Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 859105, 14 pages
doi:10.1155/2008/859105

Research Article

A Class of Integral Operators Preserving Subordination and Superordination

Nak Eun Cho and In Hwa Kim

Department of Applied Mathematics, Pukyong National University, Pusan 608-737, South Korea

Correspondence should be addressed to Nak Eun Cho, necho@pknu.ac.kr

Received 30 October 2007; Accepted 8 January 2008

Recommended by Donal O'Regan

We give some subordination- and superordination-preserving properties of certain nonlinear integral operators defined on the space of normalized analytic functions in the open unit disk. The sandwich-type theorems for these integral operators are also obtained. Moreover, we consider an application of the subordination and superordination theorems to the Gauss hypergeometric function.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \ldots\}$, let

$$\mathcal{H}[a,n] := \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}. \tag{1.1}$$

We denote by $\mathcal{A}$ the subclass of $\mathcal{H}[a,1]$ with the usual normalization $f(0) = f'(0) - 1 = 0$, and denote by A the subclass of $\mathcal{A}$ satisfying the condition $h(z)h'(z) \neq 0$ for $z \in \mathbb{U} \setminus \{0\}$.

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$ and such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. In such a case, we write

$$f \prec F \quad \text{or} \quad f(z) \prec F(z). \tag{1.2}$$

If the function F is univalent in $\mathbb{U}$, then

$$f \prec F \Longleftrightarrow f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}) \quad (\text{cf. } [1, 2]). \tag{1.3}$$

Definition 1.1 (Miller and Mocanu [1]). Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the (first-order) differential subordination

$$\phi(p(z), zp'(z)) \prec h(z), \tag{1.4}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1.4). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.4) is said to be the best dominant.

Recently, Miller and Mocanu [3] introduced the following differential superordinations, as the dual concept of differential subordinations.

Definition 1.2 (Miller and Mocanu [3]). Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the (first-order) differential superordination

$$h(z) \prec \varphi(p(z), zp'(z)), \tag{1.5}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.5). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.5) is said to be the best subordinant.

Definition 1.3 (Miller and Mocanu [3]). Denote by Q the set of functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty\right\}, \tag{1.6}$$

and are such that $f'(\zeta) \neq 0$ for $z \in \zeta \in \partial\mathbb{U} \setminus E(f)$.

Now we introduce the following integral operators $I_{h;\beta}$ defined by

$$I_{h;\beta}(f)(z) := \left[\beta \int_0^z f^{\beta}(t) h^{-1}(t) h'(t)\, dt\right]^{1/\beta}$$
$$\left(\operatorname{Re}\{\beta\} > 0,\ f \in \mathcal{A},\ h \in A,\ z \in \mathbb{U}\right). \tag{1.7}$$

The integral operators defined by (1.7) have been extensively studied by Bulboacă [4]. Also Miller et al. [5] investigated some subordination-preserving properties involving certain integral operators for analytic functions in $\mathbb{U}$ (see, also [6, 7]). Moreover, Bulboacă [8] studied a class of superordination-preserving integral operators. In the present paper, we obtain the subordination- and superordination-preserving properties of the integral operator $I_{h;\beta}$ defined by (1.7) with the sandwich-type theorems. We also consider applications of our main results to the Gauss hypergeometric function.

2. A set of lemmas

The following lemmas will be required in our present investigation.

Lemma 2.1 (Miller and Mocanu [9]). *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the following condition:*

$$\operatorname{Re}\{H(is,t)\} \leq 0 \quad \left(s \in \mathbb{R};\ t \leq \frac{-n(1+s^2)}{2};\ n \in \mathbb{N}\right). \tag{2.1}$$

If the function $p(z) = 1 + p_n z^n + \cdots$ is analytic in $\mathbb{U}$ and

$$\operatorname{Re}\{H(p(z),\ zp'(z))\} > 0 \quad (z \in \mathbb{U}), \tag{2.2}$$

then

$$\operatorname{Re}\{p(z)\} > 0 \quad (z \in \mathbb{U}). \tag{2.3}$$

Lemma 2.2 (Miller and Mocanu [10]). *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and let $h \in \mathscr{H}(\mathbb{U})$ with $h(0) = c$. If*

$$\operatorname{Re}\{\beta h(z) + \gamma\} > 0 \quad (z \in \mathbb{U}), \tag{2.4}$$

then the solution of the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathbb{U};\ q(0) = c) \tag{2.5}$$

is analytic in $\mathbb{U}$ and satisfies the inequality given by

$$\operatorname{Re}\{\beta q(z) + \gamma\} > 0 \quad (z \in \mathbb{U}). \tag{2.6}$$

Lemma 2.3 (Miller and Mocanu [1]). *Let $p \in Q$ with $p(0) = a$ and let $q(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{U}$ with $q(z) \not\equiv a$ and $n \in \mathbb{N}$. If q is not subordinate to p, then there exist points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus E(f)$ for which*

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}), \qquad q(z_0) = p(\zeta_0), \qquad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n). \tag{2.7}$$

Let $c \in \mathbb{C}$ with $\operatorname{Re}\{c\} > 0$ and let

$$N := N(c) = \frac{|c|\sqrt{1 + 2\operatorname{Re}\{c\}} + \operatorname{Im}\{c\}}{\operatorname{Re}\{c\}}. \tag{2.8}$$

If R is the univalent function in $\mathbb{U}$ defined by $R(z) := 2Nz/(1-z^2)$, then the open-door function R_c is defined by

$$R_c(z) := R\left(\frac{z+b}{1+\bar{b}z}\right) \quad (z \in \mathbb{U}), \tag{2.9}$$

where $b = R^{-1}(c)$ (cf. [1]).

Remark 2.4. The function R_c defined by (2.9) is univalent in $\mathbb{U}$, $R_c(0) = c$, and $R_c(\mathbb{U}) = R(\mathbb{U})$ is the complex plane with slits along the half-lines $\operatorname{Re}\{w\} = 0$ and $|\operatorname{Im}\{w\}| \geq N$, that is,

$$R_c(\mathbb{U}) = R(\mathbb{U}) = \mathbb{C} \setminus \{w \in \mathbb{C} : \operatorname{Re}\{w\} = 0,\ |\operatorname{Im}\{w\}| \geq N\}. \tag{2.10}$$

Lemma 2.5 (Bulboacă [4], Miller, and Mocanu [1]). *Let $\beta \in \mathbb{C}$ with $\operatorname{Re}\{\beta\} > 0$ and let $h \in A$. If $f \in \mathcal{A}_{h;\beta}$, where*

$$\mathcal{A}_{h;\beta} := \left\{ f \in \mathcal{A} : \beta \frac{zf'(z)}{f(z)} - \frac{zh'(z)}{h(z)} + 1 + \frac{zh''(z)}{h'(z)} \prec R_\beta(z),\ \beta f''(0) + h''(0) \neq 0 \right\} \tag{2.11}$$

and R_β is the open door function defined by (2.9) *with $c = \beta$, then*

$$I_{h;\beta}(f) \in \mathcal{A}, \quad \frac{I_{h;\beta}(f)(z)}{z} \neq 0 \quad (z \in \mathbb{U}), \qquad \operatorname{Re}\left\{ \beta \frac{z(I_{h;\beta}(f)(z))'}{I_{h;\beta}(f)(z)} \right\} > 0, \quad (z \in \mathbb{U}), \tag{2.12}$$

where $I_{h;\beta}$ is the integral operator defined by (1.7).

Remark 2.6. The integral operator $I_{h;1}$ defined by (1.7) with $\beta = 1$ is well defined on $\mathcal{H}[0,1]$ (see [4]).

A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is the subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$, and $L(z,s) \prec L(z,t)$ when $0 \leq s < t$.

Lemma 2.7 (Miller and Mocanu [3]). *Let $q \in \mathcal{H}[a,1]$ and $\varphi : \mathbb{C}^2 \to \mathbb{C}$. Also set $\varphi(q(z), zq'(z)) \equiv h(z)$ $(z \in \mathbb{U})$. If*

$$L(z,t) := \varphi\big(q(z), tzq'(z)\big) \quad (z \in \mathbb{U};\ 0 \leq t < \infty) \tag{2.13}$$

is a subordination chain and $p \in \mathcal{H}[a,1] \cap Q$, then

$$h(z) \prec \varphi\big(p(z), zp'(z)\big) \tag{2.14}$$

implies that

$$q(z) \prec p(z). \tag{2.15}$$

Furthermore, if

$$\varphi\big(q(z), zp'(z)\big) = h(z) \tag{2.16}$$

has a univalent solution $q \in Q$, then q is the best subordinant.

Lemma 2.8 (see [2]). *The function $L(z,t) = a_1(t)z + \cdots$ with $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if*

$$\operatorname{Re}\left\{ \frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t} \right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty). \tag{2.17}$$

3. Main results

Our first subordination theorem involving the integral operator $I_{h;\beta}$ defined by (1.7) is contained in Theorem 3.1 below.

Theorem 3.1. *Let* $f, g \in \mathcal{A}_{h;\beta}$ *with*

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U};\ \beta \neq 1), \qquad \frac{g(z)}{z} \neq 0 \quad (z \in \mathbb{U};\ \beta \neq 1), \tag{3.1}$$

and let $h \in A$. *Suppose also that*

$$\begin{gathered} \operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \\ \left(z \in \mathbb{U};\ \phi(z) := \left(\frac{\phi_1(z)}{z}\right)^{\beta};\ \phi_1(z) := g(z)\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta}\right), \end{gathered} \tag{3.2}$$

where

$$\delta = \frac{1 + |\beta|^2 - \left|1 - \beta^2\right|}{4\operatorname{Re}\{\beta\}} \quad \left(\operatorname{Re}\{\beta\} > 0\right). \tag{3.3}$$

Then the subordination condition

$$\left(\frac{f(z)}{z}\right)^{\beta} \frac{zh'(z)}{h(z)} \prec \left(\frac{g(z)}{z}\right)^{\beta} \frac{zh'(z)}{h(z)} \tag{3.4}$$

implies that

$$\left(\frac{I_{h;\beta}(f)(z)}{z}\right)^{\beta} \prec \left(\frac{I_{h;\beta}(g)(z)}{z}\right)^{\beta}, \tag{3.5}$$

where $I_{h;\beta}$ *is the integral operator defined by* (1.7). *Moreover, the function* $(I_{h;\beta}(g)(z)/z)^{\beta}$ *is the best dominant.*

Proof. Let us define the functions F and G by

$$F(z) := \left(\frac{I_{h;\beta}(f)(z)}{z}\right)^{\beta}, \qquad G(z) := \left(\frac{I_{h;\beta}(g)(z)}{z}\right)^{\beta}, \tag{3.6}$$

respectively. We note that F and G are well defined by Lemma 2.5.

We first show that if the function q is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{3.7}$$

then

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}). \tag{3.8}$$

From the definition of (1.7), we obtain

$$g(z) = I_{h;\beta}(g)(z)\left[\frac{h(z)}{zh'(z)}\frac{z\left(I_{h;\beta}(g)(z)\right)'}{I_{h;\beta}(g)(z)}\right]^{1/\beta}. \tag{3.9}$$

We also have

$$\beta\frac{z\left(I_{h;\beta}(g)(z)\right)'}{I_{h;\beta}(g)(z)} = \beta + \frac{zG'(z)}{G(z)}. \tag{3.10}$$

By a simple calculation in conjuction with (3.9) and (3.10), we obtain the following relationship:

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z)+\beta} \equiv Q(z). \tag{3.11}$$

We also note from (3.2) that

$$\operatorname{Re}\{Q(z)+\beta\} > 0 \quad (z \in \mathbb{U}) \tag{3.12}$$

and, by using Lemma 2.2, we conclude that the differential equation (3.11) has a solution $q \in \mathscr{H}(\mathbb{U})$ with $q(0) = Q(0) = 1$. Let us put

$$H(u,v) = u + \frac{v}{u+\beta} + \delta, \tag{3.13}$$

where δ is given by (3.3). From (3.2), (3.11), and (3.13), we obtain

$$\operatorname{Re}\{H(q(z), zq'(z))\} > 0 \quad (z \in \mathbb{U}). \tag{3.14}$$

Now we proceed to show that

$$\operatorname{Re}\{H(is,t)\} \leq 0 \quad \left(s \in \mathbb{R};\ t \leq \frac{-n\left(1+s^2\right)}{2}\right). \tag{3.15}$$

Indeed, from (3.13), we have

$$\operatorname{Re}\{H(is,t)\} = \operatorname{Re}\left\{is + \frac{t}{is+\beta} + \delta\right\} = \frac{t\operatorname{Re}\{\beta\}}{|\beta+is|^2} + \delta \leq -\frac{E_\delta(s)}{2|\beta+is|^2}, \tag{3.16}$$

where

$$E_\delta(s) := \left(\operatorname{Re}\{\beta\} - 2\delta\right)s^2 - 4\delta\left(\operatorname{Im}\{\beta\}\right)s - 2\delta|\beta|^2 + \operatorname{Re}\{\beta\}. \tag{3.17}$$

For δ given by (3.3), the coefficient of s^2 in the quadratic expression $E_\delta(s)$ given by (3.17) is positive or equal to zero. Moreover, the quadratic expression $E_\delta(s)$ by s in (3.17) is a perfect square for the assumed value of δ given by (2.11). Hence, from (3.16), we obtain the inequality given by (3.15). Thus, by using Lemma 2.1, we conclude that

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}), \tag{3.18}$$

that is, G defined by (3.6) is convex(univalent) in $\mathbb{U}$.

Next, we prove that the subordination condition (3.4) implies that

$$F(z) \prec G(z) \tag{3.19}$$

for the functions F and G defined by (3.6). Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and that $G'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}$. Now we consider the function $L(z,t)$ given by

$$L(z,t) := G(z) + \frac{1+t}{\beta} zG'(z) \quad (z \in \mathbb{U};\ 0 \leq t < \infty). \tag{3.20}$$

Since G is convex and $\operatorname{Re}\{\beta\} > 0$, we obtain

$$\begin{gathered}
\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(\frac{\beta+1+t}{\beta}\right) \neq 0 \quad (0 \leq t < \infty;\ \operatorname{Re}\{\beta\} > 0), \\
\operatorname{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \operatorname{Re}\left\{\beta + (1+t)\left(1 + \frac{zG''(z)}{G'(z)}\right)\right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty).
\end{gathered} \tag{3.21}$$

Therefore, by virtue of Lemma 2.8, $L(z,t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}) \quad (\zeta \in \partial\mathbb{U};\ 0 \leq t < \infty). \tag{3.22}$$

Now suppose that F is not subordinate to G, then by Lemma 2.3, there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\zeta_0) \qquad z_0F'(z_0) = (1+t)\zeta_0G'(\zeta_0) \quad (0 \leq t < \infty). \tag{3.23}$$

Hence, we have

$$\begin{aligned}
L(\zeta_0,t) &= G(\zeta_0) + \frac{1+t}{\beta}\zeta_0G'(\zeta_0) \\
&= F(z_0) + \frac{1}{\beta}z_0F'(z_0) \\
&= \left(\frac{f(z_0)}{z_0}\right)^{\beta}\frac{z_0h'(z_0)}{h(z_0)} \in \phi(\mathbb{U}),
\end{aligned} \tag{3.24}$$

by virtue of the subordination condition (3.4). This contradicts the above observation that

$$L(\zeta_0,t) \notin \phi(\mathbb{U}) \quad (\zeta_0 \in \partial\mathbb{U};\ 0 \leq t < \infty). \tag{3.25}$$

Therefore, the subordination condition (3.4) must imply the subordination given by (3.19). Considering $F(z) = G(z)$, we see that the function G is the best dominant. This evidently completes the proof of Theorem 3.1. □

Remark 3.2. We note that δ given by (3.3) in Theorem 3.1 satisfies the inequality $0 < \delta \le 1/2$.

We next provide a solution to a dual problem of Theorem 3.1, in the sense that the subordinations are replaced by superordinations.

Theorem 3.3. *Let* $f, g \in \mathcal{A}_{h;\beta}$ *with*

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U};\ \beta \neq 1), \qquad \frac{g(z)}{z} \neq 0 \quad (z \in \mathbb{U};\ \beta \neq 1), \tag{3.26}$$

and $h \in A$. *Suppose also that*

$$\begin{gathered} \operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \\ \left(z \in \mathbb{U};\ \phi(z) := \left(\frac{\varphi(z)}{z}\right)^{\beta},\ \varphi(z) := g(z)\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta}\right), \end{gathered} \tag{3.27}$$

where δ *is given by* (3.3), *the function* $(f(z)/z)^{\beta}(zh'(z)/h(z))$ *is univalent in* $\mathbb{U}$ *and* $(I_{h;\beta}(f)(z)/z)^{\beta} \in Q$, *where* $I_{h;\beta}$ *is the integral operator defined by* (1.7). *Then the superordination condition*

$$\left(\frac{g(z)}{z}\right)^{\beta}\frac{zh'(z)}{h(z)} \prec \left(\frac{f(z)}{z}\right)^{\beta}\frac{zh'(z)}{h(z)} \tag{3.28}$$

implies that

$$\left(\frac{I_{h;\beta}(g)(z)}{z}\right)^{\beta} \prec \left(\frac{I_{h;\beta}(f)(z)}{z}\right)^{\beta}. \tag{3.29}$$

Moreover, the function $(I_{h;\beta}(g)(z)/z)^{\beta}$ *is the best subordinant.*

Proof. The first part of the proof is similar to that of Theorem 3.1 and so we will use the same notation as in the proof of Theorem 3.1.

Let us define the functions F and G, respectively, by (3.6). We first note that from (3.9) and (3.10), we obtain

$$\begin{aligned} \phi(z) &= G(z) + \frac{1}{\beta}zG'(z) \\ &=: \varphi\big(G(z), zG'(z)\big). \end{aligned} \tag{3.30}$$

Then by using the same method as in the proof of Theorem 3.1, we can prove that

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}), \tag{3.31}$$

where the function q is defined by (3.7), that is, G defined by (3.6) is convex(univalent) in $\mathbb{U}$. Now we consider the function $L(z,t)$ defined by

$$L(z,t) := G(z) + \frac{t}{\beta}zG'(z) \quad (z \in \mathbb{U};\ 0 \le t < \infty). \tag{3.32}$$

Then we see that $L(z,t)$ is a subordination chain as in the proof of Theorem 3.1. Therefore, according to Lemma 2.7, we conclude that the superordination condition (3.28) must imply the superordination given by (3.29). Furthermore, since the differential equation (3.30) has the univalent solution G, it is the best subordinant of the given differential superordination. Therefore, we complete the proof of Theorem 3.3. $\square$

If we combine Theorems 3.1 and 3.3, then we obtain the following sandwich-type theorem.

Theorem 3.4. *Let $f, g_k \in \mathcal{A}_{h;\beta}$ $(k = 1,2)$ with*

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U};\ \beta \neq 1), \qquad \frac{g_k(z)}{z} \neq 0 \quad (z \in \mathbb{U};\ \beta \neq 1), \tag{3.33}$$

and $h \in A$. Suppose also that

$$\operatorname{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta$$
$$\left(z \in \mathbb{U};\ \phi_k(z) := \left(\frac{\varphi_k(z)}{z}\right)^{\beta},\ \varphi_k(z) := g_k(z)\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta},\ k = 1,2\right), \tag{3.34}$$

where δ is given by (3.3), the function $(f(z)/z)^{\beta}(zh'(z)/h(z))$ is univalent in $\mathbb{U}$ and $(I_{h;\beta}(f)(z)/z)^{\beta} \in Q$, where $I_{h;\beta}$ is the integral operator defined by (1.7). Then the subordination relation

$$\left(\frac{g_1(z)}{z}\right)^{\beta}\frac{zh'(z)}{h(z)} \prec \left(\frac{f(z)}{z}\right)^{\beta}\frac{zh'(z)}{h(z)} \prec \left(\frac{g_2(z)}{z}\right)^{\beta}\frac{zh'(z)}{h(z)} \tag{3.35}$$

implies that

$$\left(\frac{I_{h;\beta}(g_1)(z)}{z}\right)^{\beta} \prec \left(\frac{I_{h;\beta}(f)(z)}{z}\right)^{\beta} \prec \left(\frac{I_{h;\beta}(g_2)(z)}{z}\right)^{\beta}. \tag{3.36}$$

Moreover, the functions $(I_{h;\beta}(g_1)(z)/z)^{\beta}$ and $(I_{h;\beta}(g_2)(z)/z)^{\beta}$ are the best subordinant and the best dominant, respectively.

The assumption of Theorem 3.4, that the functions $(f(z)/z)^{\beta}zh'(z)/h(z)$ and $(I_{h;\beta}(f)(z)/z)^{\beta}$ need to be univalent in $\mathbb{U}$, may be replaced by another condition in the following result.

Corollary 3.5. *Let $f, g_k \in \mathcal{A}_{h;\beta}$ $(k = 1,2)$ with*

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U};\ \beta \neq 1), \qquad \frac{g_k(z)}{z} \neq 0 \quad (z \in \mathbb{U};\ \beta \neq 1), \tag{3.37}$$

and $h \in A$. Suppose also that the condition (3.34) is satisfied and that

$$\operatorname{Re}\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\delta$$
$$\left(z \in \mathbb{U};\ \psi(z) := \left(\frac{\varphi(z)}{z}\right)^{\beta};\ \varphi(z) := f(z)\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta};\ \varphi \in Q\right), \tag{3.38}$$

where δ is given by (3.3). Then the subordination relation

$$\left(\frac{g_1(z)}{z}\right)^{\beta}\frac{zh'(z)}{h(z)} \prec \left(\frac{f(z)}{z}\right)^{\beta}\frac{zh'(z)}{h(z)} \prec \left(\frac{g_2(z)}{z}\right)^{\beta}\frac{zh'(z)}{h(z)} \tag{3.39}$$

implies that

$$\left(\frac{I_{h;\beta}(g_1)(z)}{z}\right)^{\beta} \prec \left(\frac{I_{h;\beta}(f)(z)}{z}\right)^{\beta} \prec \left(\frac{I_{h;\beta}(g_2)(z)}{z}\right)^{\beta}, \tag{3.40}$$

where $I_{h;\beta}$ is the integral operator defined by (1.7)*. Moreover, the functions $(I_{h;\beta}(g_1)(z)/z)^{\beta}$ and $(I_{h;\beta}(g_2)(z)/z)^{\beta}$ are the best subordinant and the best dominant, respectively.*

Proof. In order to prove Corollary 3.5, we have to show that the condition (3.38) implies the univalence of $\varphi(z)$ and

$$F(z) := \left(\frac{I_{h;\beta}(f)(z)}{z}\right)^{\beta} \quad (z \in \mathbb{U}). \tag{3.41}$$

Since $0 < \delta \leq 1/2$ from Remark 3.2, the condition (3.38) means that φ is a close-to-convex function in $\mathbb{U}$ (see [11]) and hence φ is univalent in $\mathbb{U}$. Furthermore, by using the same techniques as in the proof of Theorem 3.4, we can prove the convexity(univalence) of F and so the details may be omitted. Therefore, by applying Theorem 3.4, we obtain Corollary 3.5. □

By setting $\beta = 1$ in Theorem 3.4, we have the following consequence of Theorem 3.4.

Corollary 3.6. *Let $f, g_k \in \mathcal{A}_{h;1}$ $(k = 1, 2)$ and $h \in A$. Suppose that*

$$\operatorname{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\frac{1}{2}$$
$$\left(z \in \mathbb{U};\ \phi_k(z) := \frac{\varphi_k(z)}{z};\ \varphi_k(z) := g_k(z)\frac{zh'(z)}{h(z)};\ k = 1, 2\right), \tag{3.42}$$

the function $(f(z)/z)(zh'(z)/h(z))$ is univalent in $\mathbb{U}$, and $I_{h;1}f(z)/z \in Q$, where $I_{h;1}$ is the integral operator defined by (1.7) *with $\beta = 1$. Then the subordination relation*

$$g_1(z)\frac{h'(z)}{h(z)} \prec f(z)\frac{h'(z)}{h(z)} \prec g_2(z)\frac{h'(z)}{h(z)} \tag{3.43}$$

implies that

$$\frac{I_{h;1}(g_1)(z)}{z} \prec \frac{I_{h;1}(f)(z)}{z} \prec \frac{I_{h;1}(g_2)(z)}{z}. \tag{3.44}$$

Moreover, the functions $I_{h;1}(g_1)(z)/z$ and $I_{h;1}(g_2)(z)/z$ are the best subordinant and the best dominant, respectively.

If we take $\beta = 1 + i$ in Theorem 3.4, then we are easily led to the following result.

Corollary 3.7. *Let $f, g_k \in \mathcal{A}_{h;1+i}$ $(k = 1, 2)$ with*

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U}), \qquad \frac{g_k(z)}{z} \neq 0 \quad (z \in \mathbb{U}), \tag{3.45}$$

and $h \in A$. Suppose that

$$\operatorname{Re}\left\{1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\frac{3-\sqrt{5}}{4}$$
$$\left(z \in \mathbb{U};\ \phi_k(z) := \left(\frac{\varphi_k(z)}{z}\right)^{\beta};\ \varphi_k(z) := g_k(z)\left[\frac{zh'(z)}{h(z)}\right]^{1/(1+i)};\ k=1,2\right), \tag{3.46}$$

the function $(f(z)/z)^{1+i}(zh'(z)/h(z))$ is univalent in $\mathbb{U}$, and $(I_{h;1+i}f(z)/z)^{1+i} \in \mathcal{Q}$, where $I_{h;1+i}$ is the integral operator defined by (1.7) *with $\beta = 1+i$. Then the subordination relation*

$$\left(\frac{g_1(z)}{z}\right)^{1+i}\frac{zh'(z)}{h(z)} \prec \left(\frac{f(z)}{z}\right)^{1+i}\frac{zh'(z)}{h(z)} \prec \left(\frac{g_2(z)}{z}\right)^{1+i}\frac{zh'(z)}{h(z)} \tag{3.47}$$

implies that

$$\left(\frac{I_{h;1+i}(g_1)(z)}{z}\right)^{1+i} \prec \left(\frac{I_{h;1+i}(f)(z)}{z}\right)^{1+i} \prec \left(\frac{I_{h;1+i}(g_1)(z)}{z}\right)^{1+i}. \tag{3.48}$$

Moreover, the functions $(I_{h;1+i}(g_1)(z)/z)^{1+i}$ and $(I_{h;1+i}(g_2)(z)/z)^{1+i}$ are the best subordinant and the best dominant, respectively.

4. Applications to the gauss hypergeometric function

We begin by recalling that the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by (see, for details, [12, Chapter 14])

$$ {}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$
$$\left(z \in \mathbb{U};\ b \in \mathbb{C};\ c \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0,-1,-2,\dots\}\right), \tag{4.1}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol (or the shifted factorial) defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+\nu-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases} \tag{4.2}$$

For this useful special function, the following Eulerian integral representation is fairly well known [12, page 293]:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}dt$$
$$\left(\operatorname{Re}\{c\} > \operatorname{Re}\{a\} > 0;\ |\arg(1-z)| \le \pi-\epsilon;\ 0<\epsilon<\pi\right). \tag{4.3}$$

In view of (4.3), we set

$$h(z)=\frac{z}{1-z},\qquad g(z)=\frac{z}{(1-z)^{\kappa}}\quad(\kappa>0),\tag{4.4}$$

so that the definition (1.7) yields

$$\begin{aligned}I_{h;\beta}(g)(z)&=\left(\beta\int_0^z t^{\beta-1}(1-t)^{-(\beta\kappa+1)}dt\right)^{1/\beta}\\&=\left(\beta z^{\beta}\int_0^1 u^{\beta-1}(1-zu)^{-(\beta\kappa+1)}du\right)^{1/\beta}\\&=z\left[{}_2F_1(\beta,\beta\kappa+1;\beta+1;z)\right]^{1/\beta}\quad(\beta>1/2).\end{aligned}\tag{4.5}$$

Moreover, we note from the definition (4.4) that

$$\frac{g(z)}{z}=\frac{1}{(1-z)^{\kappa}}\neq 0\quad(\kappa>0;\ z\in\mathbb{U}).\tag{4.6}$$

Thus, by applying Theorem 3.1, we obtain the following results involving the Gauss hypergeometric function.

Theorem 4.1. *Let* $f\in\mathcal{A}_{z/(1-z);\beta}$ *with* $f(z)/z\neq 0$ $(z\in\mathbb{U};\ \beta\neq 1)$ *and*

$$0<\kappa<\min\left\{\frac{2\beta-1}{\beta},\frac{2\delta}{\beta}\right\}\quad\left(\beta>\frac{1}{2}\right),\tag{4.7}$$

where δ *is given by* (3.3). *Then the subordination condition*

$$\frac{1}{1-z}\left(\frac{f(z)}{z}\right)^{\beta}\prec\frac{1}{(1-z)^{\kappa\beta+1}}\tag{4.8}$$

implies that

$$\left(\frac{I_{z/(1-z);\beta}(f)(z)}{z}\right)^{\beta}\prec{}_2F_1(\beta,\kappa\beta+1;\beta+1;z),\tag{4.9}$$

where $I_{z/(1-z);1}$ *is the integral operator defined by* (1.7) *with* $h(z)=z/(1-z)$. *Moreover, the function* ${}_2F_1(\beta,\kappa\beta+1;\beta+1;z)$ *is the best dominant.*

By setting $\beta=1$ in Theorem 4.1, we are led to the following Corollary 4.2.

Corollary 4.2. *Let* $f\in\mathcal{A}_{z/(1-z);1}$ *and* $0<\kappa\leq 1$. *Then the subordination condition*

$$\frac{1}{1-z}\frac{f(z)}{z}\prec\frac{1}{(1-z)^{\kappa+1}}\tag{4.10}$$

implies that

$$\frac{I_{z/(1-z);1}(f)(z)}{z}\prec{}_2F_1(1,\kappa+1;2;z),\tag{4.11}$$

where $I_{z/(1-z);1}$ *is the integral operator defined by* (1.7) *with* $h(z)=z/(1-z)$. *Moreover, the function* ${}_2F_1(1,\kappa+1;2;z)$ *is the best dominant.*

If we take $\kappa=\beta=1$ in Theorem 4.1, we are led to the following corollary.

Corollary 4.3. *Let* $f \in \mathcal{A}_{z/(1-z);1}$. *Then the subordination condition*

$$\frac{1}{1-z}\frac{f(z)}{z} \prec \frac{1}{(1-z)^2} \tag{4.12}$$

implies that

$$\frac{1}{z}\int_0^z \frac{f(t)}{t(1-t)}dt \prec \frac{1}{1-z}. \tag{4.13}$$

Moreover, the function $1/(1-z)$ *is the best dominant.*

We also state the following Theorem 4.4 below as a dual result of Theorem 4.1, which can be obtained by applying Theorem 3.3.

Theorem 4.4. *Under the assumption of Theorem 4.1, suppose also that the function* $1/(1-z)(f(z)/z)^\beta$ *is univalent in* $\mathbb{U}$ *and* $(I_{z/(1-z);\beta}(f)(z)/z)^\beta \in Q$, *where* $I_{z/(1-z);\beta}$ *is the integral operator defined by* (1.7). *Then the superordination condition*

$$\frac{1}{(1-z)^{\kappa\beta+1}} \prec \frac{1}{1-z}\left(\frac{f(z)}{z}\right)^\beta \tag{4.14}$$

implies that

$${}_2F_1(\beta,\kappa\beta+1;\beta+1;z) \prec \left(\frac{I_{z/(1-z);\beta}(f)(z)}{z}\right)^\beta. \tag{4.15}$$

Moreover, the function ${}_2F_1(\beta,\kappa\beta+1;\beta+1;z)$ *is the best subordinant.*

Acknowledgment

This work was supported by Pukyong National University Research Fund in 2007 (PK-2007-013).

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations, Theory and Applications*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.
[2] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, Germany, 1975.
[3] S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations," *Complex Variables and Elliptic Equations*, vol. 48, no. 10, pp. 815–826, 2003.
[4] T. Bulboacă, "A class of double subordination-preserving integral operators," *Pure Mathematics and Applications*, vol. 15, no. 2-3, pp. 87–106, 2004.
[5] S. S. Miller, P. T. Mocanu, and M. O. Reade, "Subordination-preserving integral operators," *Transactions of the American Mathematical Society*, vol. 283, no. 2, pp. 605–615, 1984.
[6] T. Bulboacă, "Integral operators that preserve the subordination," *Bulletin of the Korean Mathematical Society*, vol. 34, no. 4, pp. 627–636, 1997.
[7] S. Owa and H. M. Srivastava, "Some subordination theorems involving a certain family of integral operators," *Integral Transforms and Special Functions*, vol. 15, no. 5, pp. 445–454, 2004.

[8] T. Bulboacă, “A class of superordination-preserving integral operators,” *Indagationes Mathematicae. New Series*, vol. 13, no. 3, pp. 301–311, 2002.

[9] S. S. Miller and P. T. Mocanu, “Differential subordinations and univalent functions,” *Michigan Mathematical Journal*, vol. 28, no. 2, pp. 157–172, 1981.

[10] S. S. Miller and P. T. Mocanu, “Univalent solutions of Briot-Bouquet differential equations,” *Journal of Differential Equations*, vol. 56, no. 3, pp. 297–309, 1985.

[11] W. Kaplan, “Close-to-convex schlicht functions,” *Michigan Mathematical Journal*, vol. 1, pp. 169–185, 1952.

[12] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Process and of Analytic Functions: with an Account of the Principal Transcental Functions*, Cambridge University Press, Cambridge, UK, 4th edition, 1927.