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Research Article

The Weighted Square Integral Inequalities for the First Derivative of the Function of a Real Variable

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We generalize the square integral estimate for the derivative of the convex function by Shashiashvili (2005) to the case of the family of the weight functions, satisfying certain conditions. This kind of generalization is especially valuable in the problems of mathematical finance for construction of the discrete time hedging strategies.

1. Introduction

The role of mathematical inequalities within the mathematical branches as well as in its enormous applications should not be underestimated. The appearance of the new mathematical inequality often puts on firm foundation for the heuristic algorithms and procedures used in applied sciences.

The present paper considers new type of weighted square integral inequalities for the first derivative of the convex function, the particular case which has been originally established by K. Shashiashvili and M. Shashiashvili [1] and subsequently applied to the hedging problems of mathematical finance by Hussain and Shashiashvili [2].

The convexity property of the value functions of the various problems in finance leads to deep and unexpected results of great practical importance for the traders and practitioners dealing with the real-world financial markets. For example, it is shown in the article [3] by El Karoui et al. (see also Hobson [4]) that the value functions of the European as well as American options are convex with respect to the underlying stock price; and the latter property gives us the following remarkable robustness result. Even if the writer of the option uses incorrect mathematical model to describe the dynamics of stock prices, he is able to carry out his liabilities if only the incorrectly chosen volatility dominates the true volatility function.

The present paper is organized as follows. In Section 2 we prove the weighted square integral estimates for the first derivative of a function that is assumed to be twice continuously differentiable. Afterwards in Section 3 we consider more general case of the arbitrary convex functions which are not supposed even one time continuously differentiable. We emphasize the fact that the latter case can be directly applied to problems of mathematical finance, especially to discrete time hedging of the European as well as American call options.

2. The weighted square integral estimates for the first derivative of a twice continuously differentiable function

In this section, we consider the pair of twice continuously differential functions $f(x)$ and $g(x)$ defined on the closed bounded interval $[a,b]$. We assume that the function $g(x)$ is convex (that is $g''(x) \geq 0$) and the following requirement is satisfied:

$$|f''(x)| \leq g''(x), \quad a \leq x \leq b. \tag{2.1}$$

Introduce the family of nonnegative twice continuously differentiable weight functions $H(x),\ a \leq x \leq b$, which satisfy the condition

$$H(a) = H(b) = 0, \qquad H'(a) = H'(b) = 0. \tag{2.2}$$

Theorem 2.1. *Let $f(x)$ and $g(x)$ be two twice continuously differentiable functions defined on the bounded interval $[a,b]$, which satisfy the requirement* (2.1) *and let$H(x),\ a \leq x \leq b$ be arbitrary nonnegative weight function such that condition* (2.2) *is fulfilled. Then the following inequality is valid:*

$$\int_a^b (f'(x))^2 H(x)dx \leq \int_a^b \left[\frac{f^2(x)}{2} + \left(\sup_{a\leq x\leq b}|f(x)|\right)g(x)\right]H''(x)dx. \tag{2.3}$$

Proof. Using the integration-by-parts formula in the integral below, we have

$$\begin{aligned}\int_a^b (f'(x))^2 H(x)dx &= f(x)\,f'(x)\,H(x)\Big|_a^b - \int_a^b (f'H)'(x)\,f(x)dx \\ &= -\int_a^b f(x)\,f'(x)\,H'(x)dx - \int_a^b f(x)\,f''(x)\,H(x)dx \\ &= -\frac{1}{2}\int_a^b (f^2)'(x)\,H'(x)dx - \int_a^b f(x)\,f''(x)\,H(x)dx\end{aligned} \tag{2.4}$$

as $H(a) = H(b) = 0$.

Let us transform the integral

$$\begin{aligned}\frac{1}{2}\int_a^b (f^2)'(x)\,H'(x)dx &= \frac{1}{2}f^2(x)H'(x)\Big|_a^b - \frac{1}{2}\int_a^b f^2(x)H''(x)dx \\ &= -\frac{1}{2}\int_a^b f^2(x)H''(x)dx\end{aligned} \tag{2.5}$$

using the condition $H'(a) = H'(b) = 0$.

Inserting the latter expression in equality (2.4), we obtain the estimate

$$\begin{aligned}\int_a^b (f'(x))^2 H(x)dx &= \frac{1}{2}\int_a^b f^2(x)H''(x)dx - \int_a^b f(x)\, f''(x)\, H(x)dx \\ &\le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \int_a^b |f(x)|\,|f''(x)|H(x)dx \\ &\le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \sup_{a\le x\le b}|f(x)|\int_a^b |f''(x)|H(x)dx.\end{aligned} \tag{2.6}$$

Taking into account requirement (2.1), we get from the latter inequality (2.6)

$$\int_a^b (f'(x))^2 H(x)dx \le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \sup_{a\le x\le b}|f(x)|\int_a^b g''(x)H(x)dx. \tag{2.7}$$

Now we use twice the integration-by-parts formula and obtain

$$\begin{aligned}\int_a^b g''(x)H(x)dx &= g'(x)\, H(x)\Big|_a^b - \int_a^b g'(x)\, H'(x)dx \\ &= -g(x)\, H'(x)\Big|_a^b + \int_a^b g(x)H''(x)dx \\ &= \int_a^b g(x)H''(x)dx.\end{aligned} \tag{2.8}$$

The latter equality together with the previous estimate (2.7) give us the required inequality (2.3). □

Applying Hölder inequality to the right-hand side of estimate (2.3), we get the following.

Corollary 2.2. *For the functions $f(x)$, $g(x)$, and the weight function $H(x)$, satisfying the same conditions as in Theorem 2.1, the following bound is valid*

$$\int_a^b (f'(x))^2 H(x)dx \le \|f\|_\infty \left(\frac{1}{2}\|f\|_p + \|g\|_p\right)\|H''\|_q, \tag{2.9}$$

where $1 \le p \le \infty$, and p and q are conjugate exponents, and

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}, \quad 1 \le p < \infty, \qquad \|f\|_\infty = \sup_{a\le x\le b}|f(x)|. \tag{2.10}$$

Remark 2.3. Let us notice that assumption (2.1), $|f''(x)| \le g''(x)$, is equivalent to the existence of the decomposition of the function $f(x)$ as the difference of two twice continuously differentiable convex functions $f(x) = f_1(x) - f_2(x)$, $a \le x \le b$ such that $f_1(x) + f_2(x) = g(x)$. Indeed the inequality $|f''(x)| \le g''(x)$ is the same as $-g''(x) \le f''(x) \le g''(x)$, that is,

$$f''(x) + g''(x) \ge 0, \qquad g''(x) - f''(x) \ge 0. \tag{2.11}$$

The latter means that the functions

$$f_1(x) = \frac{1}{2}(f(x) + g(x)), \qquad f_2(x) = \frac{1}{2}(g(x) - f(x)) \tag{2.12}$$

are two convex functions such that

$$f(x) = f_1(x) - f_2(x), \qquad g(x) = f_1(x) + f_2(x). \tag{2.13}$$

This remark suggests to write inequality (2.9) in a different form. Take two arbitrary twice continuously differentiable convex functions $f_1(x)$ and $f_2(x)$ and define

$$f(x) = f_1(x) - f_2(x), \qquad g(x) = f_1(x) + f_2(x), \tag{2.14}$$

$(f''(x) = f_1''(x) - f_2''(x)$, and $g''(x) = f_1''(x) + f_2''(x)$, it is obvious that $|f''(x)| \le g''(x))$, then the inequality in Corollary 2.2 will take the following form:

$$\int_a^b (f_1'(x) - f_2'(x))^2 H(x)dx \le \|f_1 - f_2\|_\infty \left[\frac{1}{2}\|f_1 - f_2\|_p + \|f_1 + f_2\|_p\right] \|H''\|_q, \tag{2.15}$$

where $1 \le p \le \infty$.

Consider the special case of the latter inequality when $p = \infty$ and $H(x)$ is of the particular form

$$H(x) = (x-a)^2 (b-x)^2, \quad a \le x \le b. \tag{2.16}$$

Corollary 2.4. *Let $f_1(x)$ and $f_2(x)$ be two twice continuously differentiable convex functions defined on a closed bounded interval $[a,b]$ and let the weight function $H(x)$ be equal to*

$$H(x) = (x-a)^2 (b-x)^2, \quad a \le x \le b. \tag{2.17}$$

Then the following estimate holds

$$\int_a^b (f_1'(x) - f_2'(x))^2 H(x)dx \le \|f_1 - f_2\|_\infty \left[\frac{4\sqrt{3}}{9}\|f_1 + f_2\|_\infty + \frac{2\sqrt{3}}{9}\|f_1 - f_2\|_\infty\right](b-a)^3. \tag{2.18}$$

Proof. We have

$$H''(x) = 12x^2 - 12(a+b)x + 2\left(a^2 + 4ab + b^2\right). \tag{2.19}$$

Calculate the integral

$$\int_a^b |H''(x)|dx = 2\int_a^b |6x^2 - 6(a+b)x + a^2 + 4ab + b^2|dx. \tag{2.20}$$

Let us introduce the change of variable $x = a + u(b-a)$, $0 \le u \le 1$; from the above expression, we obtain

$$\int_a^b |H''(x)|dx = 2(b-a)^3 \int_0^1 |6u^2 - 6u + 1|du = \frac{4\sqrt{3}}{9}(b-a)^3. \tag{2.21}$$

Taking into account the latter expression in estimate (2.9), we come to the desired inequality (2.18). □

Remark 2.5. Comparing the result stated in Corollary 2.4 with Theorem 2.1 from K. Shashiashvili and M. Shashiashvili [1], we come to the conclusion that the multiplier $4\sqrt{3}/9$ is twice less than obtained in the latter paper.

3. The weighted square integral estimates for the difference of derivatives of two convex functions

In this section, we consider two arbitrary finite convex functions $f(x)$ and $g(x)$ on an infinite interval $[0,\infty)$. It is well known that they are continuous and have finite left- and right-hand derivatives $f'(x-)$, $f'(x+)$ and $g'(x-)$, $g'(x+)$ inside the open interval $(0,\infty)$.

We will assume that there exists a positive number A such that if $x \geq A$, we have

$$|f'(x-)| \leq C, \qquad |g'(x-)| \leq C, \tag{3.1}$$

where C is certain positive constant.

Let us assume also that the difference of the functions $f(x)$ and $g(x)$ is bounded on the infinite interval $[0,\infty)$:

$$\sup_{x\geq 0}|f(x)-g(x)| < \infty. \tag{3.2}$$

Introduce now the family of nonnegative twice continuously differentiable weight functions $H(x)$ defined on the open interval $(0,\infty)$, which satisfy the following conditions:

$$\lim_{x\to 0+} H(x)=0, \qquad \lim_{x\to\infty} H(x)=0, \qquad \lim_{x\to 0+} H'(x)=0, \qquad \lim_{x\to\infty} x\,H'(x)=0, \tag{3.3}$$

$$\int_0^\infty (|f(x)|+|g(x)|)\,|H''(x)|dx < \infty \tag{3.4}$$

this integral is understood in the improper sense as the limit

$$\lim_{\substack{\delta\to 0+\\ b\to\infty}} \int_\delta^b (|f(x)|+|g(x)|)\,|H''(x)|dx). \tag{3.5}$$

Theorem 3.1. *For arbitrary two finite convex functions $f(x)$ and $g(x)$ defined on $[0,\infty)$ satisfying conditions* (3.1) *and* (3.2) *and for any nonnegative twice continuously differentiable weight function $H(x)$, $0<x<\infty$, which satisfy conditions* (3.3) *and* (3.4)*, the following energy estimate is valid:*

$$\int_0^\infty (f'(x-)-g'(x-))^2\,H(x)dx \leq \frac{3}{2}\sup_{x\geq 0}|f(x)-g(x)|\int_0^\infty (|f(x)|+|g(x)|)\,|H''(x)|dx, \tag{3.6}$$

where $f'(x-)$ and $g'(x-)$ denote the left derivatives of the convex functions $f(x)$ and $g(x)$, respectively.

Proof. We will prove the theorem in two stages. On first stage, we verify the validity of the statement for twice continuously differentiable convex functions satisfying conditions (3.1) and (3.2) and on second stage we approximate arbitrary convex functions satisfying the same conditions by smooth ones inside the interval $(0,\infty)$ in an appropriate manner and afterwards we pass onto limit in the previously established estimate.

Thus let us assume at first that $f(x)$ and $g(x)$ are two convex functions defined on the interval $[0,\infty)$ which are twice continuously differentiable in the open interval $(0,\infty)$ and satisfy conditions (3.1) and (3.2).

Introduce new function $F(x)$ as follows:

$$F(x) = f(x) - g(x), \qquad 0 \le x < \infty, \tag{3.7}$$

then $F(x)$ is twice continuously differentiable inside the infinite interval $(0, \infty)$ and at point zero, it has finite limit $F(0+)$.

Consider the following integral on a finite interval $[\delta, b]$ and use in it the integration by parts formula (here δ and b are arbitrary strictly positive numbers),

$$\begin{aligned}\int_\delta^b F'(x)(FH)'(x)dx &= F'(x)F(x)H(x)\Big|_\delta^b - \int_\delta^b F''(x)(F(x)H(x))dx \\ &= F(b)F'(b)H(b) - F(\delta)F'(\delta)H(\delta) - \int_\delta^b F''(x)F(x)H(x)dx.\end{aligned} \tag{3.8}$$

Now bound the absolute value of the last integral in (3.8):

$$\begin{aligned}\left|\int_\delta^b F''(x)F(x)H(x)dx\right| &\le \sup_{\delta\le x\le b}|F(x)|\int_\delta^b |f''(x) - g''(x)|H(x)dx \\ &\le \sup_{\delta\le x\le b}|F(x)|\int_\delta^b (f''(x) + g''(x))H(x)dx,\end{aligned} \tag{3.9}$$

as $f''(x) \ge 0,\ g''(x) \ge 0,\ 0 < x < \infty$.

Let us transform the integral on the right-hand side of inequality (3.9):

$$\begin{aligned}\int_\delta^b (f''(x) + g''(x))H(x)dx &= (f'(x) + g'(x))H(x)\Big|_\delta^b - \int_\delta^b (f'(x) + g'(x))H'(x)dx \\ &= (f'(b) + g'(b))H(b) - (f'(\delta) + g'(\delta))H(\delta) \\ &\quad - \left\{(f(x) + g(x))H'(x)\Big|_\delta^b - \int_\delta^b (f(x) + g(x))H''(x)dx\right\},\end{aligned} \tag{3.10}$$

which implies

$$\begin{aligned}\int_\delta^b (f''(x) + g''(x))H(x)dx &= (f'(b) + g'(b))H(b) - (f'(\delta) + g'(\delta))H(\delta) - (f(b) + g(b))H'(b) \\ &\quad + (f(\delta) + g(\delta))H'(\delta) + \int_\delta^b (f(x) + g(x))H''(x)dx.\end{aligned} \tag{3.11}$$

Using the above expression in inequality (3.9), we obtain the estimate

$$\begin{aligned}\left|\int_\delta^b F''(x)F(x)H(x)dx\right| \le \sup_{\delta\le x\le b}|F(x)|\Big\{&|f'(b) + g'(b)|H(b) + |f'(\delta) + g'(\delta)|H(\delta) \\ &+ |f(b) + g(b)||H'(b)| + |f(\delta) + g(\delta)||H'(\delta)| \\ &+ \int_\delta^b |f(x) + g(x)||H''(x)|dx\Big\}.\end{aligned} \tag{3.12}$$

Thus from equality (3.8), we come to the following bound:

$$\begin{aligned}\left|\int_\delta^b F'(x)(FH)'(x)dx\right| &\le |F(b)F'(b)|H(b) + |F(\delta)F'(\delta)|H(\delta)\\ &\quad + \sup_{\delta\le x\le b}|F(x)|\Big\{|f'(b)+g'(b)|H(b) + |f'(\delta)+g'(\delta)|H(\delta)\\ &\quad + |f(b)+g(b)||H'(b)| + |f(\delta)+g(\delta)||H'(\delta)|\\ &\quad + \int_\delta^b |f(x)+g(x)||H''(x)|dx\Big\}.\end{aligned} \tag{3.13}$$

On the other hand,

$$\int_\delta^b F'(x)(FH)'(x)dx = \int_\delta^b (F'(x))^2\, H(x)dx + \int_\delta^b F(x)\, F'(x)\, H'(x)dx. \tag{3.14}$$

From here we have the chain of equalities

$$\begin{aligned}\int_\delta^b (F'(x))^2 H(x)dx &= \int_\delta^b F'(x)(FH)'(x)dx - \frac{1}{2}\int_\delta^b (F^2)'(x)H'(x)dx\\ &= \int_\delta^b F'(x)(FH)'(x)dx - \frac{1}{2}\left\{F^2(x)H'(x)\Big|_\delta^b - \int_\delta^b F^2(x)H''(x)dx\right\}\\ &= \int_\delta^b F'(x)(FH)'(x)dx - \frac{1}{2}F^2(b)H'(b) + \frac{1}{2}F^2(\delta)H'(\delta)\\ &\quad + \frac{1}{2}\int_\delta^b F^2(x)H''(x)dx.\end{aligned} \tag{3.15}$$

Using bound (3.13) in expression (3.15), we arrive to the estimate

$$\begin{aligned}&\int_\delta^b (F'(x))^2 H(x)dx\\ &\quad\le \frac{1}{2}\,F^2(b)\,|H'(b)| + \frac{1}{2}F^2(\delta)\,|H'(\delta)| + |F(b)F'(b)|\,H(b)\\ &\qquad + |F(\delta)F'(\delta)|\,H(\delta) + \sup_{\delta\le x\le b}|F(x)|\\ &\qquad \times\left\{\frac{3}{2}\int_\delta^b (|f(x)|+|g(x)|)|H''(x)|dx + |f'(b)+g'(b)|H(b) + |f'(\delta)+g'(\delta)|\,H(\delta)\right.\\ &\qquad \left. + |f(b)+g(b)|\,|H'(b)| + |f(\delta)+g(\delta)|\,|H'(\delta)|\right\}.\end{aligned} \tag{3.16}$$

It is well known that any convex function is locally absolutely continuous (see, e.g., [5, Proposition 17 of Chapter 5]), that is,

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'(u-)du, \quad 0 < x_1 \le x_2 < \infty, \tag{3.17}$$

where $f'(u-)$ denotes the left-derivative of the convex function $f(x)$ at point u. As the left-hand derivative $f'(x-)$ of the convex function $f(x)$ is nondecreasing function, we have

$$f'(x_1-) \le f'(u-) \le f'(x_2-), \quad \text{if } 0 < x_1 \le u \le x_2 < \infty. \tag{3.18}$$

Therefore, from expression (3.17), we find

$$f'(x_1-)\,(x_2 - x_1) \le f(x_2) - f(x_1) \le f'(x_2-)(x_2 - x_1), \tag{3.19}$$

where $0 < x_1 \le x_2 < \infty$.

Taking $x_2 = 2x_1$, we get

$$f'(x_1-)\,x_1 \le f(2x_1) - f(x_1). \tag{3.20}$$

As x_1 is arbitrary positive number, we have

$$f'(x-)\,x \le f(2x) - f(x) \quad \text{for } x > 0. \tag{3.21}$$

On the other hand, letting x_1 to zero in inequality (3.19), we write

$$f(x_2) - f(0+) \le f'(x_2-)\,x_2, \tag{3.22}$$

that is,

$$f(x) - f(0+) \le f'(x-)\,x, \quad x > 0. \tag{3.23}$$

Ultimately we obtain the two-sided inequality

$$f(x) - f(0+) \le f'(x-)\,x \le f(2x) - f(x) \quad \text{for } x > 0, \tag{3.24}$$

which gives

$$\lim_{x\to 0+} x f'(x-) = 0 \quad \Bigl(\text{similarly } \lim_{x\to 0+} x g'(x-) = 0\Bigr). \tag{3.25}$$

By equality (3.17) and using condition (3.1), we obtain the bound

$$|f(b)| \le |f(A)| + C(b - A) \le |f(A)| + Cb \quad \text{for } A \le b. \tag{3.26}$$

But we have

$$|f(A)| = \frac{|f(A)|}{A} A \le \frac{|f(A)|}{A} b \quad \text{if } A \le b. \tag{3.27}$$

Therefore we can write

$$|f(b)H'(b)| \le |f(A)H'(b)| + Cb|H'(b)| \le \left(\frac{|f(A)|}{A} + C\right) b|H'(b)| \quad \text{if } A \le b \tag{3.28}$$

and similar bound is valid for $|g(b)H'(b)|$

$$|g(b)H'(b)| \le \left(\frac{|g(A)|}{A} + C\right) b|H'(b)| \quad \text{for } A \le b. \tag{3.29}$$

Using condition (3.3) and bounds (3.28) and (3.29), we get

$$\overline{\lim_{b\to\infty}} F^2(b)|H'(b)| \le \sup_{0\le x<\infty} |F(x)| \overline{\lim_{b\to\infty}} (|f(b)| + |g(b)|) |H'(b)| = 0, \tag{3.30}$$

$$\overline{\lim_{b\to\infty}} |f(b) + g(b)||H'(b)| = 0. \tag{3.31}$$

Moreover, from conditions (3.1) and (3.3), we find

$$\begin{aligned}
\lim_{\delta\to 0+} F^2(\delta)|H'(\delta)| &= (f(0+) - g(0+))^2 \lim_{\delta\to 0+} |H'(\delta)| = 0,\\
\overline{\lim_{b\to\infty}} |F(b)F'(b-)|H(b) &\le \sup_{0\le x<\infty} |F(x)| \overline{\lim_{b\to\infty}} (|f'(b-)| + |g'(b-)|) H(b)\\
&\le 2C \sup_{0\le x<\infty} |F(x)| \overline{\lim_{b\to\infty}} H(b) = 0,\\
\overline{\lim_{b\to\infty}} |f'(b-) + g'(b-)| H(b) &\le 2C \overline{\lim_{b\to\infty}} H(b) = 0,\\
\lim_{\delta\to 0+} |f(\delta) + g(\delta)| |H'(\delta)| &= |f(0+) + g(0+)| \lim_{\delta\to 0+} |H'(\delta)| = 0.
\end{aligned} \tag{3.32}$$

By the mean value theorem, we have

$$\frac{H(\delta)}{\delta} = \frac{H(\delta) - H(0+)}{\delta} = H'(\vartheta_\delta), \quad \text{where } 0 < \vartheta_\delta < \delta, \tag{3.33}$$

therefore from condition (3.3), we get

$$\lim_{\delta\to 0+} \frac{H(\delta)}{\delta} = 0, \tag{3.34}$$

using the limit relations above and (3.25) we find

$$\begin{aligned}
\lim_{\delta\to 0+} |F(\delta)F'(\delta-)|H(\delta) &\le \sup_{0\le x<\infty} |F(x)| \lim_{\delta\to 0+} |f'(\delta-) - g'(\delta-)|H(\delta)\\
&\le \sup_{0\le x<\infty} |F(x)| \lim_{\delta\to 0+} \left(|\delta f'(\delta-)| \frac{H(\delta)}{\delta} + |\delta g'(\delta-)| \frac{H(\delta)}{\delta} \right) = 0,
\end{aligned} \tag{3.35}$$

and similarly

$$\lim_{\delta\to 0+} |f'(\delta-) + g'(\delta-)|H(\delta) = 0. \tag{3.36}$$

Now we have to pass onto limit when $b \to \infty$ and $\delta \to 0$ in inequality (3.16). Obviously, the left-hand side of the inequality increases and the right-hand side is bounded, when $b \to \infty$, $\delta \to 0$, therefore the left-hand side also converges to finite limit. Passing onto limit $\delta \to 0$, $b \to \infty$ in inequality (3.16) using assumption (3.4) and the limit relations (3.30)–(3.36), we come to the required estimate (3.6).

Next we move to the second stage of the proof. Consider two arbitrary convex functions $f(x)$ and $g(x)$ defined on $[0, \infty)$, satisfying conditions (3.1) and (3.2). We have to construct the sequences of twice continuously differentiable (in the open interval $(0, \infty)$)

convex functions $f_n(x)$ and $g_n(x)$ approximating, respectively, the functions $f(x)$ and $g(x)$ inside the interval $(0,\infty)$ in an appropriate manner.

To construct such sequences, we will use the following smoothing function:

$$\rho(x) = \begin{cases} c \cdot \exp\left[\dfrac{1}{x(x-2)}\right]; & 0 < x < 2, \\ 0; & \text{otherwise}, \end{cases} \tag{3.37}$$

where the factor c is chosen to satisfy the equality

$$\int_0^2 \rho(x)dx = 1. \tag{3.38}$$

Define for $x \in [0,\infty)$

$$\begin{aligned} f_n(x) &= \int_0^\infty n\rho(n(x-y))f(y)dy, \\ g_n(x) &= \int_0^\infty n\rho(n(x-y))g(y)dy, \end{aligned} \tag{3.39}$$

where $n = 1, 2, \ldots$.

For arbitrary fixed $\delta > 0$ consider the restriction of functions $f_n(x)$ and $g_n(x)$ on the interval $[\delta, b]$ and let $n \geq 4/\delta$. Then $nx \geq 4$ for $x \in [\delta, b]$.

Perform in (3.39) the change of variable $z = n(x-y)$, then we find

$$\begin{aligned} f_n(x) &= \int_{-\infty}^{nx} \rho(z)f\left(x - \frac{z}{n}\right)dz, \\ g_n(x) &= \int_{-\infty}^{nx} \rho(z)g\left(x - \frac{z}{n}\right)dz. \end{aligned} \tag{3.40}$$

Since the function $\rho(z)$ is equal to zero outside the interval $(0,2)$, we can write

$$\begin{aligned} f_n(x) &= \int_0^2 \rho(z)f\left(x - \frac{z}{n}\right)dz, \\ g_n(x) &= \int_0^2 \rho(z)g\left(x - \frac{z}{n}\right)dz, \end{aligned} \tag{3.41}$$

if $x \in [\delta, b],\ n \geq 4/\delta$.

From definition (3.39), it is obvious that the functions $f_n(x)$ and $g_n(x)$ are infinitely differentiable, while their convexity follows from the expressions (3.41).

Now we will show the uniform convergence of the sequence $f_n(x)$ to $f(x)$ on the interval $[\delta, b]$ (similarly, the uniform convergence of $g_n(x)$ to $g(x)$). For this purpose, we use the uniform continuity of the function $f(x)$ on the interval $[\delta/2, b]$. For fix $\varepsilon > 0$ there exists $\widehat{\delta} > 0$ such that we have

$$|f(x_2) - f(x_1)| \leq \varepsilon \quad \text{if } |x_2 - x_1| < \widehat{\delta}, \quad x_1, x_2 \in \left[\frac{\delta}{2}, b\right]. \tag{3.42}$$

Take $n \geq \max\{4/\delta, 4/\widehat{\delta}\}$. Then for $0 \leq z \leq 2$ and $x \in [\delta, b]$, we get

$$\frac{z}{n} \leq \min\left\{\frac{\delta}{2}, \frac{\widehat{\delta}}{2}\right\}, \qquad x - \frac{z}{n} \geq \frac{\delta}{2}. \tag{3.43}$$

Hence

$$\left| f\left(x - \frac{z}{n}\right) - f(x) \right| \leq \varepsilon \quad \text{for } n \geq \max\left\{\frac{4}{\delta}, \frac{4}{\widehat{\delta}}\right\} \tag{3.44}$$

and consequently

$$|f_n(x) - f(x)| = \left| \int_0^2 \rho(z) \left(f\left(x - \frac{z}{n}\right) - f(x) \right) dz \right| \leq \varepsilon \tag{3.45}$$

for $x \in [\delta, b]$ and $n \geq \max\{4/\delta, 4/\widehat{\delta}\}$.

This shows the uniform convergence of the sequence $f_n(x)$ to $f(x)$ (and $g_n(x)$ to $g(x)$) on the interval $[\delta, b]$.

Next we need to differentiate (3.41). For this purpose, we will use the following inequality ([5, page 114]) concerning convex function $f(x)$ and its left-derivative $f'(x-)$

$$f'(x_1-) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2-), \quad \text{if } 0 < x_1 < x_2 < \infty. \tag{3.46}$$

Take therein

$$x_1 = \left(x - \frac{z}{n}\right) - h, \quad x_2 = x - \frac{z}{n}, \tag{3.47}$$

where $0 < h < \delta/4$.

We have

$$f'\left(\left(x - \frac{z}{n} - h\right) - \right) \leq \frac{f(x - z/n) - f(x - z/n - h)}{h} \leq f'\left(\left(x - \frac{z}{n}\right) - \right) \tag{3.48}$$

for $x \in [\delta, b],\ 0 \leq z \leq 2,\ 0 < h < \delta/4,$ and $n \geq 4/\delta$.

It is well known that the left derivative of the convex function is nondecreasing and as

$$x - \frac{z}{n} - h \geq \frac{\delta}{4}, \qquad x - \frac{z}{n} \leq b, \tag{3.49}$$

we can write

$$f'\left(\frac{\delta}{4} - \right) \leq \frac{f(x - z/n) - f(x - z/n - h)}{h} \leq f'(b-), \tag{3.50}$$

which shows that the family of functions

$$\Phi_h^{n,x}(z) = \frac{f(x - z/n) - f(x - z/n - h)}{h} \tag{3.51}$$

is uniformly bounded by the constant $D = |f'(b-)| + |f'((\delta/4)-)|$ if only $x \in [\delta, b]$, $0 \le z \le 2$, $0 < h < (\delta/4)$, and $n \ge (4/\delta)$.

Using expression (3.41), we can write

$$\frac{f_n(x) - f_n(x-h)}{h} = \int_0^2 \rho(z) \frac{f(x - z/n) - f(x - z/n - h)}{h} dz. \tag{3.52}$$

Taking limit as h tends to zero and using bounded convergence theorem, we obtain the formula

$$f_n'(x) = \int_0^2 \rho(z) f'\left(\left(x - \frac{z}{n}\right) - \right) dz \tag{3.53}$$

for $x \in [\delta, b]$ and $n \ge 4/\delta$.

Using (3.53) let us show that for fixed $x \in [\delta, b]$, the sequence $f_n'(x)$ converges to the left-derivative $f'(x-)$.

We have

$$f_n'(x) - f'(x-) = \int_0^2 \rho(z) \left(f'\left(\left(x - \frac{z}{n}\right) - \right) - f'(x-) \right) dz, \tag{3.54}$$

where $n \ge 4/\delta$. Choose arbitrary $\varepsilon > 0$. Since the left-derivative $f'(x-)$ is left continuous, we can find $N(\varepsilon)$ such that (for $0 \le z \le 2$):

$$\left| f'\left(\left(x - \frac{z}{n}\right) - \right) - f'(x-) \right| \le \varepsilon \quad \text{if only } n \ge N(\varepsilon). \tag{3.55}$$

Hence we get

$$|f_n'(x) - f'(x-)| \le \int_0^2 \rho(z)\, \varepsilon \, dz = \varepsilon \quad \text{if } x \in [\delta, b],\ n \ge \max\left\{ \frac{4}{\delta}, N(\varepsilon) \right\}, \tag{3.56}$$

that is,

$$\lim_{n \to \infty} f_n'(x) = f'(x-) \qquad \text{for fixed } x \in [\delta, b] \tag{3.57}$$

and similarly we have

$$\lim_{n \to \infty} g_n'(x) = g'(x-) \quad \text{if } x \in [\delta, b]. \tag{3.58}$$

Let us write estimate (3.16) for the function $F_n(x) = f_n(x) - g_n(x)$ restricted to the interval $[\delta, b]$,

$$\begin{aligned} \int_\delta^b (F_n'(x))^2 H(x) dx \le {} & \frac{1}{2} F_n^2(b) \, |H'(b)| + \frac{1}{2} F_n^2(\delta) \, |H'(\delta)| + |F_n(b) F_n'(b)| \, H(b) \\ & + |F_n(\delta) F_n'(\delta)| \, H(\delta) + \sup_{\delta \le x \le b} |F_n(x)| \\ & \times \left\{ \frac{3}{2} \int_\delta^b (|f_n(x)| + |g_n(x)|) |H''(x)| dx \right. \\ & \quad + |f_n'(b) + g_n'(b)| H(b) + |f_n'(\delta) + g_n'(\delta)| \, H(\delta) \\ & \quad \left. + |f_n(b) + g_n(b)| \, |H'(b)| + |f_n(\delta) + g_n(\delta)| \, |H'(\delta)| \right\}. \end{aligned} \tag{3.59}$$

For $x \in [\delta, b],\ 0 \le z \le 2,$ and $n \ge 4/\delta$, we have

$$f'\left(\frac{\delta}{2}-\right) \le f'\left(\left(x-\frac{z}{n}\right)-\right) \le f'(b-). \tag{3.60}$$

Multiplying this inequality by $\rho(z)$ and integrating by z over $(0,2)$ from expression (3.53), we obtain

$$f'\left(\frac{\delta}{2}-\right) \le f_n'(x) \le f'(b-), \tag{3.61}$$

from which it follows that

$$|f_n'(x)| \le |f'(b-)| + \left|f'\left(\frac{\delta}{2}-\right)\right|, \quad \text{if } x \in [\delta, b],\ n \ge \frac{4}{\delta}. \tag{3.62}$$

Similarly for the functions $g_n'(x)$, we can write

$$|g_n'(x)| \le |g'(b-)| + \left|g'\left(\frac{\delta}{2}-\right)\right|. \tag{3.63}$$

From the latter bounds, we obtain

$$|F_n'(x)| \le |f'(b-)| + |g'(b-)| + \left|f'\left(\frac{\delta}{2}-\right)\right| + \left|g'\left(\frac{\delta}{2}-\right)\right|, \tag{3.64}$$

if $x \in [\delta, b]$ and $n \ge 4/\delta$.

Hence the sequence of the functions $F_n'(x)$ is uniformly bounded on the interval $[\delta, b]$ for $n \ge 4/\delta$. Thus we can apply the bounded convergence theorem in the left-hand side of inequality (3.59) tending n to infinity, we will have

$$\begin{aligned}
&\int_\delta^b (F'(x-))^2 H(x)dx \\
&\quad \le \frac{1}{2}F^2(b)\,|H'(b)| + \frac{1}{2}F^2(\delta)|H'(\delta)| + |F(b)F'(b-)|H(b) \\
&\qquad + |F(\delta)F'(\delta-)|H(\delta) + \sup_{\delta \le x \le b}|F(x)| \\
&\qquad \times \left\{\frac{3}{2}\int_\delta^b (|f(x)| + |g(x)|)|H''(x)|dx + |f'(b-) + g'(b-)|H(b) + |f'(\delta-) + g'(\delta-)|H(\delta) \right. \\
&\qquad\qquad \left. + |f(b) + g(b)||H'(b)| + |f(\delta) + g(\delta)||H'(\delta)| \right\}.
\end{aligned} \tag{3.65}$$

Finally, it remains to pass onto limit when $b \to \infty$ and $\delta \to 0$ in the latter inequality. The left-hand side of inequality (3.65) obviously increases when $b \to \infty$ and $\delta \to 0$ and the right-hand side is bounded by the assumption (3.4) and the limit relations (3.30)–(3.36). Therefore passing onto limit $b \to \infty$ and $\delta \to 0$ in inequality (3.65), we arrive to the desired estimate (3.6). □

References

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