

Research Article

A Note on $|A|_k$ Summability Factors for Infinite Series

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We obtain sufficient conditions on a nonnegative lower triangular matrix A and a sequence λ_n for the series $\sum a_n \lambda_n / na_{nn}$ to be absolutely summable of order $k \geq 1$ by A .

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A weighted mean matrix, denoted by (\bar{N}, p_n) , is a lower triangular matrix with entries p_k/P_n , where $\{p_k\}$ is a nonnegative sequence with $p_0 > 0$, and $P_n := \sum_{k=0}^n p_k$.

Mishra and Srivastava [1] obtained sufficient conditions on a sequence $\{p_k\}$ and a sequence $\{\lambda_n\}$ for the series $\sum a_n P_n \lambda_n / n p_n$ to be absolutely summable by the weighted mean matrix (\bar{N}, p_n) . Bor [2] extended this result to absolute summability of order $k \geq 1$. Unfortunately, an incorrect definition of absolute summability was used.

In this note, we establish the corresponding result for a nonnegative triangle, using the correct definition of absolute summability of order $k \geq 1$, (see [3]). As a corollary, we obtain the corrected version of Bor's result.

Let A be an infinite lower triangular matrix. We may associate with A two lower triangular matrices \bar{A} and \hat{A} , whose entries are defined by

$$\bar{a}_{nk} = \sum_{i=k}^n a_{ni}, \quad \hat{a}_{nk} = \bar{a}_{nk} - \bar{a}_{n-1,k}, \tag{1}$$

respectively. The motivation for these definitions will become clear as we proceed.

Let A be an infinite matrix. The series $\sum a_k$ is said to be absolutely summable by A , of order $k \geq 1$, written as $|A|_k$, if

$$\sum_{k=0}^{\infty} n^{k-1} |\Delta t_{n-1}|^k < \infty, \tag{2}$$

where Δ is the forward difference operator and t_n denotes the n th term of the matrix transform of the sequence $\{s_n\}$, where $s_n := \sum_{k=0}^n a_k$.

Thus

$$\begin{aligned}
 t_n &= \sum_{k=0}^n a_{nk} s_k = \sum_{k=0}^n a_{nk} \sum_{\nu=0}^k a_\nu = \sum_{\nu=0}^n a_\nu \sum_{k=\nu}^n a_{nk} = \sum_{\nu=0}^n \bar{a}_{n\nu} a_\nu, \\
 t_n - t_{n-1} &= \sum_{\nu=0}^n \bar{a}_{n\nu} a_\nu - \sum_{\nu=0}^{n-1} \bar{a}_{n-1,\nu} a_\nu = \sum_{\nu=0}^n \hat{a}_{n\nu} a_\nu,
 \end{aligned}
 \tag{3}$$

since $\bar{a}_{n-1,n} = 0$.

The result to be proved is the following.

THEOREM 1. *Let A be a triangle with nonnegative entries satisfying*

- (i) $\bar{a}_{n0} = 1, n = 0, 1, \dots$,
- (ii) $a_{n-1,\nu} \geq a_{n\nu}$ for $n \geq \nu + 1$,
- (iii) $na_{nn} \asymp O(1)$,
- (iv) $\Delta(1/a_{nn}) = O(1)$,
- (v) $\sum_{\nu=0}^n a_{\nu\nu} |a_{n,\nu+1}| = O(a_{nn})$.

If $\{X_n\}$ is a positive nondecreasing sequence and the sequences $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy

- (vi) $|\Delta\lambda_n| \leq \beta_n$,
- (vii) $\lim \beta_n = 0$,
- (viii) $|\lambda_n| X_n = O(1)$,
- (ix) $\sum_{n=1}^\infty n X_n |\Delta\beta_n| < \infty$,
- (x) $T_n := \sum_{\nu=1}^n (|s_\nu|^k / \nu) = O(X_n)$,

then the series $\sum_{\nu=1}^\infty a_n \lambda_n / na_{nn}$ is summable $|A|_k, k \geq 1$.

The proof of the theorem requires the following lemma.

LEMMA 2 (see Mishra and Srivastava [1]). *Let $\{X_n\}$ be a positive nondecreasing sequence and the sequences $\{\beta_n\}, \{\lambda_n\}$ satisfy conditions (vi)–(ix) of Theorem 1. Then*

$$nX_n\beta_n = O(1), \tag{4}$$

$$\sum_{n=1}^\infty \beta_n X_n < \infty. \tag{5}$$

Since $\{X_n\}$ is nondecreasing, $X_n \geq X_0$, which is a positive constant. Hence condition (viii) implies that λ_n is bounded. It also follows from (4) that $\beta_n = O(1/n)$, and hence that $\Delta\lambda_n = O(1/n)$ by condition (iv).

Proof. Let T_n denote the n th term of the A -transform of the series $\sum (a_n \lambda_n) / (na_{nn})$. Then we may write

$$T_n = \sum_{\nu=0}^n a_{n\nu} \sum_{i=0}^\nu \frac{a_i \lambda_i}{a_{ii} i} = \sum_{i=0}^m \frac{a_i \lambda_i}{a_{ii} i} \sum_{\nu=i}^n a_{n\nu} = \sum_{i=0}^n \bar{a}_{ni} \frac{a_i \lambda_i}{a_{ii} i}. \tag{6}$$

Thus,

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{i=0}^n \bar{a}_{ni} \frac{a_i \lambda_i}{a_{ii} i} - \sum_{i=0}^{n-1} \bar{a}_{n-1,i} \frac{a_i \lambda_i}{a_{ii} i} = \sum_{i=0}^n (\bar{a}_{ni} - \bar{a}_{n-1,i}) \frac{a_i \lambda_i}{a_{ii} i} = \sum_{i=0}^n \hat{a}_{ni} \frac{a_i \lambda_i}{a_{ii} i} \\
&= \sum_{i=0}^n \hat{a}_{ni} \frac{\lambda_i}{a_{ii} i} (s_i - s_{i-1}) = \sum_{i=0}^{n-1} \hat{a}_{ni} \frac{\lambda_i}{a_{ii} i} s_i + a_{nn} \frac{\lambda_n}{a_{nn} n} s_n - \sum_{i=0}^n \hat{a}_{ni} \frac{\lambda_i s_{i-1}}{a_{ii} i} \\
&= \sum_{i=0}^{n-1} \hat{a}_{ni} \frac{\lambda_i}{a_{ii} i} s_i + a_{nn} \frac{\lambda_n}{a_{nn} n} s_n - \sum_{i=0}^{n-1} \hat{a}_{n,i+1} \frac{\lambda_{i+1} s_i}{(i+1) a_{i+1,i+1}} \\
&= \sum_{i=0}^n \left(\hat{a}_{ni} \frac{\lambda_i}{a_{ii} i} - \hat{a}_{n,i+1} \frac{\lambda_{i+1}}{(i+1) a_{i+1,i+1}} \right) s_i + a_{nn} \frac{\lambda_n}{n a_{nn}}.
\end{aligned} \tag{7}$$

We may write

$$\begin{aligned}
\frac{\hat{a}_{ni} \lambda_i}{i a_{ii}} - \frac{\hat{a}_{n,i+1} \lambda_{i+1}}{(i+1) a_{i+1,i+1}} &= \frac{\hat{a}_{ni} \lambda_i}{i a_{ii}} - \frac{\hat{a}_{n,i+1} \lambda_{i+1}}{(i+1) a_{i+1,i+1}} + \frac{\hat{a}_{n,i+1} \lambda_i}{(i+1) a_{i+1,i+1}} - \frac{\hat{a}_{n,i+1} \lambda_i}{(i+1) a_{i+1,i+1}} \\
&= \Delta_i \left(\frac{\hat{a}_{ni}}{i a_{ii}} \right) \lambda_i + \frac{\hat{a}_{n,i+1}}{(i+1) a_{i+1,i+1}} \Delta(\lambda_i).
\end{aligned} \tag{8}$$

Also we may write

$$\begin{aligned}
\Delta_i \left(\frac{\hat{a}_{ni}}{i a_{ii}} \right) \lambda_i &= \frac{\hat{a}_{ni}}{i a_{ii}} \lambda_i - \frac{\hat{a}_{n,i+1}}{(i+1) a_{i+1,i+1}} \lambda_i - \frac{\hat{a}_{n,i+1}}{i a_{ii}} \lambda_i + \frac{\hat{a}_{n,i+1}}{i a_{ii}} \lambda_i \\
&= \frac{\Delta_i(\hat{a}_{ni}) \lambda_i}{i a_{ii}} + a_{n,i+1} \lambda_i \left(\frac{1}{i a_{ii}} - \frac{1}{(i+1) a_{i+1,i+1}} \right).
\end{aligned} \tag{9}$$

Hence,

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{i=0}^{n-1} \frac{\Delta_i(\hat{a}_{ni})}{i a_{ii}} \lambda_i s_i + \sum_{i=0}^{n-1} \hat{a}_{n,i+1} \lambda_i \left(\frac{1}{i a_{ii}} - \frac{1}{(i+1) a_{i+1,i+1}} \right) s_i \\
&\quad + \sum_{i=0}^{n-1} \frac{\hat{a}_{n,i+1}}{(i+1) a_{i+1,i+1}} \Delta_i(\lambda_i) s_i + \frac{\lambda_n}{n} s_n \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \quad \text{say.}
\end{aligned} \tag{10}$$

To finish the proof of the theorem, it will be sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} |T_{nr}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4. \tag{11}$$

Using Hölder's inequality and (iii),

$$\begin{aligned}
 I_1 &= \sum_{n=1}^{m+1} n^{k-1} |T_{n1}|^k \leq \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} \left| \frac{\Delta_i(\hat{a}_{ni})}{ia_{ii}} \lambda_i s_i \right| \right)^k \\
 &= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\Delta_i(\hat{a}_{ni}) \lambda_i s_i| \right)^k \tag{12} \\
 &= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\Delta_i(\hat{a}_{ni})| |\lambda_i|^k |s_i|^k \right) \left(\sum_{i=0}^{n-1} |\Delta_i(\hat{a}_{ni})| \right)^{k-1}.
 \end{aligned}$$

But using (ii),

$$\Delta_i(\hat{a}_{ni}) = \hat{a}_{ni} - \hat{a}_{n,i+1} = \bar{a}_{ni} - \bar{a}_{n-1,i} - \bar{a}_{n,i+1} + \bar{a}_{n-1,i+1} = a_{ni} - a_{n-1,i} \leq 0. \tag{13}$$

Thus using (i),

$$\sum_{i=0}^{n-1} |\Delta_i(\hat{a}_{ni})| = \sum_{i=0}^{n-1} |a_{n-1,i} - a_{ni}| = 1 - 1 + a_{nn} = a_{nn}. \tag{14}$$

From (viii), it follows that $\lambda_n = O(1)$. Using (iii), (vi), (x), and property (5) of Lemma 2,

$$\begin{aligned}
 I_1 &= O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=0}^{n-1} |\lambda_i|^k |s_i|^k |\Delta_i(\hat{a}_{ni})| \\
 &= O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \left(\sum_{i=0}^{n-1} |\lambda_i|^{k-1} |\lambda_i| |\Delta_i(\hat{a}_{ni})| |s_i|^k \right) \\
 &= O(1) \sum_{i=0}^m |\lambda_i| |s_i|^k \sum_{n=i+1}^{m+1} (na_{nn})^{k-1} |\Delta_i(\hat{a}_{ni})| \\
 &= O(1) \sum_{i=0}^m |\lambda_i| |s_i|^k a_{ii} = |\lambda_0| |s_0|^k a_{00} + O(1) \sum_{i=1}^m \frac{|\lambda_i| |s_i|^k}{i} \\
 &= O(1) + O(1) \sum_{i=1}^m |\lambda_i| \left[\sum_{r=1}^i \frac{|s_r|^k}{r} - \sum_{r=1}^{i-1} \frac{|s_r|^k}{r} \right] \\
 &= O(1) \left[\sum_{i=1}^m |\lambda_i| \sum_{r=1}^i \frac{|s_r|^k}{r} - \sum_{j=0}^{m-1} |\lambda_{j+1}| \sum_{r=1}^j \frac{|s_r|^k}{r} \right] \\
 &= O(1) \sum_{i=1}^{m-1} \Delta(|\lambda_i|) \sum_{r=1}^i \frac{1}{r} |s_r|^k + O(1) |\lambda_m| \sum_{i=1}^m \frac{|s_i|^k}{i}
 \end{aligned}$$

$$\begin{aligned}
&= O(1) \sum_{i=1}^{m-1} \Delta(|\lambda_i|) X_i + O(1) |\lambda_m| X_m \\
&= O(1) \sum_{i=1}^m \beta_i X_i + O(1) |\lambda_m| X_m = O(1), \\
I_2 &= \sum_{n=1}^{m+1} n^{k-1} |T_{n2}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=0}^{n-1} \hat{a}_{n,i+1} \lambda_i \Delta\left(\frac{1}{ia_{ii}}\right) s_i \right|^k \\
&= O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\lambda_i| \Delta\left(\frac{1}{ia_{ii}}\right) |s_i| \right\}^k.
\end{aligned} \tag{15}$$

Now

$$\begin{aligned}
\Delta\left(\frac{1}{ia_{ii}}\right) &= \frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}} \\
&= \frac{1}{ia_{ii}} - \frac{1}{(i+1)a_{i+1,i+1}} + \frac{1}{(i+1)a_{ii}} - \frac{1}{(i+1)a_{ii}} \\
&= \frac{1}{(i+1)} \left(\frac{1}{a_{ii}} - \frac{1}{a_{i+1,i+1}} \right) + \frac{1}{a_{ii}} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\
&= \frac{1}{(i+1)} \left[\Delta\left(\frac{1}{a_{ii}}\right) + \frac{1}{ia_{ii}} \right].
\end{aligned} \tag{16}$$

Thus using (iv) and (ii),

$$\begin{aligned}
\left| \Delta\left(\frac{1}{ia_{ii}}\right) \right| &= \left| \frac{1}{i+1} \left[\Delta\left(\frac{1}{a_{ii}}\right) + \frac{1}{ia_{ii}} \right] \right| \leq \frac{1}{i+1} \left\{ \frac{|a_{i+1,i+1} - a_{ii}|}{|a_{ii}a_{i+1,i+1}|} + \frac{1}{ia_{ii}} \right\} \\
&= \frac{1}{i+1} [O(1) + O(1)].
\end{aligned} \tag{17}$$

Hence, using Hölder's inequality, (v) and (iii),

$$\begin{aligned}
I_2 &= O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\lambda_i| \frac{1}{i+1} |s_i| \right\}^k \\
&= O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |a_{ii}| |\lambda_i| |s_i| \right\}^k \\
&= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |a_{ii}| |\lambda_i|^k |s_i|^k \right) \left(\sum_{i=0}^{n-1} |a_{ii}| |\hat{a}_{n,i+1}| \right)^{k-1} \\
&= O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |a_{ii}| |\lambda_i|^k |s_i|^k
\end{aligned}$$

$$\begin{aligned}
 &= O(1) \sum_{i=0}^m |\lambda_i|^k |s_i|^k a_{ii} \sum_{n=i+1}^{m+1} (na_{nn})^{k-1} |\hat{a}_{n,i+1}| \\
 &= O(1) \sum_{i=0}^m |\lambda_i|^k |s_i|^k a_{ii} \sum_{n=i+1}^{m+1} |\hat{a}_{n,i+1}|.
 \end{aligned}
 \tag{18}$$

From [4],

$$\sum_{n=i+1}^{m+1} |\hat{a}_{n,i+1}| \leq 1.
 \tag{19}$$

Hence,

$$I_2 = O(1) \sum_{i=1}^m |\lambda_i|^k |s_i|^k a_{ii} = O(1) \sum_{i=1}^m |\lambda_i| |\lambda_i|^{k-1} |s_i|^k \frac{1}{i} = \sum_{i=1}^m |\lambda_i| \frac{|s_i|^k}{i} = O(1),
 \tag{20}$$

as in the proof of I_1 .

Using (iii), Hölder’s inequality, and (v),

$$\begin{aligned}
 I_3 &= \sum_{n=1}^{m+1} n^{k-1} |T_{n3}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \sum_{i=0}^{n-1} \frac{\hat{a}_{n,i+1}(\Delta\lambda_i)s_i}{(i+1)a_{i+1,i+1}} \right|^k \\
 &= O(1) \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\Delta\lambda_i| |s_i| \right)^k \\
 &= O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=0}^{n-1} \frac{a_{ii}}{a_{ii}} |\hat{a}_{n,i+1}| |\Delta\lambda_i| |s_i| \right\}^k \\
 &= O(1) \sum_{n=1}^{m+1} n^{k-1} \left\{ \sum_{i=0}^{n-1} a_{ii} \frac{|\hat{a}_{n,i+1}|}{a_{ii}^k} |\Delta\lambda_i|^k |s_i|^k \right\} \left\{ \sum_{i=0}^{n-1} a_{ii} |\hat{a}_{n,i+1}| \right\}^{k-1} \\
 &= O(1) \sum_{n=1}^{m+1} (na_{nn})^{k-1} \sum_{i=0}^{n-1} a_{ii} \frac{|\hat{a}_{n,i+1}|}{a_{ii}^k} |\Delta\lambda_i|^k |s_i|^k \\
 &= O(1) \sum_{n=1}^{m+1} \sum_{i=0}^{n-1} |\hat{a}_{n,i+1}| |\Delta\lambda_i|^k |s_i|^k \frac{1}{a_{ii}^k} a_{ii} \\
 &= O(1) \sum_{i=0}^m \frac{a_{ii}}{a_{ii}^k} |\Delta\lambda_i|^k |s_i|^k \sum_{n=i+1}^{m+1} |\hat{a}_{n,i+1}| \\
 &= O(1) \sum_{i=0}^m \left(\frac{|\Delta\lambda_i|}{a_{ii}} \right)^{k-1} |\Delta\lambda_i| |s_i|^k \\
 &= O(1) \sum_{i=0}^m |\Delta\lambda_i| |s_i|^k = O(1) \sum_{i=0}^m |s_i|^k \beta_i.
 \end{aligned}
 \tag{21}$$

Since $|s_i|^k = i(T_i - T_{i-1})$ by (x), we have

$$I_3 = O(1) \sum_{i=1}^m i(T_i - T_{i-1})\beta_i. \quad (22)$$

Using Abel's transformation, (vi), and (5),

$$\begin{aligned} I_3 &= O(1) \sum_{i=1}^{m-1} T_i \Delta(i\beta_i) + O(1)mT_n\beta_n \\ &= O(1) \sum_{i=1}^{m-1} i|\Delta\beta_i|X_i + O(1) \sum_{i=1}^{m-1} X_i\beta_i + O(1)mX_n\beta_n = O(1). \end{aligned} \quad (23)$$

Using (viii) and (x),

$$\begin{aligned} I_4 &= \sum_{n=1}^{m+1} n^{k-1} |T_{n4}|^k = \sum_{n=1}^{m+1} n^{k-1} \left| \frac{s_n \lambda_n}{n} \right|^k = \sum_{n=1}^{m+1} |s_n|^k |\lambda_n|^k \frac{1}{n} \\ &= \sum_{n=1}^{m+1} \frac{|s_n|^k}{n} |\lambda_n| |\lambda_n|^{k-1} = O(1), \end{aligned} \quad (24)$$

as in the proof of I_1 . □

COROLLARY 3. Let $\{p_n\}$ be a positive sequence such that $P_n = \sum_{k=0}^n p_k \rightarrow \infty$ and satisfies

- (i) $np_n \asymp O(P_n)$;
- (ii) $\Delta(P_n/p_n) = O(1)$.

If $\{X_n\}$ is a positive nondecreasing sequence and the sequences $\{\lambda_n\}$ and $\{\beta_n\}$ are such that

- (iii) $|\Delta\lambda_n| \leq \beta_n$,
- (iv) $\beta_n \rightarrow 0$ as $n \rightarrow \infty$,
- (v) $|\lambda_n|X_n = O(1)$ as $n \rightarrow \infty$,
- (vi) $\sum_{n=1}^{\infty} nX_n|\Delta\beta_n| < \infty$,
- (vii) $T_n = \sum_{\nu=1}^n |s_\nu|^k/\nu = O(X_n)$,

then the series $\sum (a_n P_n \lambda_n)/(np_n)$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Proof. Conditions (iii)–(vii) of Corollary 3 are, respectively, conditions (vi)–(x) of Theorem 1.

Conditions (i), (ii), and (v) of Theorem 1 are automatically satisfied for any weighted mean method. Conditions (iii) and (iv) of Theorem 1 become, respectively, conditions (i) and (ii) of Corollary 3. □

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