Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 389410, 9 pages
doi:10.1155/2008/389410

Research Article

On Logarithmic Convexity for Power Sums and Related Results

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Received 28 March 2008; Revised 23 May 2008; Accepted 29 June 2008

Recommended by Martin j. Bohner

We give some further consideration about logarithmic convexity for differences of power sums inequality as well as related mean value theorems. Also we define quasiarithmetic sum and give some related results.

1. Introduction and preliminaries

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{p} = (p_1, \ldots, p_n)$ denote two sequences of positive real numbers with $\sum_{i=1}^{n} p_i = 1$. The well-known Jensen Inequality [1, page 43] gives the following, for $t < 0$ or $t > 1$:

$$\sum_{i=1}^{n} p_i x_i^t \geq \left(\sum_{i=1}^{n} p_i x_i\right)^t \tag{1.1}$$

and vice versa for $0 < t < 1$.

Simić [2] has considered the difference of both sides of (1.1). He considers the function defined as

$$\lambda_t = \begin{cases} \dfrac{\sum_{i=1}^{n} p_i x_i^t - \left(\sum_{i=1}^{n} p_i x_i\right)^t}{t(t-1)}, & t \neq 0, 1; \\ \log\left(\displaystyle\sum_{i=1}^{n} p_i x_i\right) - \displaystyle\sum_{i=1}^{n} p_i \log x_i, & t = 0; \\ \displaystyle\sum_{i=1}^{n} p_i x_i \log x_i - \left(\displaystyle\sum_{i=1}^{n} p_i x_i\right) \log\left(\displaystyle\sum_{i=1}^{n} p_i x_i\right), & t = 1; \end{cases} \tag{1.2}$$

and has proved the following theorem.

Theorem 1.1. *For $-\infty < r < s < t < +\infty$, then*

$$\lambda_s^{t-r} \leq (\lambda_r)^{t-s}(\lambda_t)^{s-r}. \tag{1.3}$$

Anwar and Pečarić [3] have considerd further generalization of Theorem 1.1. Namely, they introduced new means of Cauchy type in [4] and further proved comparison theorem for these means.

In this paper, we will give some results in the case where instead of means we have power sums.

Let $\mathbf{x}$ be positive n-tuples. The well-known inequality for power sums of order s and r, for $s > r > 0$ (see [1, page 164]), states that

$$\left(\sum_{i=1}^{n} x_i^s\right)^{1/s} < \left(\sum_{i=1}^{n} x_i^r\right)^{1/r}. \tag{1.4}$$

Moreover, if $\mathbf{p} = (p_1, \dots, p_n)$ is a positive n-tuples such that $p_i \geq 1$ $(i = 1, \dots, n)$, then for $s > r > 0$ (see [1, page 165]), we have

$$\left(\sum_{i=1}^{n} p_i x_i^s\right)^{1/s} < \left(\sum_{i=1}^{n} p_i x_i^r\right)^{1/r}. \tag{1.5}$$

Let us note that (1.5) can also be obtained from the following theorem [1, page 152].

Theorem 1.2. *Let $\mathbf{x}$ and $\mathbf{p}$ be two nonnegative n-tuples such that $x_i \in (0, a]$ $(i = 1, \dots, n)$ and*

$$\sum_{i=1}^{n} p_i x_i \geq x_j, \quad \text{for } j = 1, \dots, n, \qquad \sum_{i=1}^{n} p_i x_i \in (0, a]. \tag{1.6}$$

If $f(x)/x$ is an increasing function, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \geq \sum_{i=1}^{n} p_i f(x_i). \tag{1.7}$$

Remark 1.3. Let us note that if $f(x)/x$ is a strictly increasing function, then equality in (1.7) is valid if we have equalities in (1.6) instead of inequalities, that is, $x_1 = \cdots = x_n$ and $\sum_1^n p_i = 1$.

The following similar result is also valid [1, page 153].

Theorem 1.4. *Let $f(x)/x$ be an increasing function. If $0 < x_1 \leq \cdots \leq x_n$ and if the following hold.*

(i) *there exists an $m(\leq n)$ such that*

$$\overline{P}_1 \geq \overline{P}_2 \geq \cdots \geq \overline{P}_m \geq 1, \qquad \overline{P}_{m+1} = \cdots = \overline{P}_n = 0, \tag{1.8}$$

where $P_k = \sum_{i=1}^{k} p_i$, $\overline{P}_k = P_n - P_{k-1}$ $(k = 2, \dots, n)$ and $\overline{P}_1 = P_n$, then (1.7) holds.

(ii) *If there exists an $m(\leq n)$ such that*

$$0 \leq \overline{P}_1 \leq \overline{P}_2 \leq \cdots \leq \overline{P}_m \leq 1, \qquad \overline{P}_{m+1} = \cdots = \overline{P}_n = 0, \tag{1.9}$$

then the reverse of inequality in (1.7) holds.

In this paper, we will give some applications of power sums. That is, we will prove results similar to those shown in [2, 3], but for power sums.

2. Main results

Lemma 2.1. *Let*

$$\varphi_t(x) = \begin{cases} \dfrac{x^t}{t-1}, & t \neq 1; \\ x \log x, & t = 1. \end{cases} \tag{2.1}$$

Then $\varphi_t(x)/x$ is a strictly increasing function for $x > 0$.

Proof. Since $(\varphi_t(x)/x)' = x^{t-2} > 0$, for $x > 0$, therefore $\varphi_t(x)/x$ is a strictly increasing function for $x > 0$. □

Lemma 2.2 ([2]). *A positive function f is log convex in Jensen's sense on an open interval I, that is, for each $s, t \in I$,*

$$f(s)f(t) \geq f^2\left(\frac{s+t}{2}\right), \tag{2.2}$$

if and only if the relation

$$u^2 f(s) + 2uwf\left(\frac{s+t}{2}\right) + w^2 f(t) \geq 0 \tag{2.3}$$

holds for each real u, w, and $s, t \in I$.

The following lemma is equivalent to the definition of convex function (see [1, page 2]).

Lemma 2.3. *If f is continuous and convex for all x_1, x_2, x_3 of an open interval I for which $x_1 < x_2 < x_3$, then*

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0. \tag{2.4}$$

Theorem 2.4. *Let $\mathbf{x}$ and $\mathbf{p}$ be two positive n-tuples ($n \geq 2$) and let*

$$\phi_t = \phi_t(\mathbf{x}; \mathbf{p}) = \varphi_t\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i \varphi_t(x_i) \tag{2.5}$$

such that condition (1.6) is satisfied and all x_i's are not equal. Then ϕ_t is log-convex. Also for $r < s < t$ where $r, s, t \in \mathbb{R}^+$, we have

$$(\phi_s)^{t-r} \leq (\phi_r)^{t-s}(\phi_t)^{s-r}. \tag{2.6}$$

Proof. Since $\varphi_t(x)/x$ is a strictly increasing function for $x > 0$ and all x_i's are not equal, therefore by Theorem 1.2 with $f = \varphi_t$, we have

$$\varphi_t\left(\sum_{i=1}^{n} p_i x_i\right) > \sum_{i=1}^{n} p_i \varphi_t(x_i) \Longrightarrow \phi_t = \varphi_t\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i \varphi_t(x_i) > 0, \tag{2.7}$$

that is, ϕ_t is a positive-valued function.

Let $f(x) = u^2\varphi_s(x) + 2uw\varphi_r(x) + w^2\varphi_t(x)$, where $r = (s+t)/2$ and $u, w \in \mathbb{R}$:

$$\left(\frac{f(x)}{x}\right)' = u^2x^{s-2} + 2uwx^{r-2} + w^2x^{t-2}, \\ = \left(ux^{(s-2)/2} + wx^{(t-2)/2}\right)^2 \geq 0. \tag{2.8}$$

This implies that $f(x)/x$ is monotonically increasing.

By Theorem 1.2, we have

$$f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) \geq 0$$
$$\Longrightarrow u^2\left(\varphi_s\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i\varphi_s(x_i)\right) + 2uw\left(\varphi_r\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i\varphi_r(x_i)\right)$$
$$+ w^2\left(\varphi_t\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i\varphi_t(x_i)\right) \geq 0$$
$$\Longrightarrow u^2\phi_s + 2uw\phi_r + w^2\phi_t \geq 0. \tag{2.9}$$

Now by Lemma 2.2, we have that ϕ_t is log-convex in Jensen sense.

Since $\lim_{t\to 1}\phi_t = \phi_1$, it follows that ϕ_t is continuous, therefore it is a log-convex function [1, page 6].

Since ϕ_t is log-convex, that is, $\log\phi_t$ is convex, we have by Lemma 2.3 that, for $r < s < t$ with $f = \log\phi$,

$$(t-s)\log\phi_r + (r-t)\log\phi_s + (s-r)\log\phi_t \geq 0, \tag{2.10}$$

which is equivalent to (2.6). □

Similar application of Theorem 1.4 gives the following.

Theorem 2.5. *Let* $\mathbf{x}$ *and* $\mathbf{p}$ *be two positive* n*-tuples* $(n \geq 2)$ *such that* $0 < x_1 \leq \cdots \leq x_n$*, all* x_i*'s are not equal and*

(i) *if* $\phi_t = \phi_t(\mathbf{x};\mathbf{p}) = \varphi_t(\sum_{i=1}^{n} p_i x_i) - \sum_{i=1}^{n} p_i\varphi_t(x_i)$ *such that condition* (1.8) *is satisfied, then* ϕ_t *is log-convex, also for* $r < s < t$*, we have*

$$(\phi_s)^{t-r} \leq (\phi_r)^{t-s}(\phi_t)^{s-r}; \tag{2.11}$$

(ii) *moreover if* $\overline{\phi}_t = -\phi_t$ *and* (1.9) *is satisfied, then we have that* $\overline{\phi}_t$ *is log-convex and*

$$\left(\overline{\phi}_s\right)^{t-r} \leq \left(\overline{\phi}_r\right)^{t-s}\left(\overline{\phi}_t\right)^{s-r}. \tag{2.12}$$

We will also use the following lemma.

Lemma 2.6. *Let f be a log-convex function and assume that if $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$. Then the following inequality is valid:*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{1/(x_2-x_1)} \leq \left(\frac{f(y_2)}{f(y_1)}\right)^{1/(y_2-y_1)}. \tag{2.13}$$

Proof. In [1, page 2], we have the following result for convex function f, with $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$:

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(y_2)-f(y_1)}{y_2-y_1}. \tag{2.14}$$

Putting $f = \log f$, we get

$$\log\left(\frac{f(x_2)}{f(x_1)}\right)^{1/(x_2-x_1)} \leq \log\left(\frac{f(y_2)}{f(y_1)}\right)^{1/(y_2-y_1)}, \tag{2.15}$$

from which (2.13) immediately follows. □

Let us introduce the following.

Definition 2.7. Let $\mathbf{x}$ and $\mathbf{p}$ be two nonnegative n-tuples ($n \geq 2$) such that $p_i \geq 1$ ($i = 1, \ldots, n$), then for $t, r, s \in \mathbb{R}^+$, we define

$$A_{t,r}^{s}(\mathbf{x};\mathbf{p}) = \left\{\frac{r-s}{t-s}\frac{\left(\sum_{i=1}^{n}p_i x_i^s\right)^{t/s} - \sum_{i=1}^{n}p_i x_i^t}{\left(\sum_{i=1}^{n}p_i x_i^s\right)^{r/s} - \sum_{i=1}^{n}p_i x_i^r}\right\}^{1/(t-r)}, \quad t \neq r, r \neq s, t \neq s,$$

$$A_{s,r}^{s}(\mathbf{x};\mathbf{p}) = A_{r,s}^{s}(\mathbf{x};\mathbf{p}) = \left\{\frac{r-s}{s}\frac{\left(\sum_{i=1}^{n}p_i x_i^s\right)\log\sum_{i=1}^{n}p_i x_i^s - s\sum_{i=1}^{n}p_i x_i^s \log x_i}{\left(\sum_{i=1}^{n}p_i x_i^s\right)^{r/s} - \sum_{i=1}^{n}p_i x_i^r}\right\}^{1/(s-r)}, \quad s \neq r,$$

$$A_{r,r}^{s}(\mathbf{x};\mathbf{p}) = \exp\left(\frac{1}{s-r} + \frac{\left(\sum_{i=1}^{n}p_i x_i^s\right)^{r/s}\log\sum_{i=1}^{n}p_i x_i^s - s\sum_{i=1}^{n}p_i x_i^r \log x_i}{s\left\{\left(\sum_{i=1}^{n}p_i x_i^s\right)^{r/s} - \sum_{i=1}^{n}p_i x_i^r\right\}}\right), \quad s \neq r,$$

$$A_{s,s}^{s}(\mathbf{x};\mathbf{p}) = \exp\left(\frac{\left(\sum_{i=1}^{n}p_i x_i^s\right)\left(\log\sum_{i=1}^{n}p_i x_i^s\right)^2 - s^2\sum_{i=1}^{n}p_i x_i^s (\log x_i)^2}{2s\left\{\left(\sum_{i=1}^{n}p_i x_i^s\right)\log\left(\sum_{i=1}^{n}p_i x_i^s\right) - s\sum_{i=1}^{n}p_i x_i^s \log x_i\right\}}\right). \tag{2.16}$$

Remark 2.8. Let us note that $A_{s,r}^{s}(\mathbf{x};\mathbf{p}) = A_{r,s}^{s}(\mathbf{x};\mathbf{p}) = \lim_{t\to s} A_{t,r}^{s}(\mathbf{x};\mathbf{p}) = \lim_{t\to s} A_{r,t}^{s}(\mathbf{x};\mathbf{p})$, $A_{r,r}^{s}(\mathbf{x};\mathbf{p}) = \lim_{t\to r} A_{t,r}^{s}(\mathbf{x};\mathbf{p})$ and $A_{s,s}^{s}(\mathbf{x};\mathbf{p}) = \lim_{r\to s} A_{r,r}^{s}(\mathbf{x};\mathbf{p})$.

Theorem 2.9. *Let $r, t, u, v \in \mathbb{R}^+$ such that $r < u,\ t < v,\ r \neq t,\ u \neq v$. Then we have*

$$A_{t,r}^{s}(\mathbf{x};\mathbf{p}) \leq A_{v,u}^{s}(\mathbf{x};\mathbf{p}). \tag{2.17}$$

Proof. Let

$$\phi_t = \phi_t(\mathbf{x};\mathbf{p}) = \begin{cases} \dfrac{1}{t-1}\left(\left(\displaystyle\sum_{i=1}^{n}p_i x_i\right)^t - \displaystyle\sum_{i=1}^{n}p_i x_i^t\right), & t \neq 1; \\ \displaystyle\sum_{i=1}^{n}p_i x_i \log\sum_{i=1}^{n}p_i x_i - \sum_{i=1}^{n}p_i x_i \log x_i, & t = 1. \end{cases} \tag{2.18}$$

Now taking $x_1 = r,\ x_2 = t,\ y_1 = u,\ y_2 = v$, where $r, t, u, v \neq 1$, and $f(t) = \phi_t$ in Lemma 2.6, we have

$$\left(\frac{r-1}{t-1}\frac{\left(\sum_{i=1}^{n} p_i x_i\right)^t - \sum_{i=1}^{n} p_i x_i^t}{\left(\sum_{i=1}^{n} p_i x_i\right)^r - \sum_{i=1}^{n} p_i x_i^r}\right)^{1/(t-r)} \leq \left(\frac{u-1}{v-1}\frac{\left(\sum_{i=1}^{n} p_i x\right)^v - \sum_{i=1}^{n} p_i x_i^v}{\left(\sum_{i=1}^{n} p_i x_i^s\right)^u - \sum_{i=1}^{n} p_i x_i^u}\right)^{1/(v-u)}. \tag{2.19}$$

Since $s > 0$ by substituting $x_i = x_i^s,\ t = t/s,\ r = r/s,\ u = u/s$ and $v = v/s$, where $r, t, u, v \neq s$, in above inequality, we get

$$\left(\frac{r-s}{t-s}\frac{\left(\sum_{i=1}^{n} p_i x_i^s\right)^{t/s} - \sum_{i=1}^{n} p_i x_i^t}{\left(\sum_{i=1}^{n} p_i x_i^s\right)^{r/s} - \sum_{i=1}^{n} p_i x_i^r}\right)^{s/(t-r)} \leq \left(\frac{u-s}{v-s}\frac{\left(\sum_{i=1}^{n} p_i x_i^s\right)^{v/s} - \sum_{i=1}^{n} p_i x_i^v}{\left(\sum_{i=1}^{n} p_i x_i^s\right)^{u/s} - \sum_{i=1}^{n} p_i x_i^u}\right)^{s/(v-u)}. \tag{2.20}$$

By raising power $1/s$, we get (2.17) for $r, t, u, v \neq s$.

From Remark 2.8, we get (2.17) is also valid for $r = s$ or $t = s$ or $r = t$ or $t = r = s$. □

Corollary 2.10. *Let*

$$\Phi_t^s = \begin{cases} \dfrac{1}{t-s}\left\{\left(\displaystyle\sum_{i=1}^{n} p_i x_i^s\right)^{t/s} - \displaystyle\sum_{i=1}^{n} p_i x_i^t\right\}, & t \neq s; \\ \dfrac{1}{s}\left\{\left(\displaystyle\sum_{i=1}^{n} p_i x_i^s\right)\log\left(\displaystyle\sum_{i=1}^{n} p_i x_i^s\right) - s\displaystyle\sum_{i=1}^{n} p_i x_i^s \log x_i\right\}, & t = s. \end{cases} \tag{2.21}$$

Then for $t, r, u \in \mathbb{R}^+$ and $t < r < u$, we have

$$\left(\Phi_r^s\right)^{u-t} \leq \left(\Phi_t^s\right)^{u-r}\left(\Phi_u^s\right)^{r-t}. \tag{2.22}$$

Proof. Taking $v = r$ in (2.17), we get (2.22). □

3. Mean value theorems

Lemma 3.1. *Let $f \in C^1(I)$, where $I = (0, a]$ such that*

$$m \leq \frac{xf'(x) - f(x)}{x^2} \leq M. \tag{3.1}$$

Consider the functions ϕ_1 and ϕ_2 defined as

$$\begin{aligned} \phi_1(x) &= Mx^2 - f(x), \\ \phi_2(x) &= f(x) - mx^2. \end{aligned} \tag{3.2}$$

Then $\phi_i(x)/x$ for $i = 1, 2$ are monotonically increasing functions.

Proof. We have that

$$\begin{aligned} \frac{\phi_1(x)}{x} &= Mx - \frac{f(x)}{x} \Longrightarrow \left(\frac{\phi_1(x)}{x}\right)' = M - \frac{xf'(x) - f(x)}{x^2} \geq 0, \\ \frac{\phi_2(x)}{x} &= \frac{f(x)}{x} - mx \Longrightarrow \left(\frac{\phi_2(x)}{x}\right)' = \frac{xf'(x) - f(x)}{x^2} - m \geq 0, \end{aligned} \tag{3.3}$$

that is, $\phi_i(x)/x$ for $i = 1, 2$ are monotonically increasing functions. □

Theorem 3.2. *Let* $\mathbf{x}$ *and* $\mathbf{p}$ *be two positive* n*-tuples* $(n \geq 2)$ *satisfy condition* (1.6), *all* x_i*'s are not equal and let* $f \in C^1(I)$, *where* $I = (0, a]$. *Then there exists* $\xi \in (0, a]$ *such that*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) = \frac{\xi f'(\xi) - f(\xi)}{\xi^2}\left\{\left(\sum_{i=1}^{n} p_i x_i\right)^2 - \sum_{i=1}^{n} p_i x_i^2\right\}. \tag{3.4}$$

Proof. In Theorem 1.2, setting $f = \phi_1$ and $f = \phi_2$, respectively, as defined in Lemma 3.1, we get the following inequalities:

$$\begin{gathered} f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) \leq M\left\{\left(\sum_{i=1}^{n} p_i x_i\right)^2 - \sum_{i=1}^{n} p_i x_i^2\right\}, \\ f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) \geq m\left\{\left(\sum_{i=1}^{n} p_i x_i\right)^2 - \sum_{i=1}^{n} p_i x_i^2\right\}. \end{gathered} \tag{3.5}$$

Now by combining both inequalities, we get,

$$m \leq \frac{f(\sum_{i=1}^{n} p_i x_i) - \sum_{i=1}^{n} p_i f(x_i)}{(\sum_{i=1}^{n} p_i x_i)^2 - \sum_{i=1}^{n} p_i x_i^2} \leq M. \tag{3.6}$$

$(\sum_{i=1}^{n} p_i x_i)^2 - \sum_{i=1}^{n} p_i x_i^2$ is nonzero, it is zero if equalities are given in conditions (1.6), that is, $x_1 = \cdots = x_n$ and $\sum_{i=1}^{n} p_i = 1$.

Now by condition (3.1), there exist $\xi \in I$, such that

$$\frac{f(\sum_{i=1}^{n} p_i x_i) - \sum_{i=1}^{n} p_i f(x_i)}{(\sum_{i=1}^{n} p_i x_i)^2 - \sum_{i=1}^{n} p_i x_i^2} = \frac{\xi f'(\xi) - f(\xi)}{\xi^2}; \tag{3.7}$$

and (3.7) implies (3.4). □

Theorem 3.3. *Let* $\mathbf{x}$ *and* $\mathbf{p}$ *be two positive* n*-tuples* $(n \geq 2)$ *satisfy condition* (1.6), *all* x_i*'s are not equal and let* $f, g \in C^1(I)$, *where* $I = (0, a]$. *Then there exists* $\xi \in I$ *such that the following equality is true:*

$$\frac{f(\sum_{i=1}^{n} p_i x_i) - \sum_{i=1}^{n} p_i f(x_i)}{g(\sum_{i=1}^{n} p_i x_i) - \sum_{i=1}^{n} p_i g(x_i)} = \frac{\xi f'(\xi) - f(\xi)}{\xi g'(\xi) - g(\xi)}, \tag{3.8}$$

provided that the denominators are nonzero.

Proof. Let a function $k \in C^1(I)$ be defined as

$$k = c_1 f - c_2 g, \tag{3.9}$$

where c_1 and c_2 are defined as

$$\begin{gathered} c_1 = g\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i g(x_i), \\ c_2 = f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i). \end{gathered} \tag{3.10}$$

Then, using Theorem 3.2 with $f = k$, we have

$$0 = \left(c_1 \frac{\xi f'(\xi) - f(\xi)}{\xi^2} - c_2 \frac{\xi g'(\xi) - g(\xi)}{\xi^2} \right) \left\{ \left(\sum_{i=1}^{n} p_i x_i \right)^2 - \sum_{i=1}^{n} p_i x_i^2 \right\}. \tag{3.11}$$

Since

$$\left(\sum_{i=1}^{n} p_i x_i \right)^2 - \sum_{i=1}^{n} p_i x_i^2 \neq 0, \tag{3.12}$$

therefore, (3.11) gives

$$\frac{c_2}{c_1} = \frac{\xi f'(\xi) - f(\xi)}{\xi g'(\xi) - g(\xi)}. \tag{3.13}$$

After putting values, we get (3.8). □

Let α be a strictly monotone continuous function then quasiarithmetic sum is defined as follows:

$$S_\alpha(\mathbf{x}; \mathbf{p}) = \alpha^{-1}\left(\sum_{i=1}^{n} p_i \alpha(x_i) \right). \tag{3.14}$$

Theorem 3.4. *Let* $\mathbf{x}$ *and* $\mathbf{p}$ *be two positive* n*-tuples* $(n \geq 2)$*, all* x_i*'s are not equal and let* $\alpha, \beta, \in C^1(I)$ *be strictly monotonic continuous functions,* $\gamma \in C^1(I)$ *be positive strictly increasing continuous function, where* $I = (0, a]$ *and*

$$\sum_{i=1}^{n} p_i \gamma(x_i) \geq \gamma(x_j), \quad \text{for } j = 1, \ldots, n, \qquad \sum_{i=1}^{n} p_i \gamma(x_i) \in (0, \gamma(a)]. \tag{3.15}$$

Then there exists η *from* $(0, \gamma(a)]$ *such that*

$$\frac{\alpha(S_\gamma(\mathbf{x}; \mathbf{p})) - \alpha(S_\alpha(\mathbf{x}; \mathbf{p}))}{\beta(S_\gamma(\mathbf{x}; \mathbf{p})) - \beta(S_\beta(\mathbf{x}; \mathbf{p}))} = \frac{\gamma(\eta)\alpha'(\eta) - \gamma'(\eta)\alpha(\eta)}{\gamma(\eta)\beta'(\eta) - \gamma'(\eta)\beta(\eta)} \tag{3.16}$$

is valid, provided that all denominators are not zero.

Proof. If we choose the functions f and g so that $f = \alpha \circ \gamma^{-1}$, $g = \beta \circ \gamma^{-1}$, and $x_i \to \gamma(x_i)$. Substituting these in (3.8),

$$\frac{\alpha(S_\gamma(\mathbf{x}; \mathbf{p})) - \alpha(S_\alpha(\mathbf{x}; \mathbf{p}))}{\beta(S_\gamma(\mathbf{x}; \mathbf{p})) - \beta(S_\beta(\mathbf{x}; \mathbf{p}))} = \frac{\xi(\alpha \circ \gamma^{-1})'(\xi) - \gamma' \circ \gamma^{-1}(\xi)\alpha \circ \gamma^{-1}(\xi)}{\xi(\beta \circ \gamma^{-1})'(\xi) - \gamma' \circ \gamma^{-1}(\xi)\beta \circ \gamma^{-1}(\xi)}. \tag{3.17}$$

Then by setting $\gamma^{-1}(\eta) = \xi$, we get (3.16). □

Corollary 3.5. *Let* $\mathbf{x}$ *and* $\mathbf{p}$ *be two nonnegative* n*-tuples and let* $t, r, s \in \mathbb{R}^+$*. Then*

$$A_{t,r}^{s}(\mathbf{x}; \mathbf{p}) = \eta. \tag{3.18}$$

Proof. If t, r, and s are pairwise distinct, then we put $\alpha(x) = x^t$, $\beta(x) = x^r$, and $\gamma(x) = x^s$ in (3.16) to get (3.18).

For other cases, we can consider limit as in Remark (2.8). □

Acknowledgment

The authors are really very thankful to Mr. Martin J. Bohner for his useful suggestions.

References

[1] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, vol. 187 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1992.

[2] S. Simić, "On logarithmic convexity for differences of power means," *Journal of Inequalities and Applications*, vol. 2007, Article ID 37359, 8 pages, 2007.

[3] M. Anwar and J. E. Pečarić, "On logarithmic convexity for differences of power means," to appear in *Mathematical Inequalities & Applications*.

[4] M. Anwar and J. E. Pečarić, "New means of Cauchy's type," *Journal of Inequalities and Applications*, vol. 2008, Article ID 163202, 10 pages, 2008.