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Research Article

q-Parametric Bleimann Butzer and Hahn Operators

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We introduce a new q-parametric generalization of Bleimann, Butzer, and Hahn operators in $C^*_{1+x}[0,\infty)$. We study some properties of q-BBH operators and establish the rate of convergence for q-BBH operators. We discuss Voronovskaja-type theorem and saturation of convergence for q-BBH operators for arbitrary fixed $0<q<1$. We give explicit formulas of Voronovskaja-type for the q-BBH operators for $0<q<1$. Also, we study convergence of the derivative of q-BBH operators.

1. Introduction

q-Bernstein polynomials

$$B_{n,q}(f)(x) := \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1}(1-q^s x) \tag{1.1}$$

were introduced by Phillips in [1]. q-Bernstein polynomials form an area of an intensive research in the approximation theory, see survey paper [2] and references therein. Nowadays, there are new studies on the q-parametric operators. Two parametric generalizations of q-Bernstein polynomials have been considered by Lewanowicz and Woźny (cf. [3]), an analog of the Bernstein-Durrmeyer operator and Bernstein-Chlodowsky operator related to the q-Bernstein basis has been studied by Derriennic [4], Gupta [5] and Karsli and Gupta [6], respectively, a q-version of the Szasz-Mirakjan operator has been investigated by Aral and Gupta in [7]. Also, some results on q-parametric Meyer-König and Zeller operators can be found in [8–11].

In [12], Bleimann et al. introduced the following operators:

$$H_n(f)(x) = \frac{1}{(1+x)^n}\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}x^k, \quad x>0,\ n\in\mathbb{N}. \tag{1.2}$$

There are several studies related to approximation properties of Bleimann, Butzer, and Hahn operators (or, briefly, BBH), see, for example, [12–18]. Recently, Aral and Doğru [19] introduced a q-analog of Bleimann, Butzer, and Hahn operators and they have established some approximation properties of their q-Bleimann, Butzer, and Hahn operators in the subspace of $C_B[0,\infty)$. Also, they showed that these operators are more flexible than classical BBH operators, that is, depending on the selection of q, rate of convergence of the q-BBH operators is better than the classical one. Voronovskaja-type asymptotic estimate and the monotonicity properties for q-BBH operators are studied in [20].

In this paper, we propose a different q-analog of the Bleimann, Butzer, and Hahn operators in $C^*_{1+x}[0,\infty)$. We use the connection between classical BBH and Bernstein operators suggested in [16] to define new q-BBH operators as follows:

$$H_{n,q}(f)(x) := (\Phi^{-1}B_{n+1,q}\Phi)(f)(x), \tag{1.3}$$

where $B_{n+1,q}$ is a q-Bernstein operator, Φ and Φ^{-1} will be defined later. Thanks to (1.3), different properties of $B_{n+1,q}$ can be transferred to $H_{n,q}$ with a little extra effort. Thus the limiting behavior of $H_{n,q}$ can be immediately derived from (1.3) and the well-known properties of $B_{n+1,q}$. It is natural that even in the classical case, when $q = 1$, to define H_n in the space $C^*_{1+x}[0,\infty)$, the limit l_f of $f(x)/(1+x)$ as $x\to\infty$ has to appear in the definition of H_n. Thus in $C^*_{1+x}[0,\infty)$ the classical BBH operator has to be modified as follows:

$$H_n(f)(x) = \frac{1}{(1+x)^n}\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}x^k + l_f\frac{x^{n+1}}{(1+x)^n}, \quad x>0,\ n\in\mathbb{N}. \tag{1.4}$$

The paper is organized as follows. In Section 2, we give construction of q-BBH operators and study some elementary properties. In Section 3, we investigate convergence properties of q-BBH, Voronovskaja-type theorem and saturation of convergence for q-BBH operators for arbitrary fixed $0<q<1$, and also we study convergence of the derivative of q-BBH operators.

2. Construction and some properties of q-BBH operators

Before introducing the operators, we mention some basic definitions of q calculus.

Let $q>0$. For any $n\in N\cup\{0\}$, the q-integer $[n]=[n]_q$ is defined by

$$[n] := 1+q+\cdots+q^{n-1}, \qquad [0] := 0; \tag{2.1}$$

and the q-factorial $[n]! = [n]_q!$ by

$$[n]! := [1][2]\cdots[n], \qquad [0]! := 1. \tag{2.2}$$

For integers $0\le k\le n$, the q-binomial is defined by

$$\begin{bmatrix} n\\ k\end{bmatrix} := \frac{[n]!}{[k]![n-k]!}. \tag{2.3}$$

Also, we use the following standard notations:

$$\begin{gathered}
(z;q)_0 := 1, \qquad (z;q)_n := \prod_{j=0}^{n-1}(1-q^jz), \qquad (z;q)_\infty := \prod_{j=0}^{\infty}(1-q^jz), \\
p_{n,k}(q;x) := \begin{bmatrix} n\\ k\end{bmatrix} x^k\prod_{s=0}^{n-k-1}(1-q^sx), \qquad p_{\infty k}(q;x) := \frac{x^k}{(1-q)^k[k]!}\prod_{s=0}^{\infty}(1-q^sx).
\end{gathered} \tag{2.4}$$

It is agreed that an empty product denotes 1. It is clear that $p_{nk}(q;x) \geq 0,\ p_{\infty k}(q;x) \geq 0\ \forall x \in [0,1]$ and

$$\sum_{k=0}^{n} p_{nk}(q;x) = \sum_{k=0}^{\infty} p_{\infty k}(q;x) = 1. \tag{2.5}$$

Introduce the following spaces.

$$\begin{aligned}
&B_\rho[0,\infty) = \{f : [0,\infty)\to R \mid \exists M_f > 0 \text{ such that } |f(x)| \leq M_f\rho(x)\ \forall x \in [0,\infty)\},\\
&C_\rho[0,\infty) = \{f \in B_\rho[0,\infty) \mid f \text{ is continuous}\},\\
&C_\rho^*[0,\infty) = \left\{f \in C_\rho[0,\infty) \mid \lim_{x\to\infty}\frac{f(x)}{\rho(x)} = l_f \text{ exists and is finite}\right\},\\
&C_\rho^0[0,\infty) = \left\{f \in C_\rho[0,\infty) \mid \lim_{x\to\infty}\frac{f(x)}{\rho(x)} = 0\right\}.
\end{aligned} \tag{2.6}$$

It is clear that $C_\rho^*[0,\infty) \subset C_\rho[0,\infty) \subset B_\rho[0,\infty)$. In each space, the norm is defined by

$$\|f\|_\rho = \sup_{x\geq 0}\frac{|f(x)|}{\rho(x)}. \tag{2.7}$$

We introduce the following auxiliary operators. Firstly, let us denote

$$\psi(y) = \frac{y}{1-y}, \quad y \in [0,1), \qquad \psi^{-1}(x) = \frac{x}{1+x}, \quad x \in [0,\infty). \tag{2.8}$$

Secondly, let $\Phi : C_\rho^*[0,\infty)\to C[0,1]$ be defined by

$$\Phi(f)(y) := \begin{cases} \dfrac{f(\psi(y))}{\rho(\psi(y))}, & \text{if } y \in [0,1),\\ l_f = \lim\limits_{x\to\infty}\dfrac{f(x)}{\rho(x)}, & \text{if } y = 1. \end{cases} \tag{2.9}$$

Then Φ is a positive linear isomorphism, with positive inverse $\Phi^{-1} : C[0,1]\to C_\rho^*[0,\infty)$ defined by

$$\Phi^{-1}(g)(x) = \rho(x)g(\psi^{-1}(x)), \quad g \in C[0,1],\ x \in [0,\infty). \tag{2.10}$$

For $f \in C[0,1],\ t > 0$, we define the modulus of continuity $\omega(f;t)$ as follows:

$$\omega(f;t) := \sup\{|f(x) - f(y)| : |x-y| \leq t,\ x,y \in [0,1]\}. \tag{2.11}$$

We introduce new Bleimann-, Butzer-, and Hahn- (BBH) type operators based on q-integers as follows.

Definition 2.1. For $f \in C_\rho^*[0,\infty)$, the q-Bleimann, Butzer, and Hahn operators are given by

$$\begin{aligned}
H_{n,q}(f)(x) &:= (\Phi^{-1}B_{n+1,q}\Phi)(f)(x)\\
&= \rho(x)\sum_{k=0}^{n}\frac{f(\psi([k]/[n+1]))}{\rho(\psi([k]/[n+1]))}p_{n+1,k}(q;\psi^{-1}(x)) + l_f\rho(x)\left(\psi^{-1}(x)\right)^{n+1}, \quad n \in N,
\end{aligned} \tag{2.12}$$

where

$$p_{n+1,k}(q;\psi^{-1}(x)) := \begin{bmatrix} n+1\\ k\end{bmatrix}\left(\psi^{-1}(x)\right)^k\prod_{s=0}^{n-k}(1-q^s\psi^{-1}(x)), \quad k = 0,1,\ldots,n. \tag{2.13}$$

Note that for $q = 1,\ \rho = 1 + x$ and $l_f = 0$, we recover the classical Bleimann, Butzer, and Hahn operators. If $q = 1,\ \rho = 1 + x$ but $l_f \neq 0$, it is new Bleimann, Butzer, and Hahn operators with additional term $l_f(x^{n+1}/(1+x)^n)$. Thus if $f \in C^0_{1+x}[0,\infty)$ then

$$H_{n,q}(f)(x) := \sum_{k=0}^{n} f\left(\frac{[k]}{q^k[n-k+1]}\right)\begin{bmatrix} n \\ k \end{bmatrix}\left(\frac{qx}{1+x}\right)^k \prod_{s=1}^{n-k}\left(1 - q^s\frac{x}{1+x}\right). \tag{2.14}$$

To present an explicit form of the limit q-BBH operators, we consider

$$p_{\infty k}(q;\psi^{-1}(x)) := \frac{(\psi^{-1}(x))^k}{(1-q)^k[k]!}\prod_{s=0}^{\infty}(1 - q^s\psi^{-1}(x)). \tag{2.15}$$

Definition 2.2. Let $0 < q < 1$. The linear operator defined on $C^*_\rho[0,\infty)$ given by

$$H_{\infty,q}(f)(x) := \rho(x)\sum_{k=0}^{\infty}\frac{f(\psi(1-q^k))}{\rho(\psi(1-q^k))}p_{\infty k}(q;\psi^{-1}(x)) \tag{2.16}$$

is called the limit q-BBH operator.

Lemma 2.3. *$H_{n,q}, H_{\infty,q} : C^*_\rho[0,\infty) \to C^*_\rho[0,\infty)$ are linear positive operators and*

$$\|H_{n,q}(f)\|_\rho \le \|f\|_\rho, \qquad \|H_{\infty,q}(f)\|_\rho \le \|f\|_\rho. \tag{2.17}$$

Proof. We prove the first inequality, since the second one can be done in a like manner. Thanks to the definition, we have

$$\begin{aligned} |H_{n,q}(f)(x)| &\le \rho(x)\|f\|_\rho\sum_{k=0}^{n}p_{n+1,k}(q;\psi^{-1}(x)) + \rho(x)|l_f|(\psi^{-1}(x))^{n+1} \\ &\le \rho(x)\|f\|_\rho\sum_{k=0}^{n}p_{n+1,k}(q;\psi^{-1}(x)) + \rho(x)\|f\|_\rho(\psi^{-1}(x))^{n+1} \\ &= \rho(x)\|f\|_\rho\sum_{k=0}^{n+1}p_{n+1,k}(q;\psi^{-1}(x)) = \rho(x)\|f\|_\rho. \end{aligned} \tag{2.18}$$

□

Lemma 2.4. *The following recurrence formula holds:*

$$H_{n,q}\left(\rho(t)\left(\frac{t}{1+t}\right)^m\right)(x) = \frac{1}{[n+1]^{m-1}}\frac{x}{1+x}\sum_{j=0}^{m-1}\binom{m-1}{j}q^j[n]^jH_{n-1,q}\left(\rho(t)\left(\frac{t}{1+t}\right)^j\right)(x). \tag{2.19}$$

In particular, we have

$$\begin{gathered} H_{n,q}(\rho)(x) = \rho(x), \qquad H_{n,q}\left(\rho(t)\frac{t}{1+t}\right)(x) = \rho(x)\frac{x}{1+x}, \qquad H_{n,q}(1)(x) = 1, \\ H_{n,q}\left(\rho(t)\left(\frac{t}{1+t}\right)^2\right)(x) = \rho(x)\left(\frac{x}{1+x}\right)^2 + \rho(x)\frac{x}{(1+x)^2}\frac{1}{[n+1]}. \end{gathered} \tag{2.20}$$

Proof. We prove only the recurrence formula, since the formulae (2.20) can easily be obtained by standard computations. Since $l_f = 1$ for $f = \rho(t)(t/(1+t))^m$, we have

$$\begin{aligned}
&H_{n,q}\left(\rho(t)\left(\frac{t}{1+t}\right)^m\right)(x)\\
&\quad= \rho(x)\sum_{k=0}^{n}\left(\frac{[k]}{[n+1]}\right)^m p_{n+1,k}\left(q;\psi^{-1}(x)\right) + \rho(x)\left(\frac{x}{1+x}\right)^{n+1}\\
&\quad= \rho(x)\sum_{k=0}^{n}\left(\frac{[k]}{[n+1]}\right)^m \begin{bmatrix} n+1\\ k\end{bmatrix}\left(\frac{x}{1+x}\right)^k\prod_{s=0}^{n-k}\left(1-q^s\frac{x}{1+x}\right) + \rho(x)\left(\frac{x}{1+x}\right)^{n+1}\\
&\quad= \rho(x)\sum_{k=0}^{n}\frac{[k]^{m-1}}{[n+1]^{m-1}} \begin{bmatrix} n\\ k-1\end{bmatrix}\left(\frac{x}{1+x}\right)^k\prod_{s=0}^{n-k}\left(1-q^s\frac{x}{1+x}\right) + \rho(x)\left(\frac{x}{1+x}\right)^{n+1}\\
&\quad= \rho(x)\sum_{k=1}^{n}\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{q^j[k-1]^j}{[n+1]^{m-1}}\\
&\qquad\times\begin{bmatrix} n\\ k-1\end{bmatrix}\left(\frac{x}{1+x}\right)^k\prod_{s=0}^{n-k}\left(1-q^s\frac{x}{1+x}\right) + \rho(x)\left(\frac{x}{1+x}\right)^{n+1}\\
&\quad= \frac{1}{[n+1]^{m-1}}\frac{x}{1+x}\sum_{j=0}^{m-1}\binom{m-1}{j}q^j[n]^j\\
&\qquad\times\left[H_{n-1,q}\left(\rho(t)\left(\frac{t}{1+t}\right)^j\right)(x) - \rho(x)\left(\frac{x}{1+x}\right)^n\right] + \rho(x)\left(\frac{x}{1+x}\right)^{n+1}\\
&\quad= \frac{1}{[n+1]^{m-1}}\frac{x}{1+x}\sum_{j=0}^{m-1}\binom{m-1}{j}q^j[n]^j H_{n-1,q}\left(\rho(t)\left(\frac{t}{1+t}\right)^j\right)(x)\\
&\qquad+\rho(x)\left(\frac{x}{1+x}\right)^{n+1}\left[1-\frac{1}{[n+1]^{m-1}}\sum_{j=0}^{m-1}\binom{m-1}{j}q^j[n]^j\right]\\
&\quad= \frac{1}{[n+1]^{m-1}}\frac{x}{1+x}\sum_{j=0}^{m-1}\binom{m-1}{j}q^j[n]^j H_{n-1,q}\left(\rho(t)\left(\frac{t}{1+t}\right)^j\right)(x).
\end{aligned} \tag{2.21}$$

□

Next theorem shows the monotonicity properties of q-BBH operators.

Theorem 2.5. *If $f \in C^*_{1+x}[0,\infty)$ is convex and*

$$l_f + \left[f\left(\frac{[n]}{q^n}\right) - f\left(\frac{[n+1]}{q^{n+1}}\right)\right]q^{n+1} \geq 0, \tag{2.22}$$

then its q-BBH operators are nonincreasing, in the sense that

$$H_{n,q}(f)(x) \geq H_{n+1,q}(f)(x), \quad n = 1,2,\ldots,\ q \in (0,1],\ x \in [0,\infty). \tag{2.23}$$

Proof. We begin by writing

$$\begin{aligned}&H_{n,q}(f)(x) - H_{n+1,q}(f)(x)\\&\quad = \sum_{k=0}^{n} f\left(\frac{[k]}{q^k[n-k+1]}\right)\begin{bmatrix} n\\ k\end{bmatrix}\left(\frac{qx}{1+x}\right)^k\prod_{s=1}^{n-k}\left(1-q^s\frac{x}{1+x}\right)\\&\qquad -\sum_{k=0}^{n+1} f\left(\frac{[k]}{q^k[n-k+2]}\right)\begin{bmatrix} n+1\\ k\end{bmatrix}\left(\frac{qx}{1+x}\right)^k\prod_{s=1}^{n-k+1}\left(1-q^s\frac{x}{1+x}\right)+l_f\frac{x^{n+1}}{(1+x)^{n+1}}.\end{aligned} \tag{2.24}$$

We now split the first of the above summations into two, writing

$$\left(\frac{x}{1+x}\right)^k\prod_{s=1}^{n-k}\left(1-q^s\frac{x}{1+x}\right) = \psi_k + q^{n-k+1}\psi_{k+1}, \tag{2.25}$$

where

$$\psi_k = \left(\frac{x}{1+x}\right)^k\prod_{s=1}^{n-k+1}\left(1-q^s\frac{x}{1+x}\right). \tag{2.26}$$

The resulting three summations may be combined to give

$$\begin{aligned}&H_{n,q}(f)(x) - H_{n+1,q}(f)(x)\\&\quad = \sum_{k=0}^{n} f\left(\frac{[k]}{q^k[n-k+1]}\right)\begin{bmatrix} n\\ k\end{bmatrix} q^k\left(\psi_k + q^{n-k+1}\psi_{k+1}\right)\\&\qquad -\sum_{k=0}^{n+1} f\left(\frac{[k]}{q^k[n-k+2]}\right)\begin{bmatrix} n+1\\ k\end{bmatrix} q^k\psi_k + l_f\left(\frac{x}{1+x}\right)^{n+1}\\&\quad = \sum_{k=0}^{n} f\left(\frac{[k]}{q^k[n-k+1]}\right)\begin{bmatrix} n\\ k\end{bmatrix} q^k\psi_k + \sum_{k=1}^{n+1} f\left(\frac{[k-1]}{q^{k-1}[n-k+2]}\right)\begin{bmatrix} n\\ k-1\end{bmatrix} q^{n+1}\psi_k\\&\qquad -\sum_{k=0}^{n+1} f\left(\frac{[k]}{q^k[n-k+2]}\right)\begin{bmatrix} n+1\\ k\end{bmatrix} q^k\psi_k + l_f\left(\frac{x}{1+x}\right)^{n+1}\\&\quad = \sum_{k=1}^{n}\begin{bmatrix} n+1\\ k\end{bmatrix} a_k q^k\psi_k + \left[f\left(\frac{[n]}{q^n}\right) - f\left(\frac{[n+1]}{q^{n+1}}\right)\right]q^{n+1}\left(\frac{x}{1+x}\right)^{n+1} + l_f\left(\frac{x}{1+x}\right)^{n+1},\end{aligned} \tag{2.27}$$

where

$$a_k = \frac{[n-k+1]}{[n+1]} f\left(\frac{[k]}{q^k[n-k+1]}\right) + \frac{q^{n-k+1}[k]}{[n+1]} f\left(\frac{[k-1]}{q^{k-1}[n-k+2]}\right) - f\left(\frac{[k]}{q^k[n-k+2]}\right). \tag{2.28}$$

By assumption, the sum of the last three terms of (2.27) is positive. Thus to show monotonicity of $H_{n,q}$ it suffices to show nonnegativity of a_k, $0 \le k \le n$. Let us write

$$\alpha = \frac{[n-k+1]}{[n+1]}, \qquad x_1 = \frac{[k]}{q^k[n-k+1]}, \qquad x_2 = \frac{[k-1]}{q^k[n-k+2]}. \tag{2.29}$$

Then it follows that

$$\begin{aligned} 1-\alpha &= \frac{q^{n-k+1}[k]}{[n+1]}, \\ \alpha x_1 + (1-\alpha)x_2 &= \frac{[k]}{q^k[n+1]}\left(1+\frac{q^{n-k+2}[k-1]}{[n-k+2]}\right) \\ &= \frac{[k]}{q^k[n+1]}\left(\frac{1-q^{n-k+2}+q^{n-k+2}(1-q^{k-1})}{1-q^{n-k+2}}\right) = \frac{[k]}{q^k[n-k+2]}, \end{aligned} \tag{2.30}$$

and we see immediately that

$$a_k = \alpha f(x_1) + (1-\alpha)f(x_2) - f(\alpha x_1 + (1-\alpha)x_2) \geq 0, \tag{2.31}$$

and so $H_{n,q}(f)(x) - H_{n+1,q}(f)(x) \geq 0$. □

Remark 2.6. It is easily seen that

$$\begin{aligned} &l_f + \left[f\left(\frac{[n]}{q^n}\right) - f\left(\frac{[n+1]}{q^{n+1}}\right)\right]q^{n+1} \\ &\quad = [n+2]\left(\frac{1}{[n+2]}(\Phi f)(1) + \frac{q[n+1]}{[n+2]}(\Phi f)\left(\frac{[n]}{[n+1]}\right) - (\Phi f)\left(\frac{[n+1]}{[n+2]}\right)\right). \end{aligned} \tag{2.32}$$

The condition (2.22) follows from convexity of Φf. On the other hand, Φf is convex if f is convex and nonincreasing, see [16].

3. Convergence properties

Theorem 3.1. *Let $q \in (0,1)$, and let $f \in C_\rho^*[0,\infty)$. Then*

$$\|H_{n,q}(f) - H_{\infty,q}(f)\|_\rho \leq C(q)\omega(\Phi f, q^{n+1}), \tag{3.1}$$

where $C(q) = (4/q(1-q))\ln(1/(1-q)) + 2$.

Proof. For all $x \in [0,\infty)$, by the definitions of $H_{n,q}(f)(x)$ and $H_{\infty,q}(f)(x)$, we have that

$$\begin{aligned} H_{n,q}(f) - H_{\infty,q}(f) &= \rho(x)\sum_{k=0}^{n}\frac{f(\psi([k]/[n+1]))}{\rho(\psi([k]/[n+1]))}p_{n+1,k}(q;\psi^{-1}(x)) \\ &\quad + l_f\rho(x)\left(\frac{x}{1+x}\right)^{n+1} - \rho(x)\sum_{k=0}^{\infty}\frac{f(\psi(1-q^k))}{\rho(\psi(1-q^k))}p_{\infty k}(q;\psi^{-1}(x)) \\ &= \rho(x)\sum_{k=0}^{n+1}\left[(\Phi f)\left(\frac{[k]}{[n+1]}\right) - (\Phi f)(1-q^k)\right]p_{n+1,k}(q;\psi^{-1}(x)) \\ &\quad + \rho(x)\sum_{k=0}^{n+1}[(\Phi f)(1-q^k) - (\Phi f)(1)](p_{n+1,k}(q;\psi^{-1}(x)) - p_{\infty k}(q;\psi^{-1}(x))) \\ &\quad - \rho(x)\sum_{k=n+2}^{\infty}[(\Phi f)(1-q^k) - (\Phi f)(1)]p_{\infty k}(q;\psi^{-1}(x)) \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{3.2}$$

First, we estimate $I_1,\ I_3$. By using the following inequalities:

$$\begin{aligned}&0 \le \frac{[k]}{[n+1]} - (1-q^k) = \frac{1-q^k}{1-q^{n+1}} - (1-q^k) = \frac{q^{n+1}(1-q^k)}{1-q^{n+1}} \le q^{n+1},\\&0 \le 1-(1-q^k) = q^k \le q^{n+1}, \quad k \ge n+2,\end{aligned} \tag{3.3}$$

we get

$$\begin{aligned}&|I_1| \le \rho(x)\omega(\Phi f, q^{n+1})\sum_{k=0}^{n+1} p_{n+1,k}(q;\psi^{-1}(x)) = \rho(x)\omega(\Phi f, q^{n+1}),\\&|I_3| \le \rho(x)\sum_{k=n+2}^{\infty}\omega(\Phi f, q^k)p_{\infty k}(q;\psi^{-1}(x)) \le \rho(x)\omega(\Phi f, q^{n+1}).\end{aligned} \tag{3.4}$$

Next, we estimate I_2. Using the well-known property of modulus of continuity

$$\omega(g,\lambda t) \le (1+\lambda)\omega(g,t), \quad \lambda > 0, \tag{3.5}$$

we get

$$\begin{aligned}|I_2| &\le \rho(x)\sum_{k=0}^{n+1}\omega(\Phi f, q^k)|p_{n+1,k}(q;\psi^{-1}(x)) - p_{\infty k}(q;\psi^{-1}(x))|\\&\le \rho(x)\omega(\Phi f, q^{n+1})\sum_{k=0}^{n+1}(1+q^{k-n-1})|p_{n+1,k}(q;\psi^{-1}(x)) - p_{\infty k}(q;\psi^{-1}(x))|\\&\le 2\rho(x)\omega(\Phi f, q^{n+1})\frac{1}{q^{n+1}}\sum_{k=0}^{n+1}q^k|p_{n+1,k}(q;\psi^{-1}(x)) - p_{\infty k}(q;\psi^{-1}(x))|\\&=: \rho(x)\frac{2}{q^{n+1}}\omega(\Phi f, q^{n+1})J_{n+1}(\psi^{-1}(x)),\end{aligned} \tag{3.6}$$

where

$$J_{n+1}(\psi^{-1}(x)) = \sum_{k=0}^{n+1}q^k|p_{n+1,k}(q;\psi^{-1}(x)) - p_{\infty k}(q;\psi^{-1}(x))|. \tag{3.7}$$

Now, using the estimation (2.9) from [21], we have

$$\begin{aligned}J_{n+1}(\psi^{-1}(x)) &\le \frac{q^{n+1}}{q(1-q)}\ln\frac{1}{1-q}\sum_{k=0}^{n+1}(p_{n+1,k}(q;\psi^{-1}(x)) + p_{\infty k}(q;\psi^{-1}(x)))\\&\le \frac{2q^{n+1}}{q(1-q)}\ln\frac{1}{1-q}.\end{aligned} \tag{3.8}$$

From (3.6) and (3.8), it follows that

$$|I_2| \le \rho(x)\frac{4}{q(1-q)}\ln\frac{1}{1-q}\omega(\Phi f, q^{n+1}). \tag{3.9}$$

From (3.4), and (3.9), we obtain the desired estimation. □

Theorem 3.2. *Let $0 < q < 1$ be fixed and let $f \in C^*_{1+x}[0,\infty)$. Then $H_{\infty,q}(f)(x) = f(x)$ $\forall x \in [0,\infty)$ if and only if f is linear.*

Proof. By definition of $H_{\infty,q}$ we have

$$H_{\infty,q}(f)(x) = (\Phi^{-1}B_{\infty,q}\Phi)(f)(x). \tag{3.10}$$

Assume that $H_{\infty,q}(f)(x) = f(x)$. Then $(B_{\infty,q}\Phi)(f)(x) = (\Phi f)(x)$. From [22], we know that $B_{\infty,q}(g) = g$ if and only if g is linear. So $(B_{\infty,q}\Phi)(f)(x) = (\Phi f)(x)$ if and only if $(\Phi f)(x) = (1-x)f(x/(1-x)) = Ax+B$. It follows that $f(x) = (1+x)(A(x/(1+x))+B) = (A+B)x+B$. The converse can be shown in a similar way. □

Remark 3.3. Let $0 < q < 1$ be fixed and let $f \in C^*_{1+x}[0,\infty)$. Then the sequence $\{H_{n,q}(f)(x)\}$ does not approximate $f(x)$ unless f is linear. It is completely in contrast to the classical case.

Theorem 3.4. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For any $x \in [0,\infty)$ and for any $f \in C^*_{\rho}[0,\infty)$, the following inequality holds:*

$$\frac{1}{\rho(x)}\left|H_{n,q_n}(f)(x) - f(x)\right| \leq 2\omega\left(\Phi f, \sqrt{\lambda_n(x)}\right), \tag{3.11}$$

where $\lambda_n(x) = (x/(1+x)^2)(1/[n+1]_{q_n})$.

Proof. Positivity of B_{n+1,q_n} implies that for any $g \in C[0,1]$

$$\left|B_{n+1,q_n}(g)(x) - g(x)\right| \leq B_{n+1,q_n}(|g(t)-g(x)|)(x). \tag{3.12}$$

On the other hand,

$$\begin{aligned} |(\Phi f)(t) - (\Phi f)(x)| &\leq \omega(\Phi f, |t-x|) \\ &\leq \omega(\Phi f, \delta)\left(1 + \frac{1}{\delta}|t-x|\right), \quad \delta > 0. \end{aligned} \tag{3.13}$$

This inequality and (3.12) imply that

$$\begin{aligned} &\left|B_{n+1,q_n}(\Phi f)(x) - (\Phi f)(x)\right| \leq \omega(\Phi f, \delta)\left(1 + \frac{1}{\delta}B_{n+1,q_n}(|t-x|)(x)\right), \\ &\left|(\Phi^{-1}B_{n+1,q_n}\Phi)(f)(x) - (\Phi^{-1}\Phi f)(x)\right| \\ &\quad \leq \omega(\Phi f, \delta)\left(\Phi^{-1}(1) + \frac{1}{\delta}\Phi^{-1}B_{n+1,q_n}(|t-x|)(x)\right) \\ &\quad \leq \rho(x)\omega(\Phi f, \delta)\left(1 + \frac{1}{\delta}\left(B_{n+1,q_n}\left(\left|t-\psi^{-1}(x)\right|^2\right)\left(\psi^{-1}(x)\right)\right)^{1/2}\right) \\ &\quad = \rho(x)\omega(\Phi f, \delta)\left(1 + \frac{1}{\delta}\left(\left(\frac{x}{1+x}\right)^2 + \frac{x}{(1+x)^2}\frac{1}{[n+1]_{q_n}} - \left(\frac{x}{1+x}\right)^2\right)^{1/2}\right) \\ &\quad = \rho(x)\omega(\Phi f, \delta)\left(1 + \frac{1}{\delta}\left(\frac{x}{(1+x)^2}\frac{1}{[n+1]_{q_n}}\right)^{1/2}\right), \end{aligned} \tag{3.14}$$

by choosing $\delta = \sqrt{\lambda_n(x)}$, we obtain desired result. □

Corollary 3.5. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For any $f \in C_\rho^*[0,\infty)$ it holds that*

$$\lim_{n\to\infty} \left\| H_{n,q_n}(f)(x) - f(x) \right\|_\rho = 0. \tag{3.15}$$

Next, we study Voronovskaja-type formulas for the q-BBH operators. For the q-Bernstein operators, it is proved in [23] that for any $f \in C^1[0,1]$,

$$\lim_{n\to\infty} \frac{[n]}{q^n}\left[B_{n,q}(f)(x) - B_{\infty,q}(f)(x)\right] = L_q(f,x) \tag{3.16}$$

uniformly in $x \in [0,1]$, where

$$L_q(f,x) := \begin{cases} \displaystyle\sum_{k=0}^{\infty} [k]\left(f'(1-q^k) - \frac{f(1-q^k) - f(1-q^{k-1})}{(1-q^k)-(1-q^{k-1})}\right)\frac{x^k}{(q;q)_k}(x;q)_\infty, & 0 \le x < 1, \\ 0, & x = 1. \end{cases} \tag{3.17}$$

Similarly, we have the following Voronovskaja-type theorem for the q-BBH operators for fixed $q \in (0,1)$. Before stating the theorem we introduce an analog of $L_q(f,x)$ for q-BBH operators

$$\begin{aligned} V_q(f,x) &:= (\Phi^{-1}L_q\Phi)(f)(x) = \left(\frac{x}{1+x},q\right)_\infty \sum_{k=0}^{\infty}[k] \\ &\quad \times \left(f'\left(\frac{1-q^k}{q^k}\right)\frac{1}{q^k} - f\left(\frac{1-q^k}{q^k}\right) - \frac{q^k f((1-q^k)/q^k) - q^{k-1}f((1-q^{k-1})/q^{k-1})}{(1-q^k)-(1-q^{k-1})}\right) \\ &\quad \times \frac{1}{(q,q)_k}\frac{x^k}{(1+x)^{k-1}} \\ &= \left(\frac{x}{1+x};q\right)_\infty \sum_{k=0}^{\infty}[k]\left(f'\left(\frac{1-q^k}{q^k}\right)\frac{1}{q^k} - q^{k-1}\frac{f((1-q^k)/q^k) - f((1-q^{k-1})/q^{k-1})}{q^{k-1}-q^k}\right) \\ &\quad \times \frac{1}{(q;q)_k}\frac{x^k}{(1+x)^{k-1}}. \end{aligned} \tag{3.18}$$

Theorem 3.6. *Let $0 < q < 1$, $f \in C_{1+x}^*[0,\infty) \cap C^1[0,\infty)$, and Φf is differentiable at $x = 1$. Then*

$$\lim_{n\to\infty} \frac{[n+1]}{q^{n+1}}\left[H_{n,q}(f)(x) - H_{\infty,q}(f)(x)\right] = V_q(f,x), \tag{3.19}$$

in $C_{1+x}^*[0,\infty)$.

Proof. We estimate the difference

$$\begin{aligned} \Delta(x) &:= \left| \frac{[n+1]}{q^{n+1}}\left(H_{n,q}(f)(x) - H_{\infty,q}(f)(x)\right) - V_q(f,x) \right| \\ &= \left| \frac{[n+1]}{q^{n+1}}\left((\Phi^{-1}B_{n+1,q}\Phi)(f)(x) - (\Phi^{-1}B_{\infty,q}\Phi)(f)(x)\right) - (\Phi^{-1}L_q\Phi)(f)(x) \right| \\ &= \left| \left(\Phi^{-1}\left[\frac{[n+1]}{q^{n+1}}(B_{n+1,q} - B_{\infty,q}) - L_q\right]\Phi\right)(f)(x) \right| \\ &= (1+x)\left| \left[\frac{[n+1]}{q^{n+1}}(B_{n+1,q} - B_{\infty,q}) - L_q\right](\Phi f)(\psi^{-1}(x)) \right|. \end{aligned} \tag{3.20}$$

Since Φf is well defined on whole $[0,1]$, from [23, Theorem 1], we get that

$$\lim_{n\to\infty}\|\Delta\|_{1+x} \le \lim_{n\to\infty}\sup_{0\le u\le 1}\left|\left[\frac{[n+1]}{q^{n+1}}(B_{n+1,q}-B_{\infty,q})-L_q\right](\Phi f)(u)\right| = 0. \tag{3.21}$$

Theorem is proved. □

Remark 3.7. It is clear that Φf is differentiable in $[0,1)$ if $f \in C^1[0,\infty)$. If Φf is not differentiable at $x=1$, then

$$\lim_{n\to\infty}\frac{[n+1]}{q^{n+1}}[H_{n,q}(f)(x)-H_{\infty,q}(f)(x)] = V_q(f,x), \tag{3.22}$$

uniformly on any $[0,A]\subset[0,\infty)$.

Theorem 3.8. *If $f\in C^2[0,\infty)$ and $q_n\to 1$ as $n\to\infty$, then*

$$\lim_{n\to\infty}[n+1]_{q_n}\{H_{n,q_n}(f)(x)-f(x)\} = \frac{1}{2}f''(x)(1+x)^2x \tag{3.23}$$

uniformly on any $[0,A]\subset[0,\infty)$.

Proof. By definition of H_{n,q_n},

$$\begin{aligned}H_{n,q_n}(f)(x)-f(x) &= (\Phi^{-1}B_{n+1,q_n}\Phi)(f)(x)-(\Phi^{-1}\Phi f)(x)\\ &= (\Phi^{-1}[B_{n+1,q_n}-I]\Phi)(f)(x)\\ &= (1+x)([B_{n+1,q_n}-I]\Phi)(f)(\varphi^{-1}(x)),\end{aligned} \tag{3.24}$$

and if $L := (1/2)f''(x)(1-x)x$, then

$$\begin{aligned}\frac{1}{2}f''(x)(1+x)^2x &= (\Phi^{-1}L\Phi)(f)(x) = (1+x)(L\Phi)(f)(\varphi^{-1}(x))\\ &= \frac{1}{2}(1+x)(\Phi f)''(\varphi^{-1}(x))\varphi^{-1}(x)(1-\varphi^{-1}(x)).\end{aligned} \tag{3.25}$$

On the other hand, by [24, Corollary 5.2] we have that

$$\lim_{n\to\infty}\sup_{0\le u\le 1}\left|[n+1]_{q_n}([B_{n+1,q_n}-I]\Phi)(f)(u)-\frac{1}{2}(\Phi f)''(u)u(1-u)\right| = 0. \tag{3.26}$$

Now, the result follows from the following inequality:

$$\begin{aligned}&\left|[n+1]_{q_n}\{H_{n,q_n}(f)(x)-f(x)\}-\frac{1}{2}f''(x)(1+x)^2x\right|\\ &\quad= \left|(1+x)[n+1]_{q_n}([B_{n+1,q_n}-I]\Phi)(f)(\varphi^{-1}(x))-(1+x)\frac{1}{2}(\Phi f)''(\varphi^{-1}(x))\varphi^{-1}(x)(1-\varphi^{-1}(x))\right|\\ &\quad\le (1+A)\sup_{0\le u\le A/(1+A)}\left|[n+1]_{q_n}([B_{n+1,q_n}-I]\Phi)(f)(u)-\frac{1}{2}(\Phi f)''(u)u(1-u)\right|.\end{aligned} \tag{3.27}$$

The theorem is proved. □

From Theorem 3.6, we have the following saturation of convergence for the q-BBH operators for fixed $q \in (0,1)$.

Corollary 3.9. *Let $0 < q < 1$ and $f \in C^*_{1+x}[0,\infty) \cap C^1[0,\infty)$. Then*

$$\|H_{n,q}(f)(x) - H_{\infty,q}(f)(x)\|_{1+x} = o(q^{n+1}) \tag{3.28}$$

if and only if $V_q(f,x) \equiv 0$, and this is equivalent to

$$f'\left(\frac{1-q^k}{q^k}\right)\left(\frac{1}{q^k} - \frac{1}{q^{k-1}}\right) = f\left(\frac{(1-q^k)}{q^k}\right) - f\left(\frac{(1-q^{k-1})}{q^{k-1}}\right), \quad k = 1,2,\ldots. \tag{3.29}$$

Theorem 3.10. *Let $0 < q < 1$ and $f \in C^*_{1+x}[0,\infty) \cap C^1[0,\infty)$. If f is a convex function, then $\|H_{n,q}(f)(x) - H_{\infty,q}(f)(x)\|_{1+x} = o(q^{n+1})$ if and only if f is a linear function.*

Proof. If $\|H_{n,q}(f) - H_{\infty,q}(f)\|_{1+x} = o(q^{n+1})$, then by Corollary 3.9

$$f'\left(\frac{1-q^k}{q^k}\right)\frac{q^{k-1} - q^k}{q^{2k-1}} = f\left(\frac{(1-q^k)}{q^k}\right) - f\left(\frac{(1-q^{k-1})}{q^{k-1}}\right), \quad k = 1,2,\ldots. \tag{3.30}$$

Hence for $k = 1,2,\ldots$

$$\int_{(1-q^{k-1})/q^{k-1}}^{(1-q^k)/q^k} \left(f'\left(\frac{1-q^k}{q^k}\right) - f'(t)\right)dt = 0. \tag{3.31}$$

Since f is convex and f' is continuous on $[0,\infty)$, we get $f'(t) = f'((1-q^k)/q^k)$ $\forall t \in [(1-q^{k-1})/q^{k-1}, (1-q^k)/q^k]$. Hence $f'(t) \equiv f'(0)$, and therefore $f(t) = At + B$. Conversely, if f is linear, then $\|H_{n,q}(f)(x) - H_{\infty,q}(f)(x)\|_{1+x} = 0$. □

One of the remarkable properties of the q-Bernstein approximation is that derivatives of $B_n(f)$ of any order converge to corresponding derivatives of f, see [25]. Next theorem shows the same property for H_{nq} for the first derivative.

Theorem 3.11. *Let $f \in C^*_{1+x}[0,\infty) \cap C^1[0,\infty)$ and let $\{q_n\}$ be a sequence chosen so that the sequence*

$$\varepsilon_n = \frac{n}{1 + q_n + q_n^2 + \cdots + q_n^{n-1}} - 1 \tag{3.32}$$

converges to zero from above faster than $\{1/3^n\}$. Then

$$\lim_{n\to\infty} [H_{n,q_n}(f)(x)]' = f'(x) \tag{3.33}$$

uniformly on any $[0,A] \subset [0,\infty)$.

Proof. By definition

$$H_{n,q_n}(f)(x) = (1+x)(B_{n+1,q_n}\Phi)f\left(\frac{x}{1+x}\right). \tag{3.34}$$

Since $H_{n,q_n}(f)(x)$ is a composition of differentiable functions, it is differentiable at any $x \in [0, A]$ and

$$\begin{aligned}\frac{d}{dx}H_{n,q_n}(f)(x) &= \frac{d}{dx}\left[(1+x)(B_{n+1,q_n}\Phi)f\left(\frac{x}{1+x}\right)\right] \\ &= (B_{n+1,q_n}\Phi)f\left(\frac{x}{1+x}\right) + \frac{1}{1+x}\frac{d}{dx}(B_{n+1,q_n}\Phi)f\left(\frac{x}{1+x}\right).\end{aligned} \tag{3.35}$$

By [24, Theorem 4.1]

$$\left|(B_{n+1,q_n}\Phi)f\left(\frac{x}{1+x}\right) - (\Phi f)\left(\frac{x}{1+x}\right)\right| \leq 2\omega\left(\Phi f, \sqrt{B_{n+1,q_n}\left(t-\frac{x}{1+x}\right)^2\left(\frac{x}{1+x}\right)}\right), \tag{3.36}$$

and by [25, Theorem 3]

$$\lim_{n\to\infty}\sup_{0\leq x\leq A}\left|\frac{d}{dx}\left(B_{n+1,q_n}\Phi\right)f\left(\frac{x}{1+x}\right) - (\Phi f)'\left(\frac{x}{1+x}\right)\right| = 0. \tag{3.37}$$

Thus the desired limit follows from the following inequality:

$$\begin{aligned}&\left|\frac{d}{dx}H_{n,q_n}(f)(x) - \frac{d}{dx}f(x)\right| \\ &= \left|\frac{d}{dx}H_{n,q_n}(f)(x) - \frac{d}{dx}(1+x)(\Phi f)\left(\frac{x}{1+x}\right)\right| \\ &\leq \left|(B_{n+1,q_n}\Phi)f\left(\frac{x}{1+x}\right) - (\Phi f)\left(\frac{x}{1+x}\right)\right| + \frac{1}{1+x}\left|\frac{d}{dx}(B_{n+1,q_n}\Phi)f\left(\frac{x}{1+x}\right) - (\Phi f)'\left(\frac{x}{1+x}\right)\right| \\ &\leq 2\omega\left(\Phi f, \sqrt{B_{n+1,q_n}\left(t-\frac{x}{1+x}\right)^2\left(\frac{x}{1+x}\right)}\right) + \left|\frac{d}{dx}(B_{n+1,q_n}\Phi)f\left(\frac{x}{1+x}\right) - (\Phi f)'\left(\frac{x}{1+x}\right)\right| \\ &= 2\omega\left(\Phi f, \sqrt{\frac{x}{(1+x)^2}\frac{1}{[n+1]_{q_n}}}\right) + \left|\frac{d}{dx}(B_{n+1,q_n}\Phi)f\left(\frac{x}{1+x}\right) - (\Phi f)'\left(\frac{x}{1+x}\right)\right| \\ &\leq 2\omega\left(\Phi f, \sqrt{\frac{A}{[n+1]_{q_n}}}\right) + \left|\frac{d}{dx}(B_{n+1,q_n}\Phi)f\left(\frac{x}{1+x}\right) - (\Phi f)'\left(\frac{x}{1+x}\right)\right|.\end{aligned} \tag{3.38}$$

□

Remark 3.12. In [1], it is shown that

$$B_{n+1,q}(f)(x) = \sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}\Delta^k f_0 x^k, \tag{3.39}$$

where

$$\begin{aligned}&f_i = f\left(\frac{[i]}{[n+1]}\right), \quad \Delta^0 f_i = f_i, \quad \Delta^{k+1}f_i = \Delta^k f_{i+1} - q^k\Delta^k f_i, \\ &\Delta^k f_i = \sum_{j=0}^{k}(-1)^j q^{j(j-1)/2}\begin{bmatrix}k\\j\end{bmatrix} f\left(\frac{[i+k-j]}{[n+1]}\right).\end{aligned} \tag{3.40}$$

Immediately from the definition of $H_{n,q}$, we get an analog of (3.39) for $H_{n,q}$:

$$\begin{aligned} H_{n,q}(f)(x) &= (\Phi^{-1}B_{n+1,q}\Phi)(f)(x) \\ &= \Phi^{-1}\sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}\Delta^k(\Phi f)_0 x^k \\ &= \sum_{k=0}^{n+1}\begin{bmatrix} n+1 \\ k \end{bmatrix}\Delta^k(\Phi f)_0 \frac{x^k}{(1+x)^{k-1}}. \end{aligned} \tag{3.41}$$

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