{gathered} \tag{1.24}$$

defined in [5].

(ii) For $\lambda = 1$, we have

$$\mathcal{KH}_k(\alpha, b, 1) \equiv \mathcal{SH}_{k+1}(\alpha, b, 1). \tag{1.25}$$

D. Breaz and N. Breaz [6] introduced and studied the integral operator

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\mu_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\mu_n} dt, \tag{1.26}$$

where $f_i \in \mathcal{A}$ and $\mu_i > 0$ for all $i \in \{1, \ldots, n\}$.

By using the Al-Oboudi differential operator, we introduce the following integral operator. So we generalize the integral operator F_n.

Definition 1.5. Let $k \in \mathbb{N}_0$, $l = (l_1, \ldots, l_n) \in \mathbb{N}_0^n$, and $\mu_i > 0$, $1 \le i \le n$. One defines the integral operator $I_{k,n,l,\mu} : \mathcal{A}^n \to \mathcal{A}$,

$$\begin{gathered} I_{k,n,l,\mu}(f_1, \ldots, f_n) = F, \\ D^k F(z) = \int_0^z \left(\frac{D^{l_1} f_1(t)}{t}\right)^{\mu_1} \cdots \left(\frac{D^{l_n} f_n(t)}{t}\right)^{\mu_n} dt, \end{gathered} \tag{1.27}$$

where $f_1, \ldots, f_n \in \mathcal{A}$ and D is the Al-Oboudi differential operator.

Remark 1.6. In Definition 1.5, if we set

(i) $\lambda = 1$, then we have [7, Definition 1].

(ii) $\lambda = 1$, $k = 0$ and $l_1 = \cdots = l_n = 0$, then we have the integral operator defined by (1.26).

(iii) $k = 0$, $l_1 = \cdots = l_n = l \in \mathbb{N}_0$, then we have [8, Definition 1.1].

2. Main Results

The following lemma will be required in our investigation.

Lemma 2.1. *For the integral operator $I_{k,n,l,\mu}(f_1, \ldots, f_n) = F$, defined by* (1.27), *one has*

$$\frac{\lambda z (D^k F(z))''}{(D^k F(z))'} = \sum_{i=1}^n \mu_i \frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - \sum_{i=1}^n \mu_i. \tag{2.1}$$

Proof. By (1.27), we get

$$(D^k F(z))' = \left(\frac{D^{l_1} f_1(z)}{z}\right)^{\mu_1} \cdots \left(\frac{D^{l_n} f_n(z)}{z}\right)^{\mu_n}. \tag{2.2}$$

Also, using (1.3) and (1.4), we obtain

$$(D^k F(z))' = \frac{D^{k+1} F(z) - (1-\lambda) D^k F(z)}{\lambda z}. \tag{2.3}$$

On the other hand, from (2.2) and (2.3), we find

$$\left(D^k F(z)\right)'' = \sum_{i=1}^{n} \mu_i \left(\frac{D^{l_i} f_i(z)}{z}\right)^{\mu_i} \left(\frac{z\left(D^{l_i} f_i(z)\right)' - D^{l_i} f_i(z)}{z D^{l_i} f_i(z)}\right) \prod_{\substack{j=1 \\ (j \neq i)}}^{n} \left(\frac{D^{l_j} f_j(z)}{z}\right)^{\mu_j}, \tag{2.4}$$

$$\left(D^k F(z)\right)'' = \frac{D^{k+2} F(z) - (2-\lambda) D^{k+1} F(z) + (1-\lambda) D^k F(z)}{\lambda^2 z^2}. \tag{2.5}$$

Thus by (2.2) and (2.4), we can write

$$\begin{aligned} \frac{\left(D^k F(z)\right)''}{\left(D^k F(z)\right)'} &= \sum_{i=1}^{n} \mu_i \left(\frac{z\left(D^{l_i} f_i(z)\right)' - D^{l_i} f_i(z)}{z D^{l_i} f_i(z)}\right) \\ &= \sum_{i=1}^{n} \mu_i \left(\frac{D^{l_i+1} f_i(z) - D^{l_i} f_i(z)}{\lambda z D^{l_i} f_i(z)}\right). \end{aligned} \tag{2.6}$$

Finally, we obtain

$$\frac{\lambda z\left(D^k F(z)\right)''}{\left(D^k F(z)\right)'} = \sum_{i=1}^{n} \mu_i \left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right), \tag{2.7}$$

which is the desired result. □

Theorem 2.2. *Let* $\alpha_i \geq 0$, $\delta_i \in [-1, 1)$, $\alpha_i + \delta_i \geq 0$ $(1 \leq i \leq n)$, *and* $b \in \mathbb{C} - \{0\}$, $\lambda \geq 0$. *Also suppose that*

$$\sum_{i=1}^{n} \mu_i \frac{1-\delta_i}{1+\alpha_i} \leq 1. \tag{2.8}$$

If $f_i \in \mathcal{US}_{l_i}(\alpha_i, \delta_i, b, \lambda)(1 \leq i \leq n)$, *then the integral operator* $I_{k,n,l,\mu} = F$, *defined by* (1.27), *is in the class* $\mathcal{K}_k(\gamma, b, \lambda)$, *where*

$$\gamma = 1 - \sum_{i=1}^{n} \mu_i \frac{1-\delta_i}{1+\alpha_i}. \tag{2.9}$$

Proof. Since $f_i \in \mathcal{US}_{l_i}(\alpha_i, \delta_i, b, \lambda)(1 \leq i \leq n)$, by (1.14) we have

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right)\right\} > \frac{\alpha_i + \delta_i}{\alpha_i + 1} \tag{2.10}$$

for all $z \in \mathbb{U}$. By (2.1), we get

$$1 + \frac{1}{b}\frac{\lambda z\left(D^k F(z)\right)''}{\left(D^k F(z)\right)'} = 1 + \sum_{i=1}^{n} \mu_i \frac{1}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right) = 1 + \sum_{i=1}^{n} \mu_i \left[1 + \frac{1}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right)\right] - \sum_{i=1}^{n} \mu_i. \tag{2.11}$$

So, (2.10) and (2.11) give us

$$\operatorname{Re}\left\{1 + \frac{1}{b}\frac{\lambda z\left(D^k F(z)\right)''}{\left(D^k F(z)\right)'}\right\} = 1 - \sum_{i=1}^{n} \mu_i + \sum_{i=1}^{n} \mu_i \operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right)\right\} > 1 - \sum_{i=1}^{n} \mu_i + \sum_{i=1}^{n} \mu_i \frac{\alpha_i + \delta_i}{\alpha_i + 1} = 1 - \sum_{i=1}^{n} \mu_i \frac{1 - \delta_i}{1 + \alpha_i} \tag{2.12}$$

for all $z \in \mathbb{U}$. Hence, we obtain $F \in \mathcal{K}_k(\gamma, b, \lambda)$, where $\gamma = 1 - \sum_{i=1}^{n} \mu_i((1 - \delta_i)/(1 + \alpha_i))$. □

Corollary 2.3. *Let $\alpha_i \geq 0$, $\delta_i \in [-1, 1)$, $\alpha_i + \delta_i \geq 0 (1 \leq i \leq n)$, and $b \in \mathbb{C} - \{0\}$. Also suppose that*

$$\sum_{i=1}^{n} \mu_i \frac{1 - \delta_i}{1 + \alpha_i} \leq 1. \tag{2.13}$$

If $f_i \in \mathcal{US}_{l_i}(\alpha_i, \delta_i, b, 1)$ $(1 \leq i \leq n)$, then the integral operator $I_{k,n,l,\mu} = F$, defined by (1.27)*, is in the class $\mathcal{S}_{k+1}(\gamma, b, 1)$, where γ is defined as in* (2.9).

Proof. In Theorem 2.2, we consider $\lambda = 1$. □

From Corollary 2.3, we immediately get Corollary 2.4.

Corollary 2.4. *Let $\alpha_i \geq 0$, $\delta_i \in [-1, 1)$, $\alpha_i + \delta_i \geq 0$ $(1 \leq i \leq n)$, and $b \in \mathbb{C} - \{0\}$. Also suppose that*

$$\sum_{i=1}^{n} \mu_i \frac{1 - \delta_i}{1 + \alpha_i} \leq 1. \tag{2.14}$$

If $f_i \in \mathcal{US}_{l_i}(\alpha_i, \delta_i, b, 1)$ $(1 \leq i \leq n)$, then the integral operator $I_{k,n,l,\mu} = F$, defined by (1.27)*, is in the class $\mathcal{S}_{k+1}(0, b, 1)$.*

Remark 2.5. If we set $b = 1$ in Corollary 2.4, then we have [7, Theorem 1]. So Corollary 2.4 is an extension of Theorem 1.

Corollary 2.6. *Let $\delta_i \in [0, 1) (1 \leq i \leq n)$ and $b \in \mathbb{C} - \{0\}$, $\lambda \geq 0$. Also suppose that*

$$\sum_{i=1}^{n} \mu_i (1 - \delta_i) \leq 1. \tag{2.15}$$

If $f_i \in \mathcal{S}_{l_i}(\delta_i, b, \lambda)(1 \le i \le n)$, *then the integral operator* $I_{k,n,l,\mu} = F$, *defined by* (1.27), *is in the class* $\mathcal{K}_k(\rho, b, \lambda)$, *where*

$$\rho = 1 - \sum_{i=1}^{n} \mu_i (1 - \delta_i). \tag{2.16}$$

Proof. In Theorem 2.2, we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. □

Corollary 2.7. *Let* $\delta_i \in [0, 1)$ $(1 \le i \le n)$ *and* $b \in \mathbb{C} - \{0\}$. *Also suppose that*

$$\sum_{i=1}^{n} \mu_i (1 - \delta_i) \le 1. \tag{2.17}$$

If $f_i \in \mathcal{S}_{l_i}(\delta_i, b, 1)$ $(1 \le i \le n)$, *then the integral operator* $I_{k,n,l,\mu} = F$, *defined by* (1.27), *is in the class* $\mathcal{S}_{k+1}(\rho, b, 1)$, *where* ρ *is defined as in* (2.16).

Proof. In Corollary 2.6, we consider $\lambda = 1$. □

Corollary 2.8 readily follows from Corollary 2.7.

Corollary 2.8. *Let* $\delta_i \in [0, 1)$ $(1 \le i \le n)$, *and* $b \in \mathbb{C} - \{0\}$. *Also suppose that*

$$\sum_{i=1}^{n} \mu_i (1 - \delta_i) \le 1. \tag{2.18}$$

If $f_i \in \mathcal{S}_{l_i}(\delta_i, b, 1)$ $(1 \le i \le n)$, *then the integral operator* $I_{k,n,l,\mu} = F$, *defined by* (1.27), *is in the class* $\mathcal{S}_{k+1}(0, b, 1)$.

Remark 2.9. If we set $b = 1$ in Corollary 2.8, then we have [7, Corollary 1].

Theorem 2.10. *Let* $\alpha_i \ge 0$, $\delta_i \in [-1, 1)$, $\alpha_i + \delta_i \ge 0$ $(1 \le i \le n)$ *and* $b \in \mathbb{C} - \{0\}$, $\lambda \ge 0$. *Also suppose that*

$$\sum_{i=1}^{n} \mu_i \le 1. \tag{2.19}$$

If $f_i \in \mathcal{US}_{l_i}(\alpha_i, \delta_i, b, \lambda)$ $(1 \le i \le n)$, *then the integral operator* $I_{k,n,l,\mu} = F$, *defined by* (1.27), *is in the class* $\mathcal{K}_k(\gamma, b, \lambda)$, *where* γ *is defined as in* (2.9).

Proof. The proof is similar to the proof of Theorem 2.2. □

Corollary 2.11. *Let* $\alpha_i \ge 0$, $\delta_i \in [-1, 1)$, $\alpha_i + \delta_i \ge 0$ $(1 \le i \le n)$ *and* $b \in \mathbb{C} - \{0\}$. *Also suppose that*

$$\sum_{i=1}^{n} \mu_i \le 1. \tag{2.20}$$

If $f_i \in \mathcal{US}_{l_i}(\alpha_i, \delta_i, b, 1)$ $(1 \le i \le n)$*, then the integral operator* $I_{k,n,l,\mu} = F$*, defined by* (1.27)*, is in the class* $\mathcal{S}_{k+1}(\gamma, b, 1)$*, where* γ *is defined as in* (2.9)*.*

Proof. In Theorem 2.10, we consider $\lambda = 1$. □

Remark 2.12. If we set $b = 1$ in Corollary 2.11, then we have [7, Theorem 2].

Corollary 2.13. *Let* $\delta_i \in [0,1)(1 \le i \le n)$ *and* $b \in \mathbb{C} - \{0\}$*,* $\lambda \ge 0$*. Also suppose that*

$$\sum_{i=1}^{n} \mu_i \le 1. \tag{2.21}$$

If $f_i \in \mathcal{S}_{l_i}(\delta_i, b, \lambda)$ $(1 \le i \le n)$*, then the integral operator* $I_{k,n,l,\mu} = F$*, defined by* (1.27)*, is in the class* $\mathcal{K}_k(\rho, b, \lambda)$*, where* ρ *is defined as in* (2.16)*.*

Proof. In Theorem 2.10, we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. □

Corollary 2.14. *Let* $\delta_i \in [0,1)(1 \le i \le n)$ *and* $b \in \mathbb{C} - \{0\}$*. Also suppose that*

$$\sum_{i=1}^{n} \mu_i \le 1. \tag{2.22}$$

If $f_i \in \mathcal{S}_{l_i}(\delta_i, b, 1)$ $(1 \le i \le n)$*, then the integral operator* $I_{k,n,l,\mu} = F$*, defined by* (1.27)*, is in the class* $\mathcal{S}_{k+1}(\rho, b, 1)$*, where* ρ *is defined as in* (2.16)*.*

Proof. In Corollary 2.13, we consider $\lambda = 1$. □

Remark 2.15. If we set $b = 1$ in Corollary 2.14, then we have [7, Corollary 2].

Theorem 2.16. *Let* $\alpha \ge 0$*,* $\delta \in [-1,1)$*,* $\alpha + \delta \ge 0$ *and* $b \in \mathbb{C} - \{0\}$*,* $\lambda \ge 0$*. Also suppose that*

$$\sum_{i=1}^{n} \mu_i \le 1. \tag{2.23}$$

If $f_i \in \mathcal{US}_{l_i}(\alpha, \delta, b, \lambda)$ $(1 \le i \le n)$*, then the integral operator* $I_{k,n,l,\mu} = F$*, defined by* (1.27)*, is in the class* $\mathcal{UK}_k(\alpha, \delta, b, \lambda)$*.*

Proof. Since $f_i \in \mathcal{US}_{l_i}(\alpha, \delta, b, \lambda)$ $(1 \le i \le n)$, by (1.14) we have

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right)\right| + \delta \tag{2.24}$$

for all $z \in \mathbb{U}$.

On the other hand, from (2.1), we obtain

$$
\begin{aligned}
1+\frac{\lambda}{b}\frac{z\left(D^kF(z)\right)''}{\left(D^kF(z)\right)'} &= 1+\sum_{i=1}^{n}\mu_i\frac{1}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\\
&= 1-\sum_{i=1}^{n}\mu_i+\sum_{i=1}^{n}\mu_i\left[1+\frac{1}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right].
\end{aligned}
\tag{2.25}
$$

Considering (1.16) with the above equality, we find

$$
\begin{aligned}
&\operatorname{Re}\left\{1+\frac{\lambda}{b}\frac{z\left(D^kF(z)\right)''}{\left(D^kF(z)\right)'}\right\}-\alpha\left|\frac{\lambda}{b}\frac{z\left(D^kF(z)\right)''}{\left(D^kF(z)\right)'}\right|-\delta\\
&= 1-\sum_{i=1}^{n}\mu_i+\sum_{i=1}^{n}\mu_i\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right\}-\alpha\left|\sum_{i=1}^{n}\mu_i\frac{1}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right|-\delta\\
&\geq 1-\sum_{i=1}^{n}\mu_i+\sum_{i=1}^{n}\mu_i\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right\}-\alpha\sum_{i=1}^{n}\mu_i\left|\frac{1}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right|-\delta\\
&> 1-\sum_{i=1}^{n}\mu_i+\sum_{i=1}^{n}\mu_i\left[\alpha\left|\frac{1}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right|+\delta\right]-\alpha\sum_{i=1}^{n}\mu_i\left|\frac{1}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right|-\delta\\
&= (1-\delta)\left(1-\sum_{i=1}^{n}\mu_i\right)\geq 0
\end{aligned}
\tag{2.26}
$$

for all $z\in\mathbb{U}$. This completes proof. □

Corollary 2.17. *Let $\alpha\geq 0$, $\delta\in[-1,1)$, $\alpha+\delta\geq 0$, and $b\in\mathbb{C}-\{0\}$. Also suppose that*

$$
\sum_{i=1}^{n}\mu_i\leq 1. \tag{2.27}
$$

If $f_i\in\mathcal{US}_{l_i}(\alpha,\delta,b,1)$ $(1\leq i\leq n)$, then the integral operator $I_{k,n,l,\mu}=F$, defined by (1.27)*, is in the class $\mathcal{US}_{k+1}(\alpha,\delta,b,1)$.*

Proof. In Theorem 2.16, we consider $\lambda=1$. □

Remark 2.18. If we set $b=1$ in Corollary 2.17, then we have [7, Theorem 3].

Theorem 2.19. *Let $\alpha\geq 0$, $b\in\mathbb{C}-\{0\}$, and $\lambda\geq 0$. Also suppose that*

$$
\sum_{i=1}^{n}\mu_i\leq 1. \tag{2.28}
$$

If $f_i\in\mathcal{SH}_{l_i}(\alpha,b,\lambda)$ $(1\leq i\leq n)$, then the integral operator $I_{k,n,l,\mu}=F$, defined by (1.27)*, is in the class $\mathcal{KH}_k(\alpha,b,\lambda)$.*

Proof. Since $f_i \in \mathcal{SH}_{l_i}(\alpha, b, \lambda)$ $(1 \le i \le n)$, by (1.22) we have

$$\operatorname{Re}\left\{\sqrt{2} + \frac{\sqrt{2}}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right)\right\} + 2\alpha(\sqrt{2}-1) - \left|1 + \frac{1}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right) - 2\alpha(\sqrt{2}-1)\right| > 0 \tag{2.29}$$

for all $z \in \mathbb{U}$. Considering this inequality and (2.1), we obtain

$$\begin{aligned}
&\operatorname{Re}\left\{\sqrt{2} + \frac{\sqrt{2}}{b}\frac{\lambda z (D^k F(z))''}{(D^k F(z))'}\right\} + 2\alpha(\sqrt{2}-1) - \left|1 + \frac{1}{b}\frac{\lambda z (D^k F(z))''}{(D^k F(z))'} - 2\alpha(\sqrt{2}-1)\right| \\
&= \operatorname{Re}\left\{\sqrt{2} + \frac{\sqrt{2}}{b}\sum_{i=1}^{n}\mu_i\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right)\right\} + 2\alpha(\sqrt{2}-1) \\
&\quad - \left|1 + \frac{1}{b}\sum_{i=1}^{n}\mu_i\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right) - 2\alpha(\sqrt{2}-1)\right| \\
&= \sqrt{2} + \sum_{i=1}^{n}\mu_i \operatorname{Re}\left\{\frac{\sqrt{2}}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right)\right\} + 2\alpha(\sqrt{2}-1) \\
&\quad - \left|1 + \sum_{i=1}^{n}\mu_i\frac{1}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right) - 2\alpha(\sqrt{2}-1)\right| \\
&= \sqrt{2} + \sum_{i=1}^{n}\mu_i \operatorname{Re}\left\{\sqrt{2} + \frac{\sqrt{2}}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right)\right\} - \sqrt{2}\sum_{i=1}^{n}\mu_i + 2\alpha(\sqrt{2}-1) \\
&\quad - \left|1+\sum_{i=1}^{n}\mu_i\left[1+\frac{1}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} -1\right)-2\alpha(\sqrt{2}-1)\right]-\sum_{i=1}^{n}\mu_i+2\alpha(\sqrt{2}-1)\sum_{i=1}^{n}\mu_i-2\alpha(\sqrt{2}-1)\right| \\
&= \sqrt{2}\left(1 - \sum_{i=1}^{n}\mu_i\right) + 2\alpha(\sqrt{2}-1) + \sum_{i=1}^{n}\mu_i \operatorname{Re}\left\{\sqrt{2} + \frac{\sqrt{2}}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right)\right\} \\
&\quad - \left|\left[1 - 2\alpha(\sqrt{2}-1)\right]\left(1 - \sum_{i=1}^{n}\mu_i\right) + \sum_{i=1}^{n}\mu_i\left[1 + \frac{1}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right) - 2\alpha(\sqrt{2}-1)\right]\right| \\
&\ge \sqrt{2}\left(1 - \sum_{i=1}^{n}\mu_i\right) + 2\alpha(\sqrt{2}-1) + \sum_{i=1}^{n}\mu_i \operatorname{Re}\left\{\sqrt{2} + \frac{\sqrt{2}}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right)\right\} \\
&\quad - \left|1 - 2\alpha(\sqrt{2}-1)\right|\left(1 - \sum_{i=1}^{n}\mu_i\right) - \sum_{i=1}^{n}\mu_i\left|1 + \frac{1}{b}\left(\frac{D^{l_i+1} f_i(z)}{D^{l_i} f_i(z)} - 1\right) - 2\alpha(\sqrt{2}-1)\right| \\
&\quad + 2\alpha(\sqrt{2}-1)\sum_{i=1}^{n}\mu_i - 2\alpha(\sqrt{2}-1)\sum_{i=1}^{n}\mu_i
\end{aligned}$$

$$
\begin{aligned}
&= \left[\sqrt{2}+2\alpha(\sqrt{2}-1)-\left|1-2\alpha(\sqrt{2}-1)\right|\right]\left(1-\sum_{i=1}^{n}\mu_i\right)\\
&\quad+\sum_{i=1}^{n}\mu_i\left[\operatorname{Re}\left\{\sqrt{2}+\frac{\sqrt{2}}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right\}+2\alpha(\sqrt{2}-1)-\left|1+\frac{1}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)-2\alpha(\sqrt{2}-1)\right|\right]\\
&> \left[\sqrt{2}+2\alpha(\sqrt{2}-1)-\left|1-2\alpha(\sqrt{2}-1)\right|\right]\left(1-\sum_{i=1}^{n}\mu_i\right)\\
&> \left(1-\sum_{i=1}^{n}\mu_i\right)\min\left\{(\sqrt{2}-1)(1+4\alpha),\sqrt{2}+1\right\}\geq 0
\end{aligned}
\tag{2.30}
$$

for all $z \in \mathbb{U}$. Hence by (1.23), we have $F \in \mathcal{KH}_k(\alpha, b, \lambda)$. □

Corollary 2.20. *Let $\alpha \geq 0$ and $b \in \mathbb{C} - \{0\}$. Also suppose that*

$$\sum_{i=1}^{n}\mu_i \leq 1. \tag{2.31}$$

If $f_i \in \mathcal{SH}_{l_i}(\alpha, b, 1)$ $(1 \leq i \leq n)$, then the integral operator $I_{k,n,l,\mu} = F$, defined by (1.27), is in the class $\mathcal{SH}_{k+1}(\alpha, b, 1)$.

Proof. In Theorem 2.19, we consider $\lambda = 1$. □

Remark 2.21. If we set $b = 1$ in Corollary 2.20, then we have [7, Theorem 4].

Theorem 2.22. *Let $\alpha \geq 0$, $b \in \mathbb{C} - \{0\}$ and $\lambda \geq 0$. Also suppose that*

$$\left(1+\sqrt{2}\alpha(\sqrt{2}-1)\right)\sum_{i=1}^{n}\mu_i < 1. \tag{2.32}$$

If $f_i \in \mathcal{SH}_{l_i}(\alpha, b, \lambda)$ $(1 \leq i \leq n)$, then the integral operator $I_{k,n,l,\mu} = F$, defined by (1.27), is in the class $\mathcal{K}_k(0, b, \lambda)$.

Proof. Since $f_i \in \mathcal{SH}_{l_i}(\alpha, b, \lambda)$ $(1 \leq i \leq n)$, by (1.22) we have

$$\operatorname{Re}\left\{\sqrt{2}+\frac{\sqrt{2}}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right\}+2\alpha(\sqrt{2}-1) > \left|1+\frac{1}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)-2\alpha(\sqrt{2}-1)\right| \tag{2.33}$$

for all $z \in \mathbb{U}$. Considering this inequality and (2.1), we obtain

$$\begin{aligned}
&\sqrt{2}\operatorname{Re}\left\{1+\frac{\lambda}{b}\frac{z(D^k f(z))''}{(D^k f(z))'}\right\} \\
&= \operatorname{Re}\left\{\sqrt{2}+\frac{\sqrt{2}}{b}\sum_{i=1}^{n}\mu_i\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right\} \\
&= \sqrt{2}-\sqrt{2}\sum_{i=1}^{n}\mu_i+\sum_{i=1}^{n}\mu_i\operatorname{Re}\left\{\sqrt{2}+\frac{\sqrt{2}}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right\} \\
&= \sqrt{2}-\sqrt{2}\sum_{i=1}^{n}\mu_i-2\alpha(\sqrt{2}-1)\sum_{i=1}^{n}\mu_i+\sum_{i=1}^{n}\mu_i\left[\operatorname{Re}\left\{\sqrt{2}+\frac{\sqrt{2}}{b}\left(\frac{D^{l_i+1}f_i(z)}{D^{l_i}f_i(z)}-1\right)\right\}+2\alpha(\sqrt{2}-1)\right] \\
&> \sqrt{2}\left(1-(1+\sqrt{2}\alpha(\sqrt{2}-1))\sum_{i=1}^{n}\mu_i\right)>0
\end{aligned} \tag{2.34}$$

for all $z \in \mathbb{U}$. Hence, by (1.8), we have $F \in \mathcal{K}_k(0,b,\lambda)$. □

Corollary 2.23. *Let $\alpha \geq 0$ and $b \in \mathbb{C}-\{0\}$. Also suppose that*

$$(1+\sqrt{2}\alpha(\sqrt{2}-1))\sum_{i=1}^{n}\mu_i < 1. \tag{2.35}$$

If $f_i \in \mathcal{SH}_{l_i}(\alpha,b,1)$ $(1 \leq i \leq n)$, then the integral operator $I_{k,n,l,\mu} = F$, defined by (1.27), is in the class $\mathcal{S}_{k+1}(0,b,1)$.

Proof. In Theorem 2.22, we consider $\lambda = 1$. □

Remark 2.24. If we set $b = 1$ in Corollary 2.23, then we have [7, Theorem 5].

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